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ALMOST CONTRACTIVE RETRACTIONS IN ORLICZ SPACES

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Let B_k denote the Euclidean unit ball in $\mathbb{R}^k$ equipped with the k-dimensional Lebesgue measure and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function satisfying $\phi(0) = 0$, $\phi(t) > 0$ for some $t > 0$. Denote by $E_\phi = E_\phi(B_k)$ the Orlicz space of finite elements (see (1.6)) generated by ϕ. The aim of this paper is to show that there exists a retraction of the closed unit ball in E_ϕ onto the unit sphere in E_ϕ being a $(2+\varepsilon)\gamma_\phi$-set contraction (Theorem 3.6), which generalises [**9**, Corollary 6] proved for the case of $L_p[-1,1]$, $1 \leqslant p < \infty$. Here γ_ϕ denote the Hausdorff measure of noncompactness. This theorem is proved both for the Amemiya and the Luxemburg norms. Also some related results concerning the case of s-convex $(0 < s \leqslant 1)$ functions are presented.

1. INTRODUCTION

Let X be a Banach space with the closed unit ball B and the unit sphere S. A continuous mapping $R : B \to S$ is called *a retraction* if $Rx = x$ for any $x \in S$. Let ψ be a measure of noncompactness defined of X. A mapping $T : X \supset D(T) \to X$ is called a ψ *k-set contraction* if there exists $k \geqslant 0$ such that

$$\psi\big(T(A)\big) \leqslant k\psi(A)$$

for any bounded set $A \subset D(T)$. Set

$$k_L(X) = \inf\{k \geqslant 1 : \text{ there exists a } k\text{-Lipschizian retraction } R : B \to S\}. \tag{1.1}$$

and

$$k_\psi(X) = \inf\big\{k \geqslant 1 : \text{ there exists a } (\psi) \text{ k-set contractive retraction } R : B \to S\big\}. \tag{1.2}$$

By [**2**] and [**6**] for any infinite-dimensional Banach space X, $k_L(X) < \infty$. By [**3**], $k_L(X) \geqslant 3$ for any Banach space X. Also it is easy to see that $k_\gamma(X) \leqslant k_L(X)$, where γ denotes the Hausdorff measure of noncompactness, that is,

$$\gamma(A) = \inf\left\{r > 0 : A \subset \bigcup_{i=1}^{k} B(x_i, r), x_1, \dots, x_k \in X\right\}, \tag{1.3}$$

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where A is a bounded subset of X and $B(x_i, r)$ denote the closed ball with a centre x_i and radius r. For more complete information about measures of noncompactness and (ψ) k-sets contractions the reader is referred to [**1, 3, 7, 8, 10**]. Moreover, it has been shown in [**10**] that $k_\gamma(C_{\mathbb{R}}[0,1]) \leqslant 1$ and in [**9**, Corollary 6] that $k_\gamma(L_p[-1,1]) \leqslant 2$.

The aim of this paper is to generalise the above mentionned results to the case of Orlicz spaces (see Theorem 3.6). We consider both the Luxemburg and Amemiya norms (see (1.7) and (1.8)).

Also we prove some results for Orlicz spaces generated by s-convex functions. In particular, in s-convex case, we introduce a kind of measure of non-compactness ω_ϕ^s and $\omega_{\phi,A}^s$ (see Definition 2.8) which is an analogue of the measure of noncompactness ω_p considered in [**9**]. We show that in the convex case $(s = 1)$

$$\omega_\phi^1/2 \leqslant \gamma_\phi^1 \leqslant \omega_\phi^1 \text{ and } \omega_{\phi,A}^1/2 \leqslant \gamma_{\phi,A}^1 \leqslant \omega_{\phi,A}^1,$$

where γ_ϕ^1 ($\gamma_{\phi,A}^1$, respectively) is the Hausdorff measure of noncompactness associated with the Luxemburg norm (with the Amemiya norm, respectively), which generalises the classical result concerning L_p-spaces (see for example, [**4**]). Also we show that $k_{\omega_\phi^s}(E_\phi) \leqslant 1$ (Theorem 3.3) and $K_{\omega_{\phi,A}^s}(E_\phi) \leqslant 1$, (Theorem 3.4), which generalises [**9**, Theorem 5].

Now we present some basic facts concerning Orlicz spaces. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous, s-convex function such that $\phi(0) = 0$ and $\phi(t) > 0$ for some $t > 0$. Recall that a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called s-convex for some $0 < s \leqslant 1$, if

$$\phi(ax + by) \leqslant a^s\phi(x) + b^s\phi(y)$$

for $x, y \in \mathbb{R}^+$, $a, b \geqslant 0$ $a^s + b^s = 1$. Observe that for $s = 1$, we get the class of convex functions. Let (Ω, Σ, μ) be a measure space. Denote by $\mathcal{M} = \mathcal{M}(\Omega, \mathbb{R})$ the set of all real-valued μ-measurable functions defined on Ω. For $f \in \mathcal{M}$ set

$$\rho_\phi(f) = \int_\Omega \phi\Big(\big|f(t)\big|\Big)\, d\mu(t). \tag{1.4}$$

By $L_\phi = L_\phi(\Omega, \Sigma, \mu)$ we denote the Orlicz space generated by ϕ, that is,

$$L_\phi = \big\{f \in \mathcal{M} : \lim_{\lambda\to 0} \rho_\phi(\lambda f) = 0\big\}. \tag{1.5}$$

By E_ϕ we denote the space of finite elements, that is,

$$E_\phi = \big\{f \in \mathcal{M} : \rho_\phi(\lambda f) < \infty \text{ for any } \lambda > 0\big\}. \tag{1.6}$$

It is well-known that E_ϕ is a closed subspace of L_ϕ. Moreover, $L_\phi = E_\phi$ if and only if the appropriate Δ_2 condition holds true (see for example, [**5**, Theorem 8.14, p. 53]). If

ϕ is an s-convex function, we can equip L_ϕ with an s-convex norm (norm if $s = 1$), given by

$$\|f\|_\phi^s = \inf\left\{u > 0 : \rho_\phi(f/u^{1/s}) \leqslant 1\right\}. \tag{1.7}$$

named the Luxemburg s-norm (norm if $s = 1$).

Observe that, if $\phi(t) = t^p$ where $1 \leqslant p < \infty$, then $L_\phi = E_\phi = L_p$ and $\|f\|_\phi^1 = \|f\|_p$. If $0 < p < 1$, then ϕ is a p-convex function and $\|f\|_\phi^p = \int_\Omega |f(t)|^p d\mu(t)$.

It is also well-known, (see for example, [**5**,Theorem 1.10, p. 6]) that in s-convex case, we can consider in L_ϕ the other s-norm (norm if $s = 1$), called the Amemiya s-norm defined by

$$|f|_\phi^s = \inf\left\{\left(1 + \rho_\phi(u^{1/s} f)\right)/u : u > 0\right\}. \tag{1.8}$$

For more detailed information about Orlicz spaces see for example, [**5**].

2. Technical lemmas

DEFINITION 2.1: Let B_k denote the unit Euclidean ball in $\mathbb{R}^k$. For $x \in \mathbb{R}^k$ denote by $\|x\|_e$ the Euclidean norm of x. Let Σ_k be the set of all Borel subsets of B_k equipped with the Lebesgue measure μ_k. Define for $f \in L_\phi = L_\phi(B_k, \Sigma_k, \mu_k)$ $\|f\|_\phi^s \leqslant 1$,

$$(Q_\phi^s f)(t) = \begin{cases} f\left(\dfrac{2}{1 + \|f\|_\phi^s} t\right) & \text{if } \|t\|_e \leqslant \dfrac{1 + \|f\|_\phi^s}{2} \\ 0 & \text{if } \|t\|_e > \dfrac{1 + \|f\|_\phi^s}{2} \end{cases}$$

Analogously, for $f \in L_\phi$, $|f|_\phi^s \leqslant 1$, set

$$(Q_{\phi,A}^s f)(t) = \begin{cases} f\left(\dfrac{2}{1 + |f|_\phi^s} t\right) & \text{if } \|t\|_e \leqslant \dfrac{1 + |f|_\phi^s}{2} \\ 0 & \text{if } \|t\|_e > \dfrac{1 + |f|_\phi^s}{2} \end{cases}$$

LEMMA 2.2. *For any $\lambda > 0$, $f \in L_\phi$ with $\|f\|_\phi^s \leqslant 1$*

$$\rho_\phi(\lambda Q_\phi^s(f)) = \frac{\left(1 + \|f\|_\phi^s\right)^k}{2^k} \rho_\phi(\lambda f). \tag{2.1}$$

Moreover, for any $b \geqslant 1$,

$$\rho_\phi(b\lambda Q_\phi^s(f)) \geqslant b^s \frac{\left(1 + \|f\|_\phi^s\right)^k}{2^k} \rho_\phi(\lambda f). \tag{2.2}$$

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The same results hold true for the Amemiya s-norm $|\cdot|_\phi^s$.

PROOF: Observe that for any $\lambda > 0$

$$\rho_\phi(\lambda Q_\phi^s(f)) = \int_{\|t\|_e \leqslant (1+\|f\|_\phi^s)/2} \phi\Big(\lambda \Big| f\Big(\frac{2}{1+\|f\|_\phi^s} t\Big)\Big|\Big)\, d\mu_k(t).$$

Set $g(t) = \big(2/(1+\|f\|_\phi^s)\big)t$. Note that by changing variables from t to $g(t)$

$$\begin{aligned}\rho_\phi(\lambda Q_\phi^s(f)) &= \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \int_{g^{-1}(B_k)} \phi\Big(\lambda\Big|f\Big(\frac{2}{1+\|f\|_\phi^s}t\Big)\Big|\Big)|\det g'(t)|d\mu_k(t)\\ &= \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \int_{B_k} \phi\big(\lambda|f(s)|\big)d\mu_k(s) = \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \rho_\phi(\lambda f),\end{aligned}$$

which proves our claim. The same reasoning applies to the operator $Q_{\phi,A}^s$ and any function $f \in L_\phi$, $|f|_\phi^s \leqslant 1$.

Now fix $b \geqslant 1$. Then $\phi(bx) \geqslant b^s\phi(x)$ for any $x \in \mathbb{R}^+$. Hence

$$\rho_\phi(b\lambda Q_\phi^s(f)) = \int_{B_k} \phi\Big(b\lambda\Big|f\Big(\frac{2}{1+\|f\|_\phi^s}t\Big)\Big|\Big)\, d\mu_k(t) \geqslant b^s\rho_\phi(Q_\phi^s(\lambda f)).$$

By the previous part of the proof,

$$b^s\rho_\phi(Q_\phi^s(\lambda f)) = b^s\Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \rho_\phi(\lambda f),$$

which shows (2.2).

Reasoning in the same way, we can show that

$$\rho_\phi(b\lambda Q_{\phi,A}^s(f)) \geqslant b^s \frac{\big(1+|f|_\phi^s\big)^k}{2^k}\rho_\phi(\lambda f).$$

□

LEMMA 2.3. *For any* $f \in L_\phi$, $\|f\|_\phi^s \leqslant 1$,

$$\Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \|f\|_\phi^s \leqslant \|Q_\phi^s(f)\|_\phi^s \leqslant \|f\|_\phi^s.$$

The same result holds true for $|\cdot|_\phi^s$ *and* $Q_{\phi,A}^s$.

PROOF: Take any $u > \|f\|_\phi^s$. By (1.7) and Lemma 2.2 applied to $\lambda = 1/u^{1/s}$, we get

$$\rho_\phi(Q_\phi^s(f)/u^{1/s}) = \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \rho_\phi\big(f/u^{1/s}\big) \leqslant \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \leqslant 1.$$

Hence $\|Q_\phi^s(f)\|_\phi^s \leqslant \|f\|_\phi^s$.

Now take any $f \in L_\phi$, $|f|_\phi^s \leqslant 1$. By Lemma 2.2 and (1.8) one can easily show that $|Q_{\phi,A}^s(f)|_\phi^s \leqslant |f|_\phi^s$.

Now we prove the second inequality. Let $u = d\Big(\big(1+\|f\|_\phi^s\big)/2\Big)^k \|f\|_\phi^s$ for some $d \in (0,1)$. Then by Lemma 2.2, and s-convexity of ϕ,

$$\begin{aligned}\rho_\phi\Big(Q_\phi^s(f)/u^{1/s}\Big) &= \rho_\phi\left(\Big(\frac{2}{1+\|f\|_\phi^s}\Big)^{k/s} Q_\phi^s(f)/\big(d\|f\|_\phi^s\big)^{1/s}\right)\\ &\geqslant \Big(\frac{2}{1+\|f\|_\phi^s}\Big)^k \rho_\phi\Big(Q_\phi^s(f)/\big(d\|f\|_\phi^s\big)^{1/s}\Big)\\ &= \rho_\phi\Big(f/\big(d\|f\|_\phi^s\big)^{1/s}\Big) > 1.\end{aligned}$$

Since d can be an arbitrary number from $(0,1)$,

$$\|Q_\phi^s(f)\|_\phi^s \geqslant \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \|f\|_\phi^s,$$

as required.

Finally, we consider the case of $|\cdot|_\phi^s$. Take any $u > 0$. Observe that by Lemma 2.2,

$$\begin{aligned}\frac{1+\rho_\phi\big(u^{1/s}Q_{\phi,A}^s(f)\big)}{u} &= \frac{1+\Big(\big(1+|f|_\phi^s\big)/2\Big)^k \rho_\phi\big(u^{1/s}f\big)}{u}\\ &\geqslant \Big(\frac{1+|f|_\phi^s}{2}\Big)^k\Big(\frac{1+\rho_\phi\big(u^{1/s}f\big)}{u}\Big) \geqslant \Big(\frac{1+|f|_\phi^s}{2}\Big)^k |f|_\phi^s.\end{aligned}$$

Hence taking infimum over $u > 0$, we get

$$|Q_{\phi,A}^s(f)|_\phi^s \geqslant \Big(\frac{1+|f|_\phi^s}{2}\Big)^k |f|_\phi^s,$$

which completes the proof. ∎

Lemma 2.4. *Let (f_n) be a sequence of functions from E_ϕ (see (1.6)) and let $f \in E_\phi$, $\|f_n\|_\phi^s \leqslant 1$ for any $n \in \mathbb{N}$ and $\|f\|_\phi^s \leqslant 1$. If for any $\lambda > 0$,*

$$\rho_\phi\big(\lambda(f_n - f)\big) \to 0,$$

then

$$\rho_\phi\Big(\lambda\big(Q_\phi^s(f_n) - Q_\phi^s(f)\big)\Big) \to 0$$

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for any $\lambda > 0$. *Analogously, if* $|f_n|^s_\phi \leqslant 1$ *for any* $n \in \mathbb{N}$ *and* $|f|^s_\phi \leqslant 1$, *then*

$$\rho_\phi\Big(\lambda\big(Q^s_{\phi,A}(f_n) - Q^s_{\phi,A}(f)\big)\Big) \to 0$$

for any $\lambda > 0$.

PROOF: Note that by [**5**, Theorem 6, p. 3],

$$\rho_\phi(\lambda(f_n - f)) \to 0,$$

if and only if $\|f_n - f\|^s_\phi \to 0$ and

$$\rho_\phi\Big(\lambda\big(Q^s_\phi(f_n) - Q^s_\phi(f)\big)\Big) \to 0$$

if and only if $\left\|Q^s_\phi(f_n) - Q^s_\phi(f)\right\|^s_\phi \to 0$. Observe that

$$\left\|Q^s_\phi(f_n) - Q^s_\phi(f)\right\|^s_\phi \leqslant \left\|Q^s_\phi(f_n) - Q^s_{\phi,n}(f)\right\|^s_\phi + \left\|Q^s_{\phi,n}(f) - Q^s_\phi(f)\right\|^s_\phi,$$

where

$$(Q^s_{\phi,n}f)(t) = \begin{cases} f\Big(\dfrac{2}{1+\|f_n\|^s_\phi}t\Big) & \text{if } \|t\|_e \leqslant \dfrac{1+\|f_n\|^s_\phi}{2} \\ 0 & \text{if } \|t\|_e > \dfrac{1+\|f_n\|^s_\phi}{2} \end{cases}$$

By Lemma 2.2, for any $\lambda > 0$,

$$\rho_\phi\Big(\lambda\big(Q^s_\phi(f_n) - Q^s_{\phi,n}(f)\big)\Big) = \Big(\frac{1+\|f_n\|^s_\phi}{2}\Big)^k \rho_\phi(\lambda(f_n - f)).$$

Hence $\left\|Q^s_\phi(f_n) - Q^s_{\phi,n}(f)\right\|^s_\phi \to 0$. To end the proof, it is sufficient to show that $\left\|Q^s_{\phi,n}(f) - Q^s_\phi(f)\right\|^s_\phi \to 0$. To do this, fix $\varepsilon > 0$. Since $f \in E_\phi$, there exists a continuous function $g : B_k \to \mathbb{R}$, such that $\|g\|^s_\phi = \|f\|^s_\phi$ and $\|g - f\|^s_\phi \leqslant \varepsilon$. Observe that

$$\begin{aligned}\left\|Q^s_{\phi,n}(f) - Q^s_\phi(f)\right\|^s_\phi &\leqslant \left\|Q^s_{\phi,n}(f) - Q^s_{\phi,n}(g)\right\|^s_\phi \\ &\quad + \left\|Q^s_{\phi,n}(g) - Q^s_\phi(g)\right\|^s_\phi + \left\|Q^s_\phi(g) - Q^s_\phi(f)\right\|^s_\phi.\end{aligned}$$

Note that by Lemma 2.2, for any $\lambda > 0$ and $n \in \mathbb{N}$,

$$\rho_\phi\Big(\lambda\big(Q^s_{\phi,n}(f) - Q^s_{\phi,n}(g)\big)\Big) = \Big(\frac{1+\|f_n\|^s_\phi}{2}\Big)^k \rho_\phi(\lambda(f - g)).$$

Hence, by (1.7),

$$\left\|Q^s_{\phi,n}(f) - Q^s_{\phi,n}(g)\right\|^s_\phi \leqslant \|f - g\|^s_\phi \leqslant \varepsilon.$$

Analogously, since $\|f\|_\phi^s = \|g\|_\phi^s$,

$$\left\|Q_\phi^s(f) - Q_\phi^s(g)\right\|_\phi^s \leqslant \|f-g\|_\phi^s \leqslant \varepsilon.$$

Now we show that

$$\left\|Q_{\phi,n}^s(g) - Q_\phi^s(g)\right\|_\phi^s \to 0.$$

Let $\|g\|_\infty$ denote the supremum norm of g. Fix $\lambda > 0$ and set

$$D_n = \left\{t \in S_k : \frac{1+\|f\|_\phi^s}{2} \leqslant \|t\|_e \leqslant \frac{1+\|f_n\|_\phi^s}{2}\right\}.$$

If $\|f_n\|_\phi^s \geqslant \|f\|_\phi^s$, then by Lemma 2.2,

$$\begin{aligned}
&\int_{B_k} \phi\Big(\lambda\Big|(Q_{\phi,n}^s(g)) - (Q_\phi^s(g))(t)\Big|\Big)d\mu_k(t)\\
&\quad = \int_{\|t\|_e \leqslant \frac{1+\|f\|_\phi^s}{2}} \phi\Big(\lambda\Big|g\Big(\frac{2}{1+\|f_n\|_\phi^s}t\Big)\\
&\qquad - g\Big(\frac{2}{1+\|f\|_\phi^s}t\Big)\Big|\Big)d\mu_k(t) + \int_{D_n} \phi\Big(\lambda\Big|g\Big(\frac{2}{1+\|f_n\|_\phi^s}t\Big)d\mu_k(t)\Big|\Big)\\
&\quad \leqslant \Big(\frac{1+\|f\|_\phi^s}{2}\Big)^k \int_{B_k} \phi\Big(\lambda\Big|g\Big(\frac{1+\|f\|_\phi^s}{1+\|f_n\|_\phi^s}t\Big) - g(t)\Big|\Big)d\mu_k(t) + \mu_k(D_n)\phi(\lambda\|g\|_\infty).
\end{aligned} \tag{2.3}$$

Analogously, if $\|f_n\|_\phi^s < \|f\|_\phi^s$, then

$$\begin{aligned}
&\int_{B_k} \phi\Big(\lambda\Big|(Q_{\phi,n}^s(g)) - (Q_\phi^s(g))(t)\Big|\Big)d\mu_k(t)\\
&\quad \leqslant \Big(\frac{1+\|f_n\|_\phi^s}{2}\Big)^k \int_{B_k} \phi\Big(\lambda\Big|g\Big(\frac{1+\|f_n\|_\phi^s}{1+\|f\|_\phi^s}t\Big) - g(t)\Big|\Big)d\mu_k(t) + \mu_k(C_n)\phi(\lambda\|g\|_\infty),
\end{aligned} \tag{2.4}$$

where

$$C_n = \left\{t \in S_k : \frac{1+\|f_n\|_\phi^s}{2} \leqslant \|t\|_e \leqslant \frac{1+\|f\|_\phi^s}{2}\right\}.$$

Set for $n \in \mathbb{N}$

$$h_n(t) = \begin{cases} \phi\Big(\lambda\Big|g\Big(\frac{1+\|f\|_\phi^s}{1+\|f_n\|_\phi^s}(t)\Big) - g(t)\Big|\Big) & \text{if } \|f_n\|_\phi^s \geqslant \|f\|_\phi^s \\ \phi\Big(\lambda\Big|g\Big(\frac{1+\|f_n\|_\phi^s}{1+\|f\|_\phi^s}(t)\Big) - g(t)\Big|\Big) & \text{if } \|f_n\|_\phi^s < \|f\|_\phi^s \end{cases}$$

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Note that by the continuity of g, $h_n(t) \to 0$ for any $t \in B_k$. Moreover, $|h_n(t)| \leqslant \phi(2\lambda\|g\|_\infty)$ for any $t \in B_k$. Since $\mu_k(B_k) < \infty$, by the Lebesgue dominated convergence theorem

$$\int_{B_k} h_n(t)\, d\mu_k(t) \to 0. \tag{2.5}$$

Since, $\mu_k(C_n) \to 0$ and $\mu_k(D_n) \to 0$, by (2.3) - (2.5), for any $\lambda > 0$,

$$\rho_\phi\Big(\lambda\big(Q^s_{\phi,n}(g) - Q^s_\phi(g)\big)\Big) \to 0$$

and consequently $\big\|Q^s_{\phi,n}(g) - Q^s_\phi(g)\big\|^s_\phi \to 0$.

Reasoning in the same way, we can show that for any $\lambda > 0$,

$$\rho_\phi\Big(\lambda\big(Q^s_{\phi,A}(f_n) - Q^s_{\phi,A}(f)\big)\Big) \to 0.$$

The proof is complete. ∎

By Lemma 2.4 and [**5**, Theorem 6, p. 3] one can easily deduce

Corollary 2.5. *Let $f_n, f \in E_\phi$, $\|f_n\|^s_\phi \leqslant 1$ for any $n \in \mathbb{N}$ and $\|f\|^s_\phi \leqslant 1$. If $\|f_n - f\|^s_\phi \to 0$ then*

$$\big\|Q^s_\phi(f_n) - Q^s_\phi(f)\big\|^s_\phi \to 0.$$

Analogously if, $|f_n|^s_\phi \leqslant 1$ for any $n \in \mathbb{N}$, $|f|^s_\phi \leqslant 1$ and $|f_n - f|^s_\phi \to 0$ then

$$\big|Q^s_{\phi,A}(f_n) - Q^s_{\phi,A}(f)\big|^s_\phi \to 0.$$

Definition 2.6: Let $f \in L_\phi(B_k) \cap L_1(B_k)$. Set for $r > 0$, $B_k(r) = \Big\{t \in \mathbb{R}^k : \|t\|_e \leqslant r\Big\}$. For any $h > 0$, by f_h we denote the Steklov function of f, that is

$$f_h(t) = \left(\int_{t+B_k(h)} f(u)\, d\mu_k(u)\right) \Big/ \mu_k\big(B_k(h)\big).$$

Remark 2.7. Observe that if $f \in L_1(B_k)$, then f_h is well-defined and continuous for any $h > 0$. Hence $f_h \in E_\phi(B_k)$. Moreover, if ϕ is a convex function then $L_\phi(B_k) \subset L_1(B_k)$, (as a subset) since $\mu_k(B_k) < \infty$. Hence in this case $f_h \in E_\phi$ for any $h > 0$ and $f \in L_\phi$.

Definition 2.8: Let ϕ be an s-convex function and let $B \subset L_\phi(B_k) \cap L_1(B_k)$ be a bounded set with respect to the Luxemburg s-convex norm $\|\cdot\|^s_\phi$. Set

$$\omega^s_\phi(B) = \lim_{\delta \to 0}\Big(\sup_{f \in B}\ \sup_{0 < h \leqslant \delta} \|f - f_h\|^s_{\phi,\mathbb{R}^k}\Big),$$

where $\|\cdot\|^s_{\phi,\mathbb{R}^k}$ denotes the s-convex Luxemburg norm associated with a modular

$$\rho_{\phi,\mathbb{R}^k}(f) = \int_{\mathbb{R}^k} \phi\Big(\big|f(t)\big|\Big)\, d\mu_k(t).$$

In the case of the Amemiya s-norm $|\cdot|^s_\phi$ (see (1.8)) we put

$$\omega^s_{\phi,A}(B) = \lim_{\delta\to 0}\Big(\sup_{f\in B}\sup_{0<h\leqslant\delta} |f - f_h|^s_{\phi,\mathbb{R}^k}\Big).$$

Proposition 2.9. *Let ϕ be an s-convex function such that*

$$\phi^*(v) = \sup_{u>0}\big(uv - \phi(u)\big) \tag{2.6}$$

is finite for any $v \geqslant 0$. Let γ^s_ϕ ($\gamma^s_{\phi,A}$ respectively) denote the Hausdorff measure of noncompactness in $L_\phi(B_k, \Sigma_k, \mu_k)$ with respect to the Luxemburg s-norm (to the Amemiya s-norm respectively). Let $B \subset E_\phi$ (see (1.6)) be a bounded set. Then

$$\gamma^s_\phi(B) \leqslant \omega^s_\phi(B)$$

and

$$\gamma^s_{\phi,A}(B) \leqslant \omega^s_{\phi,A}(B).$$

Moreover, if $s = 1$, that is, ϕ is a convex function, then

$$\gamma^1_\phi(B) \geqslant \frac{\omega^1_\phi(B)}{2}$$

and

$$\gamma^1_{\phi,A}(B) \geqslant \frac{\omega^1_{\phi,A}(B)}{2}.$$

Proof: First we consider the case of the Luxemburg s-norm. Since $\phi^*(v) < +\infty$, for any $v \geqslant 0$, by (2.6),

$$\big|f(t)\big|/u^{1/s} \leqslant \phi\Big(\big|f(t)/u^{1/s}\big|\Big) + \phi^*(1),$$

for any $f \in L_\phi$, $t \in B_k$ and $u > \|f\|^s_\phi$. By integrating the above inequality, and (1.4) we easily get that $f \in L_1(B_k)$. Hence by Remark 2.7, $\omega^s_\phi(B)$ is well-defined for any bounded set $B \subset E_\phi(B_k)$ (see Definition 2.8).

Now fix $b > \omega^s_\phi(B)$ and take $\varepsilon \in \Big(0, \big(b - \omega^s_\phi(B)\big)/2\Big)$. Then there exists $\delta > 0$ such that $0 < h < \delta$,

$$\sup_{f\in B}\big(\|f - f_h\|^s_{\phi,\mathbb{R}^k}\big) < b - \varepsilon.$$

Fix $0 < h < \delta$ and let $S_h = \{f_h : f \in B\}$. Observe that by (2.6), for any $k \in \mathbb{N}$ and $u > 0$,

$$u \leqslant \phi(u)/k + \phi^*(k)/k.$$

Since $\phi^*(k) < +\infty$ for any $k \in \mathbb{N}$, reasoning as in [**5**, Theorem 9.11, p. 61], we can show that S_h is a conditionaly compact set in E_ϕ. Hence there exist $g_1, \dots, g_{n(\varepsilon)} \in E_\phi$ such that for any $f \in B$, $\|f_h - g_i\|_\phi^s \leqslant \varepsilon$ for some $i \in \{1, \dots, n(\varepsilon)\}$. Hence by the triangle inequality for any $f \in B$ there exists $i \in \{1, \dots, n(\varepsilon)\}$ such that

$$\|f - g_i\|_\phi^s \leqslant \|f - f_h\|_\phi^s + \|f_h - g_i\|_\phi^s \leqslant b.$$

Hence $\gamma_\phi^s(B) \leqslant b$ and consequently, $\gamma_\phi^s(B) \leqslant \omega_\phi^s(B)$, as required.

The same reasoning applies to the case of the Amemiya norm.

To prove the second part, assume that ϕ is a convex function. By [**5**,Theorem 9.10, p. 61] for any $f \in L_\phi(B_k)$, and $h > 0$,

$$\|f_h\|_{\phi,\mathbb{R}^k}^1 \leqslant \|f\|_{\phi,\mathbb{R}^k}^1 = \|f\|_\phi^1 \tag{2.7}$$

Fix $b > \gamma_\phi^1(B)$ and take $\varepsilon \in \left(0, \left(b - \gamma_\phi^1(B)\right)/2\right)$. Then we can find $g_1, \dots, g_{n(b)} \in E_\phi(B_k)$ such that for any $f \in B$, $\|f - g_i\|_\phi^1 \leqslant b - \varepsilon$, for some $i \in \{1, \dots, n(b)\}$. Since $B \subset E_\phi$, we can assume that g_i are continuous functions. Since B_k is a compact set, there exists $\delta > 0$ such that for any $h < \delta$ and $i = 1, \dots, n(b)$

$$\left\|g_i - (g_i)_h\right\|_{\phi,\mathbb{R}^k}^1 \leqslant \varepsilon. \tag{2.8}$$

Hence, by (2.7) and (2.8), for any $f \in B$ and $0 < h < \delta$

$$\begin{aligned}(\|f - f_h\|_{\phi,\mathbb{R}^k}^1)/2 &\leqslant \left(\|f - g_i\|_{\phi,\mathbb{R}^k}^1 + \left\|f_h - (g_i)_h\right\|_{\phi,\mathbb{R}^k}^1 + \left\|g_i - (g_i)_h\right\|_{\phi,\mathbb{R}^k}^1\right)/2 \\ &\leqslant \|f - g_i\|_\phi^1 + + \left(\left\|g_i - (g_i)_h\right\|_{\phi,\mathbb{R}^k}^1\right)/2 \leqslant b - \varepsilon + \varepsilon/2 < b.\end{aligned}$$

Consequently

$$\gamma_\phi^1(B) \geqslant \frac{\omega_\phi^1(B)}{2}$$

as required. The same reasoning applies to the case of the Amemiya norm. ∎

PROPOSITION 2.10. *For any s-convex function ϕ, satisfying the assumptions of Proposition 2.9, $\lambda \in \mathbb{R}$ and bounded subsets $C, D \subset E_\phi$ the following conditions are satisfied:*

(2.9) $\omega_\phi^s(C) = 0$ *implies C is conditionally compact* ;

(2.10) $\omega_\phi^s(\mathrm{cl}^s(C)) = \omega_\phi^s(C)$,

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where

$$\mathrm{cl}^s(C) = \mathrm{cl}\Big(C^s = \Big\{f \in E_\phi : f = \sum_{j=1}^{k} a_j f_j : f_j \in C, a_1, \dots ; a_k \geqslant 0, \sum_{j=1}^{k} a_j^s = 1\Big\}\Big);$$

$$\omega_\phi^s(C \cup D) = \max\{\omega_\phi^s(C), \omega_\phi^s(D)\}; \tag{2.11}$$

$$\omega_\phi^s(C + D) \leqslant \omega_\phi^s(C) + \omega_\phi^s(D); \tag{2.12}$$

$$\omega_\phi^s(\lambda C) = |\lambda|^s \omega_\phi^s(C). \tag{2.13}$$

Moreover, if ϕ is convex, then $\omega_\phi^1(C) = 0$ for any conditionally compact set C. The same properties holds true for $\omega_{\phi,A}^s$.

PROOF: Suppose that $\omega_\phi^s(C) = 0$. Then by Proposition 2.9, $\gamma_\phi^s(C) = 0$ and consequently $\mathrm{cl}(C)$ is a compact set. If ϕ is convex, then again by Proposition 2.9 $\omega_\phi^1(C) = 0$ for any conditionally compact set C.

To prove (2.10), first we show that for any fixed $h > 0$, if $\|f_n\|_{\phi,\mathbb{R}^k}^s \to 0$, then $\|(f_n)_h\|_{\phi,\mathbb{R}^k}^s \to 0$. Suppose, this is not true. Without loss of generality, we can assume $\|f_n\|_{\phi,\mathbb{R}^k}^s \to 0$, and $\|(f_n)_h\|_{\phi,\mathbb{R}^k}^s \geqslant d > 0$. By our assumptions, reasoning as in [**5**, Theorem 9.11, p. 61], we can show that the sequence $\{(f_n)_h\}$ contains a subsequence (we denote it again by $(f_n)_h$), tending uniformly on $\mathbb{R}^k$ to a function g continuous on $B_k(1+h)$ and equal 0 outside this set. By our assumptions on $\{(f_n)_h\}$, $g(t) \neq 0$ for some $t \in B_k(1+h)$. Observe that

$$\mu_k\Big(C_n = \Big\{s \in B_k(h) : |f_n(s+t) - g(t)| \geqslant |g(t)|/2\Big\}\Big) \to 0.$$

Consequently, $\mu_k(B_k(h) \setminus C_n) \to \mu_k(B_h)$. Note that

$$B_k(h) \setminus C_n \subset \Big\{s \in \mathbb{R}^k : |f_n(s+t)| \geqslant |g(t)|/2\Big\}.$$

Hence $\{f_n\}$ does not converge to 0 in measure. By [**5**, Lemma 9.2, p. 56], $\|f_n\|_{\phi,\mathbb{R}^k}^s$ does not converge to 0, a contradiction.

By the previous part of the proof, we get immediately for any bounded set $C \subset E_\phi$

$$\omega_\phi^s(\mathrm{cl}^s(C)) = \omega_\phi^s(C).$$

Now take any $f \in C^s, f = \sum_{j=1}^{n} a_j f_j$, where for $j = 1, \dots, n$ $f_j \in C$, $a_j \geqslant 0$, $\sum_{j=1}^{n} a_j^s = 1$. Observe that for any $h > 0$,

$$\|f - f_h\|_{\phi,\mathbb{R}^k}^s \leqslant \sum_{j=1}^{n} a_j^s \|f_j - (f_j)_h\|_{\phi,\mathbb{R}^k}^s \leqslant \max_{j=1,\dots,n} \|f_j - (f_j)_h\|_{\phi,\mathbb{R}^k}^s.$$

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Consequently,

$$\omega^s_\phi(\mathrm{cl}\,(C^s)) = \omega^s_\phi(C^s) \leqslant \omega^s_\phi(C).$$

Since the opposite inequality is obvious, (2.10) is proved. The other properties follow immediately from Definition 2.8.

The same proof applies to the case of $\omega^s_{\phi,A}$. ∎

Lemma 2.11. *For any $f \in L_\phi(B_k) \cap L_1(B_k)$, $\|f\|^s_\phi = \|f\|^s_{\phi,\mathbb{R}^k} \leqslant 1$, (we put $f(t) = 0$ for $\|t\|_e > 1$), $\lambda, h > 0$*

$$\int_{\mathbb{R}^k} \Big(\phi\Big(\lambda\big|(Q^s_\phi f)(t) - (Q^s_\phi f)_h(t)\big|\Big)\, d\mu_k(t)\Big) = \Big(\frac{1+\|f\|^s_\phi}{2}\Big)^k \int_{\mathbb{R}^k} \phi\Big(\lambda\big|(f - f_{h/a})(t)\big|\Big)\, d\mu_k(t),$$

where $a = \Big((1+\|f\|^s_\phi)/2\Big)$. The same result holds true for any $f \in L_\phi(B_k)$, $|f|^s_\phi \leqslant 1$ and $Q^s_{\phi,A}$ with $a = \Big((1+|f|^s_\phi)/2\Big)$.

Proof: Fix $f \in L_\phi$, $\|f\|^s_\phi \leqslant 1$ and $\lambda > 0$. Set for any $t \in R^k$, $u(t) = t/a$. Then after changing variables from t to u, we get

$$\begin{aligned}\int_{\mathbb{R}^k} \phi\Big(\lambda\big|(Q^s_\phi f)(t) - (Q^s_\phi f)_h(t)\big|\, d\mu_k(t)\Big) &= a^k \int_{\mathbb{R}^k} \phi\Big(\lambda\big|(Q^s_\phi f)(au) - (Q^s_\phi f)_h(au)\big|\, d\mu_k(u)\Big)\\ &= a^k \int_{\mathbb{R}^k} \phi\Big(\lambda\big|f(u) - (Q^s_\phi f)_h(au)\big| d\mu_k(u)\Big).\end{aligned}$$

To end the proof of the lemma, we show that $\Big(Q^s_\phi f\Big)_h(au) = f_{h/a}(u)$. Observe that

$$(Q^s_\phi f)_h(au) = \Big(\int_{au+B_k(h)} f(s/a)\chi_{B_k}(s/a)d\mu_k(s)\Big)\Big/\mu_k(B_k(h)),$$

where χ_{B_k} denotes the characteristic function of B_k. Set $z(s) = s/a$. After changing variables from s to z we get

$$\begin{aligned}(Q^s_\phi f)_h(au) &= a^k\Big(\int_{u+B_k(h/a)} f(z)\chi_{B_k}(z)d\mu_k(z)\Big)\Big/\mu_k(B_k(h))\\ &= \Big(\int_{u+B_k(h/a)} f(z)\chi_{B_k}(z)d\mu_k(z)\Big)\Big/\mu_k(B_k(h/a)) = (f_{h/a})(u).\end{aligned}$$

The same reasoning applies to the Amemiya s-norm. The proof is complete. ∎

Applying Lemma 2.11 and the definitions of the Luxemburg and the Amemiya s-norms one can easily get

COROLLARY 2.12. *For any* $f \in L_\phi \cap L_1(B_k)$, $\|f\|_\phi^s = \|f\|_{\phi,\mathbb{R}^k}^s \leqslant 1$,

$$\left\|Q_\phi^s(f) - \left(Q_\phi^s f\right)_h\right\|_{\phi,\mathbb{R}^k}^s \leqslant \|f - f_{h/a}\|_{\phi,\mathbb{R}^k}^s.$$

If $|f|_\phi^s \leqslant 1$, *then*

$$\left|Q_{\phi,A}^s(f) - \left(Q_{\phi,A}^s f\right)_h\right|_{\phi,\mathbb{R}^k}^s \leqslant |f - f_{h/a}|_{\phi,\mathbb{R}^k}^s.$$

THEOREM 2.13. *For any set* $B \subset \left\{f \in L_\phi \cap L_1(B_k) : \|f\|_\phi^s \leqslant 1\right\}$,

$$\omega_\phi^s\big(Q_\phi^s(B)\big) \leqslant \omega_\phi^s(B)$$

and for any set $B \subset \left\{f \in L_\phi \cap L_1(B_k) : |f|_\phi^s \leqslant 1\right\}$,

$$\omega_{\phi,A}^s\big(Q_{\phi,A}^s(B)\big) \leqslant \omega_{\phi,A}^s(B).$$

PROOF: Follows immediately from Definition 2.8 and Corollary 2.12. ∎

3. MAIN RESULTS.

Let B_ϕ ($B_{\phi,A}$ respectively) denote the unit ball in $E_\phi = E_\phi(B_k, \Sigma_k, \mu_k)$ (see (1.6)) with respect to the Luxemburg s-norm (with respect to the Amemiya s-norm respectively). For any $u > 0$ define $P_{\phi,u} : B_\phi \to E_\phi$ by

$$(P_{\phi,u} f)(t) = \max\left\{0, u\big(2\|t\|_e - \|f\|_\phi^s - 1\big)\right\}. \tag{3.1}$$

Analogously, for any $u > 0$ define $P_{\phi,u,A} : B_{\phi,A} \to E_\phi$ by

$$(P_{\phi,u,A} f)(t) = \max\left\{0, u\big(2\|t\|_e - |f|_\phi^s - 1\big)\right\}. \tag{3.2}$$

Set

$$T_{\phi,u}(f) = Q_\phi^s(f) + P_{\phi,u}(f) \tag{3.3}$$

and

$$T_{\phi,u,A}(f) = Q_{\phi,A}^s(f) + P_{\phi,u,A}(f), \tag{3.4}$$

where Q_ϕ^s and $Q_{\phi,A}^s$ are given by Definition 2.1. Observe that $\left(Q_\phi^s f\right)(t) = 0$ if and only if $\|t\|_e > \left(1 + \|f\|_\phi^s\right)/2$ and for any $u > 0$ $\left(P_{\phi,u}^f\right)(t) = 0$ if and only if $\|t\|_e \leqslant \left(1 + \|f\|_\phi^s\right)/2$. Hence for any $u, \lambda > 0$ and $f \in B_\phi$

$$\rho_\phi\big(\lambda(Q_\phi^s f)\big) \leqslant \rho_\phi\big(\lambda(T_{\phi,u} f)\big).$$

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Consequently, for any $u > 0$, $f \in B_\phi$

(3.5) $$\|Q^s_\phi f\|^s_\phi \leqslant \|T_{\phi,u} f\|^s_\phi.$$

Analogously, for any $u > 0$, $f \in B_{\phi,A}$

(3.6) $$|Q^s_{\phi,A} f|^s_\phi \leqslant |T_{\phi,u,A} f|^s_\phi.$$

Observe that for any $u > 0$, $f \in B_\phi$ and $z, t \in B_k$,

$$\left|P_{\phi,u} f(t) - P_{\phi,u} f(z)\right| \leqslant 2u\|t - z\|_e.$$

Consequently for any $\varepsilon > 0$ there exist $\delta > 0$ such that for any $f \in B_\phi$ and $0 < h < \delta$,

$$\left\|P_{\phi,u} f - (P_{\phi,u} f)_h\right\|_\infty = \sup_{t \in B_k} \left(\left|P_{\phi,u} f(t) - (P_{\phi,u} f)_h(t)\right|\right) \leqslant \varepsilon.$$

This implies immediately that for any $B \subset E_\phi \cap L_1(B_k)$, $\omega^s_\phi(P_{\phi,u}(B)) = 0$. Hence by Proposition 2.10 and Theorem 2.13, for any $B \subset B_\phi \cap L_1(B_k)$

(3.7) $$\omega^s_\phi(T_{\phi,u}(B)) \leqslant \omega^s_\phi(Q^s_\phi(B)) + \omega^s_\phi(P_{\phi,u}(B)) \leqslant \omega^s_\phi(B).$$

Analogously, for any $B \subset B_{\phi,A} \cap L_1(B_k)$,

(3.8) $$\omega^s_{\phi,A}(T_{\phi,u,A}(B)) \leqslant \omega^s_{\phi,A}(Q^s_{\phi,A}(B)) + \omega^s_{\phi,A}(P_{\phi,u,A}(B)) \leqslant \omega^s_{\phi,A}(B).$$

Lemma 3.1. *For any $\varepsilon > 0$ there esists $u_\varepsilon > 0$ such that for any $f \in B_\phi$, $\|T_{\phi,u_\varepsilon}\|^s_\phi \geqslant 1 - \varepsilon$.*

Proof: Fix $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\frac{(1-\delta)(2-\delta)^k}{2^k} > 1 - \varepsilon.$$

If $\|f\|^s_\phi \geqslant 1 - \delta$, then by Lemma 2.3 and (3.5), for any $u > 0$

(3.9) $$\|T_{\phi,u} f\|^s_\phi \geqslant \|Q^s_\phi f\|^s_\phi \geqslant \|f\|^s_\phi \left(\left(1 + \|f\|^s_\phi\right)/2\right)^k \geqslant \frac{(1-\delta)(2-\delta)^k}{2^k} > 1 - \varepsilon.$$

Now suppose $\|f\|^s_\phi < 1 - \delta$. Since $\operatorname{supp}\left(Q^s_\phi f\right) \cap \operatorname{supp}\left(P^s_{\phi,u} f\right) = \emptyset$ for any $u > 0$,

$$\begin{aligned}
\rho_\phi(T_{\phi,u} f) &= \rho_\phi(Q^s_\phi f) + \rho_\phi(P_{\phi,u} f) \\
&\geqslant \rho_\phi(Q^s_\phi f) + \int_{B_k \setminus B_k(1-\delta/4)} \phi\left(u(2\|t\|_e - \|f\|^s_\phi - 1)\right) d\mu_k(t) \\
&\geqslant \rho_\phi(Q^s_\phi f) + \mu_k(B_k \setminus B_k(1-\delta/4))\phi\left(u(2 - \delta/2 - (1-\delta) - 1)\right) \\
&= \rho_\phi(Q^s_\phi f) + \mu_k(B_k \setminus B_k(1-\delta/4))\phi(u\delta/2).
\end{aligned}$$

Hence, we can find $u_\varepsilon > 0$ such that for any $f \in B_\phi$, $\|f\|_\phi^s < 1-\delta$, $\rho_\phi(T_{\phi,u}f) > 1$ and consequently,

$$\left\|(T_{\phi,u}f)\right\|_\phi^s \geqslant 1 > 1-\varepsilon,$$

which completes the proof. ∎

Lemma 3.2. *For any $\varepsilon > 0$ there exists $u_\varepsilon > 0$ such that for any $f \in B_{\phi,A}$, $|T_{\phi,u_\varepsilon,A}|_\phi^s \geqslant 1-\varepsilon$.*

Proof: Fix $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\frac{(1-\delta)(2-\delta)^k}{2^k} > 1-\varepsilon.$$

If $|f|_\phi^s \geqslant 1-\delta$, then reasoning as in Lemma 3.1, we get $|T_{\phi,u,A}|_\phi^s > 1-\varepsilon$ for any $u > 0$. So assume $|f|_\phi^s < 1-\delta$. Fix $k > 0$. If $k \leqslant 1$, then

$$\frac{1+\rho_\phi\big(k^{1/s}T_{\phi,u,A}f\big)}{k} \geqslant 1/k \geqslant 1.$$

for any $u > 0$. If $k > 1$, then by s-convexity of ϕ,

$$\frac{1+\rho_\phi\big(k^{1/s}T_{\phi,u,A}f\big)}{k} \geqslant \frac{\rho_\phi\big(k^{1/s}T_{\phi,u,A}f\big)}{k} \geqslant \rho_\phi(T_{\phi,u,A}f)$$

By the proof of Lemma 3.1, $\rho_\phi(T_{\phi,u,A}f) \geqslant \phi(u\delta/2)$. Hence there exists $u_\varepsilon > 0$ such that for any $k > 0$, $f \in B_\phi$, $|f|_\phi^s < 1-\delta$

$$\frac{1+\rho_\phi\big(k^{1/s}T_{\phi,u,A}f\big)}{k} \geqslant 1$$

and consequently, $|T_{\phi,u_\varepsilon,A}f|_\phi^s > 1-\varepsilon$. ∎

Theorem 3.3. *Let S_ϕ denote the unit sphere in E_ϕ with respect to the Luxemburg s-convex norm $\|\cdot\|_\phi^s$. For any $\varepsilon > 0$ there exists a retraction $R_\phi : B_\phi \to S_\phi$, such that for any $B \subset B_\phi \cap L_1(B_k)$*

$$\omega_\phi^s\big(R_\phi(B)\big) \leqslant (1+\varepsilon)\omega_\phi^s(B).$$

In particular, $k_{\omega_\phi^s}(E_\phi) \leqslant 1$.

Proof: Fix $\varepsilon > 0$. Choose $\varepsilon_1 > 0$ such that $\big(1/(1-\varepsilon_1)\big)^s < 1+\varepsilon$. Let $u_1 > 0$ be a positive number corresponding to ε_1 by Lemma 3.1. Set for $f \in B_\phi$

$$R_\phi f = \frac{T_{\phi,u_1}f}{\big(\|T_{\phi,u_1}f\|_\phi^s\big)^{1/s}}.$$

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Observe that $\|R_\phi f\|_\phi^s = 1$ for any $f \in B_\phi$. Moreover, by (3.1) and (3.3), $R_\phi f = f$ for any $f \in S_\phi$. By Corollary 2.5 and (3.3), R_ϕ is a continuous mapping. Now fix $B \subset B_\phi \cap L_1(B_k)$. By the choice of u_1 and Lemma 3.1, for any $f \in B$ and $h > 0$,

$$\begin{aligned}\left\|R_\phi f - (R_\phi f)_h\right\|_\phi^s &= \left(\left\|T_{\phi,u_1} f - (T_{\phi,u_1} f)_h\right\|_\phi^s\right)/(\|T_{\phi,u_1} f\|_\phi^s) \\ &\leqslant \left(\left\|T_{\phi,u_1} f - (T_{\phi,u_1} f)_h\right\|_\phi^s\right)/(1-\varepsilon_1).\end{aligned}$$

Consequently, by (3.7),

$$\begin{aligned}\omega_\phi^s\big(R_\phi(B)\big) &\leqslant \omega_\phi^s\left(\frac{T_{\phi,u_1}(B)}{1-\varepsilon_1}\right) = \frac{1}{(1-\varepsilon_1)^s}\omega_\phi^s\big(T_{\phi,u_1}(B)\big) \\ &\leqslant (1+\varepsilon)\omega_\phi^s\big(T_{\phi,u_1}(B)\big) \leqslant (1+\varepsilon)\omega_\phi^s(B).\end{aligned}$$

The proof is complete. ∎

Replacing Lemma 3.1 by Lemma 3.2 and reasoning as in Theorem 3.3, one can show

Theorem 3.4. *Let $S_{\phi,A}$ denote the unit sphere in E_ϕ with respect to the Amemiya s-convex norm $|\cdot|_\phi^s$. For any $\varepsilon > 0$ there exists a retraction $R_{\phi,A} : B_{\phi,A} \to S_{\phi,A}$, such that for any $B \subset B_{\phi,A} \cap L_1(B_k)$*

$$\omega_{\phi,A}^s\big(R_\phi(B)\big) \leqslant (1+\varepsilon)\omega_{\phi,A}^s(B).$$

In particular, $k_{\omega_{\phi,A}^s}(E_\phi) \leqslant 1$.

Remark 3.5. If ϕ is a convex function, then, by Remark 2.7, Theorem 3.3 (Theorem 3.4 respectively) holds true for any $B \subset B_\phi$ ($B \subset B_{\phi,A}$ respectively). If in the definition of ω_ϕ^s for any $r > 1$, we replace $\|\cdot\|_{\phi,\mathbb{R}^k}^s$ by $\|\cdot\|_{\phi,B_k(r)}^s$, where $\|\cdot\|_{\phi,B_k(r)}^s$, is the Luxemburg s-norm associated with a modular $\rho_{\phi,B_k(r)}(f) = \int_{B_k(r)} \phi\Big(\big|f(t)\big|\Big)\, d\mu_k(t)$ Theorem 3.3 remains true. In the case of the Amemiya norm the same applies to Theorem 3.4.

If ϕ is a convex function, by Remark 2.7, Proposition 2.9, Theorems 3.3 and 3.4, we easily get.

Theorem 3.6. *Let ϕ be a convex function such that $\phi^*(v)$ (see (2.6)) is finite for any $v > 0$. Then for any $\varepsilon > 0$ there exists a retraction $R_\phi : B_\phi \to S_\phi$ such that for any $B \subset B_\phi$*

$$\gamma_\phi^1\big(R_\phi(B)\big) \leqslant (2+\varepsilon)\gamma_\phi^1(B).$$

In particular, $k_{\gamma_\phi^1}(E_\phi) \leqslant 2$. The same result holds true in the case of the Amemiya norm.

Remark 3.7. In the case of $\phi(t) = t^p$ $(1 \leqslant p < \infty)$ and $k = 1$, that is, $E_\phi = L_p[-1,1]$, Theorem 3.6 has been proven in [**9**, Corollary 6].

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