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Products and Direct Sums in Locally Convex Cones

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Abstract. In this paper we define lower, upper, and symmetric completeness and discuss closure of the sets in products and direct sums. In particular, we introduce suitable bases for these topologies, which leads us to investigate completeness of the direct sum and its components. Some results obtained about X-topologies and polars of the neighborhoods.

1 Introduction

An *ordered cone* is a set $\mathcal{P}$ on which two operations, addition and scalar multiplication for non-negative real numbers $\lambda \geq 0$, are defined. The addition is assumed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for the scalar multiplication the usual associative and distributive properties hold. In addition, the cone $\mathcal{P}$ carries a preorder, *i.e.*, a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ and $\lambda a \leq \lambda b$ for all $a, b, c \in \mathcal{P}$ and $\lambda \geq 0$. For example, $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a preordered cone with respect to the usual addition, multiplication, and order on $\overline{\mathbb{R}}$. In any cone $\mathcal{P}$, equality is obviously a preorder, hence all results about ordered cones apply to cones without order structures as well.

The subset $\mathcal{V} \subset (\mathcal{P}, \leq)$, where $\leq$ is a preorder relation on $\mathcal{P}$, is called an *(abstract) 0-neighborhood system* if it satisfies the following requirements:
(v_1) $0 < v$ for all $v \in \mathcal{V}$.
(v_2) For all $u, v \in \mathcal{V}$ there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$.
(v_3) $u + v \in \mathcal{V}$ and $\lambda v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\lambda > 0$.

An (abstract) 0-neighborhood system $\mathcal{V}$ induces three topologies on $\mathcal{P}$, called the *upper*, *lower*, and *symmetric topologies*. For $v \in \mathcal{V}$, the *neighborhoods* of an element $a \in \mathcal{P}$ with respect to these topologies are defined as

$$v(a) = \{b \in \mathcal{P} : b \leq a + v\}, \quad (a)v = \{b \in \mathcal{P} : a \leq b + v\},$$

and $v(a)v = v(a) \cap (a)v$, respectively. We remark that the symmetric topology is the common refinement of the upper and lower topologies.

If we assume that all elements of $\mathcal{P}$ are *bounded below*, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$, then the pair $(\mathcal{P}, \mathcal{V})$ is called a *full locally convex cone*. A *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the (abstract) 0-neighborhood system $\mathcal{V}$.

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We will also deal with another equivalent structure, called a *convex quasi-uniform structure*, that is defined as a collection $\mathbf{U}$ of convex subsets of $\mathcal{P}^2$ with the following conditions:

(u_1) $\Delta \subset \mathtt{U}$ for all $\mathtt{U} \in \mathbf{U}$; $\Delta = \{(a,a) : a \in \mathcal{P}\}$.

(u_2) For all $\mathtt{U}, \mathtt{V} \in \mathbf{U}$ there is a $\mathtt{W} \in \mathbf{U}$ such that $\mathtt{W} \subseteq \mathtt{U} \cap \mathtt{V}$.

(u_3) $\lambda\mathtt{U} \circ \mu\mathtt{U} \subseteq (\lambda + \mu)\mathtt{U}$ for all $\lambda, \mu > 0$ and $\mathtt{U} \in \mathbf{U}$, where $\lambda\mathtt{U} \circ \mu\mathtt{U} = \{(a,b) \in \mathcal{P}^2 : \exists c \in \mathcal{P}$ with $(a,c) \in \lambda\mathtt{U}$ and $(c,b) \in \mu\mathtt{U}\}$.

(u_4) $\lambda\mathtt{U} \in \mathbf{U}$, for all $\mathtt{U} \in \mathbf{U}$ and $\lambda > 0$.

Each convex quasi-uniform structure $\mathbf{U}$ on $\mathcal{P}$ induces a preorder relation where $a \leq b$ if and only if $(a,b) \in \mathtt{U}$ for all $\mathtt{U} \in \mathbf{U}$ as well as three topologies: The neighborhood bases for an element $a \in \mathcal{P}$ in the *upper*, *lower*, and *symmetric* topologies are given respectively by the sets

$$\mathtt{U}(a) = \{b \in \mathcal{P} : (b,a) \in \mathtt{U}\}, \quad (a)\mathtt{U} = \{b \in \mathcal{P} : (a,b) \in \mathtt{U}\}, \quad \mathtt{U} \in \mathbf{U},$$

and $\mathbf{U}_s = \{\mathtt{U} \cap \mathtt{U}^{-1} : \mathtt{U} \in \mathbf{U}\}$, where $\mathtt{U}^{-1} = \{(b,a) : (a,b) \in \mathtt{U}\}$.

The notions of (abstract) 0-neighborhood system $\mathcal{V}$ and convex quasi-uniform structure $\mathbf{U}$ on a cone $\mathcal{P}$ are equivalent in the following sense.

For a locally convex cone $(\mathcal{P}, \mathcal{V})$ and each $v \in \mathcal{V}$, we put

$$\tilde{v} = \{(a,b) \in \mathcal{P} \times \mathcal{P} : a \leq b + v\}.$$

The collection $\widetilde{\mathcal{V}} = \{\tilde{v} : v \in \mathcal{V}\}$ is a convex quasi-uniform structure on $\mathcal{P}$ that induces the same upper, lower and symmetric topologies. On the other hand, if $\mathcal{P}$ is a cone with a convex quasi-uniform structure $\mathbf{U}$, then one can find a preorder and an (abstract) 0-neighborhood system $\mathcal{V}$ such that the convex quasi-uniform structure $\widetilde{\mathcal{V}}$ is equivalent to $\mathbf{U}$ [1, Chapter I, 5.5].

If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone, the condition that every element $a \in \mathcal{P}$ has to be bounded below translates into "for each $\tilde{v} \in \widetilde{\mathcal{V}}$ there is some $\rho > 0$ such that $(0,a) \in \rho\tilde{v}$". On the other hand, if a convex quasi-uniform structure $\mathbf{U}$ on a cone $\mathcal{P}$ has the extra property

(u_5) for all $a \in \mathcal{P}$ and $\mathtt{U} \in \mathbf{U}$, there is some $\rho > 0$ such that $(0,a) \in \rho\mathtt{U}$,

then the resulting cone will be locally convex, that is, every element $a \in \mathcal{P}$ will be bounded below.

For locally convex cones $\mathcal{P}$ and $\mathcal{Q}$, with convex quasi-uniform structures $\mathbf{U}$ and $\mathbf{V}$ respectively, a linear mapping $t\colon \mathcal{P} \to \mathcal{Q}$ is called *uniformly continuous (u-continuous)* if for every $\mathtt{V} \in \mathbf{V}$, there is some $\mathtt{U} \in \mathbf{U}$ such that $(a,b) \in \mathtt{U}$ implies $\big(t(a), t(b)\big) \in \mathtt{V}$, *i.e.*, $T(\mathtt{U}) \subseteq \mathtt{V}$, $T = t \times t$. If $\mathcal{V}$ and $\mathcal{W}$ are (abstract) 0-neighborhood systems on $\mathcal{P}$ and $\mathcal{Q}$, t is u-continuous if and only if for every $w \in \mathcal{W}$ there is some $v \in \mathcal{V}$, such that $(a,b) \in \widetilde{v}$ implies $\big(t(a), t(b)\big) \in \widetilde{w}$ or equivalently; $t(a) \leq t(b) + w$ whenever $a \leq b + v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$. The set of all u-continuous linear functionals $\mu\colon \mathcal{P} \to \overline{\mathbb{R}}$ is a cone called the *dual cone* of $\mathcal{P}$ and is denoted by $\mathcal{P}^*$. In a locally convex cone $(\mathcal{P}, \mathcal{V})$ the *polar* v° of $v \in \mathcal{V}$ is defined by $v^\circ = \{\mu \in \mathcal{P}^* : a \leq b + v$ implies $\mu(a) \leq \mu(b) + 1\}$. Obviously we have $\mathcal{P}^* = \bigcup_{v \in \mathcal{V}} v^\circ$.

Let $\mathbf{U}_1$ and $\mathbf{U}_2$ be convex quasi-uniform structures on $\mathcal{P}$. We say that $\mathbf{U}_1$ is finer than $\mathbf{U}_2$ (or $\mathbf{U}_2$ is coarser than $\mathbf{U}_1$) if the identity mapping $i\colon (\mathcal{P},\mathbf{U}_1) \to (\mathcal{P},\mathbf{U}_2)$ is uniformly continuous.

In Section 2, we define notions of lower, upper, and symmetric completeness and study closure of the sets in products.

In Section 3, we introduce a base to each of the lower, upper, and symmetric topologies, which help us to study the completeness of the direct sum and its components. Finally we obtain some results about X-topologies and polars of neighborhoods.

2 Products

For each $\gamma \in \Gamma$, let $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ be a locally convex cone and $\phi_\gamma\colon \times_{\gamma\in\Gamma} \mathcal{P}_\gamma \to \mathcal{P}_\gamma$ be the projection mapping. The product $\mathcal{P} = \times_\gamma \mathcal{P}_\gamma$ is the projective limit of the locally convex cones $\mathcal{P}_\gamma$ by the mappings ϕ_γ and is called the *product locally convex cone*, denoted by $(\mathcal{P},\mathcal{V})$. If $\mathcal{V}_\gamma = \{v_{\gamma_\delta} : \delta \in \mathcal{I}_\gamma\}$, $\gamma \in \Gamma$, then each $\tilde{v} \in \widetilde{\mathcal{V}}$ is a finite intersection of the sets $\Phi_\gamma^{-1}(\tilde{v}_{\gamma_\delta})$, where $\Phi_\gamma = \phi_\gamma \times \phi_\gamma$, and $\tilde{v} = \cap_{i=1}^{n} \Phi_{\gamma_i}^{-1}(\tilde{v}_{\gamma_{i\delta_i}})$, say.

Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone and $A \subseteq \mathcal{P}$. The closure of A with respect to the lower topology is defined by

$$\begin{aligned}\overline{A} &= \{b \in \mathcal{P} : \text{for every } v \in \mathcal{V}, \text{ there is some } a \in A \text{ with } b \le a + v\}\\ &= \{b \in \mathcal{P} : \text{for every } v \in \mathcal{V}, (b)v \cap A \neq \varnothing\}.\end{aligned}$$

For $a \in \mathcal{P}$, we put $\overline{a} = \overline{\{a\}}$. In particular $\overline{a} = \bigcap_{v\in\mathcal{V}} v(a)$. The closures of A with respect to the upper and symmetric topologies, denoted by $\overline{\overline{A}}$ and $\overline{A}^s$ respectively, are defined in a similar way; for details, see [1].

Proposition 2.1 *Let $A = \times_{\gamma\in\Gamma} A_\gamma \subseteq (\mathcal{P},\mathcal{V})$. Then*

$$\overline{A} = \times_{\gamma\in\Gamma}\overline{A}_\gamma, \quad \overline{\overline{A}} = \times_{\gamma\in\Gamma}\overline{\overline{A}}_\gamma, \quad \textit{and} \quad \overline{A}^s = \times_{\gamma\in\Gamma}\overline{A}^s_\gamma.$$

Proof We prove the first equality. Let $t = (t_\gamma) \in \overline{(\times A_\gamma)}$. Fix $\gamma \in \Gamma$ and let $v_{\gamma_\delta} \in \mathcal{V}_\gamma$, $\delta \in \mathcal{I}_\gamma$, be arbitrary. Since ϕ_γ is u-continuous, we have $\Phi_\gamma^{-1}(\tilde{v}_{\gamma_\delta}) \in \widetilde{\mathcal{V}}$. Hence there is an $x = (x_\gamma) \in \times A_\gamma$ such that $(t,x) \in \Phi_\gamma^{-1}(\tilde{v}_{\gamma_\delta})$ or $(t_\gamma, x_\gamma) = \Phi_\gamma(t,x) \in \tilde{v}_{\gamma_\delta}$ for each $v_{\gamma_\delta} \in \mathcal{V}_\gamma$. Thus for each $\gamma \in \Gamma$, $t_\gamma \in \overline{A}_\gamma$ or $t \in \times_{\gamma\in\Gamma}\overline{A}_\gamma$.

Now let $t = (t_\gamma) \in \times_{\gamma\in\Gamma}\overline{A}_\gamma$ and $v \in \mathcal{V}$ be arbitrary. There are $v_{\gamma_{i\delta_i}} \in \mathcal{V}_{\gamma_i}$ ($i = 1,2,\dots,n$) such that $\tilde{v} = \bigcap_{i=1}^{n} \Phi_{\gamma_i}^{-1}(\tilde{v}_{\gamma_{i\delta_i}})$. Since $t_{\gamma_i} \in \overline{A}_{\gamma_i}$, there is $x_{\gamma_i} \in (t_{\gamma_i})v_{\gamma_{i\delta_i}} \cap A_{\gamma_i}$ ($i = 1,2,\dots,n$). Put $x = (x_\gamma) \in \times A_\gamma$, where $x_\gamma = x_{\gamma_i}$ for $\gamma = \gamma_i$ and $x_\gamma \in A_\gamma$ otherwise. Then $(t_{\gamma_i}, x_{\gamma_i}) = \Phi_{\gamma_i}(t,x) \in \tilde{v}_{\gamma_{i\delta_i}}$, hence $(t,x) \in \tilde{v}$. This implies that $t \in \overline{A}$. ∎

Definition 2.2 A locally convex cone $(\mathcal{P},\mathcal{V})$ is called *separated* if $\overline{a} = \overline{b}$ implies $a = b$ for all $a, b \in \mathcal{P}$.

The locally convex cone $(\mathcal{P},\mathcal{V})$ is separated if and only if the symmetric topology on $\mathcal{P}$ is Hausdorff [1, Chapter I, Proposition 3.9]. This implies that $(\mathcal{P},\mathcal{V})$ is separated if and only if for each $a \in \mathcal{P}$, $\overline{a}^s = \{a\}$.

Proposition 2.3 *Let* $(\mathcal{P},\mathcal{V}) = \times_{\gamma\in\Gamma}(\mathcal{P}_\gamma\mathcal{V}_\gamma)$. *Then*
(a) *for each* $x = (x_\gamma) \in (\mathcal{P},\mathcal{V})$, *we have*

$$\overline{x} = \times_{\gamma\in\Gamma}\overline{x}_\gamma, \quad \overline{\overline{x}} = \times_{\gamma\in\Gamma}\overline{\overline{x}}_\gamma, \quad \textit{and} \quad \overline{x}^s = \times_{\gamma\in\Gamma}\overline{x}^s_\gamma;$$

(b) *the product locally convex cone* $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$ *is separated if and only if each* $\mathcal{P}_\gamma$ *is separated;*
(c) *if each* $\mathcal{P}_\gamma$ *is separated, then for each* γ, $j_\gamma(\mathcal{P}_\gamma)$ *is closed with respect to the symmetric topology in* $(\mathcal{P},\mathcal{V})$.

Proof Parts (a) and (c) follow from Proposition 2.1. For (b), let $x = (x_\gamma)$. Each $\mathcal{P}_\gamma$ is separated if and only if $\overline{x}^s_\gamma = \{x_\gamma\}$, $\gamma \in \Gamma$, and this is the case if and only if $\overline{x}^s = \{x\}$. ■

Example 2.4 (i) A subset A of a preordered cone is called *decreasing* if $a \in A$ and $b \le a$ for some $b \in A$ imply $b \in A$. For a subset B of $\mathcal{P}$, we denote by

$$\downarrow B = \{a \in \mathcal{P} : a \le b \text{ for some } b \in B\}$$

the decreasing subset generated by B. In a same way one defines the notion of an *increasing* subset and $\uparrow B$, the increasing subset generated by B. We denote by $\operatorname{Conv}(\mathcal{P})$, the set of all nonempty convex subsets of $\mathcal{P}$ which is a cone with usual addition and scalar multiplication.

Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone. If we identify the elements of $\mathcal{V}$ with singleton sets $\bar{v} = \{v\}$, then $\overline{\mathcal{V}} = \{\bar{v} : v \in \mathcal{V}\}$ is a subset of $\operatorname{Conv}(\mathcal{P})$, which can be preordered using the preorder of $\mathcal{P}$. For $A, B \in \operatorname{Conv}(\mathcal{P})$, we define

$$A \le B \text{ if for each } a \in A \text{ there is some } b \in B \text{ such that } a \le b,$$

then $\big(\operatorname{Conv}(\mathcal{P}),\overline{\mathcal{V}}\big)$ becomes a full locally convex cone. Also, we denote by $\operatorname{DConv}(\mathcal{P})$ ($\overline{\operatorname{DConv}}(\mathcal{P})$), the set of all decreasing convex subsets (respectively, closed decreasing convex subsets) of $\mathcal{P}$, where closure is meant with respect to the lower topology on $\mathcal{P}$. If we modify the addition, both sets will become cones as well:

$$A \oplus B = \downarrow (A+B) \quad \text{for } A, B \in \operatorname{DConv}(\mathcal{P}),$$

$$A \overline{\oplus} B = \overline{\{\downarrow (A+B)\}} \quad \text{for } A, B \in \overline{\operatorname{DConv}}(\mathcal{P}).$$

With the preorder and (abstract) 0-neighborhood system induced by $\operatorname{Conv}(\mathcal{P})$, both $\big(\operatorname{DConv}(\mathcal{P}),\overline{\mathcal{V}}\big)$ and $\big(\overline{\operatorname{DConv}}(\mathcal{P}),\overline{\mathcal{V}}\big)$ are locally convex cones. In particular, $\overline{P} = \{\overline{a} : a \in \mathcal{P}\}$ as a subcone of $\overline{\operatorname{DConv}}(\mathcal{P})$ is a locally convex cone. For details see [1].

Now, if $(\mathcal{P},\mathcal{V}) = \times_{\gamma\in\Gamma}(\mathcal{P}_\gamma,\mathcal{V}_\gamma)$, then $\big(\overline{\operatorname{DConv}}(\mathcal{P}),\overline{\mathcal{V}}\big)$ is the product locally convex cone of the locally convex cones $\big(\overline{\operatorname{DConv}}(\mathcal{P}_\gamma),\overline{\mathcal{V}}_\gamma\big)$; in particular, $(\overline{\mathcal{P}},\overline{\mathcal{V}})$ is the product locally convex cone $(\overline{\mathcal{P}}_\gamma,\overline{\mathcal{V}}_\gamma)$, $\gamma \in \Gamma$. For, by Proposition 2.1, the set $A \subset (\mathcal{P},\mathcal{V})$ is closed decreasing convex if and only if each A_γ is closed decreasing convex in $(\mathcal{P}_\gamma,\mathcal{V}_\gamma)$. Hence

$$\overline{\operatorname{DConv}}(\mathcal{P}) = \times_{\gamma\in\Gamma}\overline{\operatorname{DConv}}(\mathcal{P}_\gamma),$$

and the projection mappings

$$\overline{\phi}_\gamma : \overline{\mathrm{DConv}}(\mathcal{P}) \to \overline{\mathrm{DConv}}(\mathcal{P}_\gamma), \quad \gamma \in \Gamma,$$
$$\overline{\phi}_\gamma(A) = \overline{\downarrow A_\gamma}, \quad \text{where } A \in \overline{\mathrm{DConv}}(\mathcal{P})$$

are well defined and u-continuous by [1, II, 1.6].

Suppose that $v \in \mathcal{V}$, $\tilde{v} = \bigcap_{i=1}^{n} \Phi_{\gamma_i}^{-1}(\tilde{v}_{\gamma_i \delta_i})$ and $A, B \in \overline{\mathrm{DConv}}(\mathcal{P})$. Then

$$A \leq B + \tilde{v} \quad \text{if and only if} \quad \phi_{\gamma_i}(A) \leq \phi_{\gamma_i}(B) + \tilde{v}_{\gamma_i \delta_i} \quad (i = 1, 2, \ldots, n).$$

Thus

$$\widetilde{\bar{v}} = \bigcap_{i=1}^{n} \overline{\Phi}_{\gamma_i}^{-1}(\widetilde{\bar{v}}_{\gamma_i \delta_i}), \text{ where } \overline{\Phi}_{\gamma_i} = \overline{\phi}_{\gamma_i} \times \overline{\phi}_{\gamma_i}.$$

Hence the product convex quasi-uniform structure on $\mathcal{P}$, induced by the (abstract) 0-neighborhood systems $\overline{\mathcal{V}}_\gamma$, is identical to $\widetilde{\overline{\mathcal{V}}}$; where $\widetilde{\overline{\mathcal{V}}} = \{\widetilde{\bar{v}} : \bar{v} \in \overline{\mathcal{V}}\}$.

Likewise, $\big(\mathrm{Conv}(\mathcal{P}), \overline{\mathcal{V}}\big)$ and $\big(\mathrm{DConv}(\mathcal{P}), \overline{\mathcal{V}}\big)$ are the product locally convex cones of $\big(\mathrm{Conv}(\mathcal{P}_\gamma), \overline{\mathcal{V}}_\gamma\big)$ and $\big(\mathrm{DConv}(\mathcal{P}_\gamma), \overline{\mathcal{V}}_\gamma\big)$, respectively.

(ii) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. $A \subset \mathcal{P}$ is called *bounded* if for every $v \in \mathcal{V}$ there exists $\lambda > 0$ such that

$$a \leq \lambda v \quad \text{and} \quad 0 \leq a + \lambda v \quad \text{for all } a \in A.$$

Also $A \subset \mathcal{P}$ is called *internally bounded* if for every $v \in \mathcal{V}$ there exists $\lambda > 0$ such that $a \leq b + \lambda v$ for all $a, b \in A$ [4]. The set of all bounded $(\mathcal{B})$ or internally bounded $(\mathcal{IB})$ sets of a locally convex cone is a subcone. For $A \subset (\mathcal{P}, \mathcal{V})$, we have

$$A \in \mathcal{B} \quad \text{if and only if} \quad A_\gamma \in \mathcal{B}_\gamma.$$

and

$$A \in \mathcal{IB} \quad \text{if and only if} \quad A_\gamma \in (\mathcal{IB})_\gamma.$$

Therefore $(\mathcal{B}, \mathcal{V}) = \times_{\gamma \in \Gamma}(\mathcal{B}_\gamma \mathcal{V}_\gamma)$ and $(\mathcal{IB}, \mathcal{V}) = \times_{\gamma \in \Gamma}\big((\mathcal{IB})_\gamma, \mathcal{V}_\gamma\big)$; in particular, each $x = (x_\gamma) \in \mathcal{P}$ is bounded if and only if each x_γ is bounded. Hence $\mathtt{B}$, the subcone of all bounded elements of $\mathcal{P}$, is the product locally convex cone of $\mathtt{B}_\gamma$, $\gamma \in \Gamma$, where each $\mathtt{B}_\gamma$ is the subcone of all bounded elements of $\mathcal{P}_\gamma$.

Definition 2.5 Let $(x_\alpha)_{\alpha \in \mathcal{I}}$ be a net in $(\mathcal{P}, \mathcal{V})$ and $x \in \mathcal{P}$. We write $x_\alpha \downarrow x$ ($x_\alpha \uparrow x$) if $(x_\alpha)_{\alpha \in \mathcal{I}}$ converges to x with respect to the lower (respectively, upper) topology. Also $x_\alpha \to x$ means that $x_\alpha \uparrow x$ and $x_\alpha \downarrow x$, *i.e.*, $(x_\alpha)_{\alpha \in \mathcal{I}}$ converges to x with respect to the symmetric topology.

We define $(x_\alpha)_{\alpha \in \mathcal{I}}$ in $(\mathcal{P}, \mathcal{V})$ to be *lower (upper) Cauchy* if for every $v \in \mathcal{V}$ there is some $\alpha_v \in \mathcal{I}$ such that $x_\beta \leq x_\alpha + v$ (respectively, $x_\alpha \leq x_\beta + v$) for all α, β with $\beta \geq \alpha \geq \alpha_v$. Also $(x_\alpha)_{\alpha \in \mathcal{I}}$ is called *symmetric Cauchy* if it is lower and upper Cauchy, *i.e.*, if for each $v \in \mathcal{V}$ there is some $\alpha_v \in \mathcal{I}$ such that $x_\beta \leq x_\alpha + v$ and $x_\alpha \leq x_\beta + v$ for all α, β with $\alpha, \beta \geq \alpha_v$.

Definition 2.6 The locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *lower* (*upper* and *symmetric*) complete if every lower (respectively, upper and symmetric) Cauchy net converges in lower (respectively, upper and symmetric) topology. In general, the set $A \subset \mathcal{P}$ is called lower (upper and symmetric) complete if every lower (respectively, upper and symmetric) Cauchy net is convergent to an element of A in corresponding topology.

Proposition 2.7 *In a separated locally convex cone $(\mathcal{P}, \mathcal{V})$, a net $(x_\alpha)_{\alpha\in\mathcal{I}}$ cannot converge to more than one point.*

Proof Suppose that $(x_\alpha)_{\alpha\in\mathcal{I}}$ converges to a and b in $\mathcal{P}$ in the symmetric topology. Then for each $v \in \mathcal{V}$, there is some $\alpha_v \in \mathcal{I}$ such that $x_\alpha \in v(a)v \cap v(b)v$ for all $\alpha \geq \alpha_v$, which is a contradiction, since the symmetric topology is Hausdorff. ■

Proposition 2.8 *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Then*

(a) *if $B \subset \mathcal{P}$ is lower (upper and symmetric) complete, then every closed subset of B, with respect to the lower (respectively, upper and symmetric) topology, is lower (respectively, upper and symmetric) complete,*

(b) *if $(\mathcal{P}, \mathcal{V})$ is separated, then a symmetric complete subset of $\mathcal{P}$ is closed with respect to the symmetric topology.*

Proof For (a), let A be lower complete and B be a closed subset of A with respect to the lower topology. If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a lower Cauchy net in B, then it is also lower Cauchy in A. Hence $x_\alpha \downarrow a \in A$ and a is a limit point of B, in lower topology. So $a \in \overline{B} = B$. For part (b), let A be symmetric complete and $a \in \overline{A}^s$. Let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a net in A, such that $x_\alpha \to a$. Clearly, $(x_\alpha)_{\alpha\in\mathcal{I}}$ is symmetric Cauchy in A and x_α converges to a point of A in symmetric topology. Since $(\mathcal{P}, \mathcal{V})$ is separated, by Proposition 2.7, this point is a. Thus $a \in A$ and A is closed with respect to the symmetric topology. ■

Note that only decreasing sets in $(\mathcal{P}, \mathcal{V})$ may be closed with respect to the lower topology, and only increasing sets could be closed with respect to the upper topology. Hence the symmetric complete subsets of a separated locally convex cone need not be closed with respect to upper or lower topology. For each $a \in (\mathcal{P}, \mathcal{V})$, the singleton set $\{a\}$ is lower (upper) complete but not closed with respect to lower (respectively, upper) topology, so the lower (upper) complete set need not be closed with respect to lower (respectively, upper) topology. Also, if a locally convex cone is symmetric complete, it is not necessary to be both lower and upper complete.

Example 2.9 $\mathrm{Conv}(\overline{\mathbb{R}})$ with set inclusion as order and $\mathcal{V} = \{(-\epsilon, \epsilon) : \epsilon > 0\}$ as the 0-neighborhood system, is symmetric and upper complete but not lower complete. Let $(c_\alpha)_{\alpha\in\mathcal{I}}$ be a symmetric Cauchy net in $\mathrm{Conv}(\overline{\mathbb{R}})$. Let $\varepsilon \in \mathcal{V}, \varepsilon = (-\epsilon, \epsilon)$. There is $\alpha_\varepsilon \in \mathcal{I}$ such that

$$c_\alpha \leq c_\beta + \varepsilon \quad \text{and} \quad c_\beta \leq c_\alpha + \varepsilon \quad \text{for all } \alpha, \beta \text{ with } \alpha, \beta \geq \alpha_\varepsilon.$$

If for all $\alpha \geq \alpha_\epsilon$, $c_\alpha = (-\infty, +\infty]$, then $c_\alpha \to (-\infty, +\infty] \in \mathrm{Conv}(\overline{\mathbb{R}})$. For every $\alpha \geq \alpha_\varepsilon$, let c_α be a finite open, semi-open or closed interval in $\mathbb{R}$. Let $(i_\alpha)_{\alpha\in\mathcal{I}}$ and $(e_\alpha)_{\alpha\in\mathcal{I}}$ be the nets of the initial and end points of these intervals, respectively. Then for all $\alpha, \beta \geq \alpha_\varepsilon$, we have

$$i_\beta \leq i_\alpha + \epsilon \quad \text{and} \quad e_\alpha \leq e_\beta + \epsilon.$$

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Put $i = \inf_{\alpha \geq \alpha_\varepsilon} i_\alpha$ and $e = \sup_{\alpha \geq \alpha_\varepsilon} e_\alpha$. Then $c_\alpha \to ([i, e] \in \mathrm{Conv}(\overline{\mathbb{R}})$. The upper completeness is proved in a similar way.

The sequence $\big((-\infty, n]\big)_{n \in \mathbb{N}}$ is lower Cauchy but not convergent in lower topology. So $\big(\mathrm{Conv}(\overline{\mathbb{R}}), \mathcal{V}\big)$ is not lower complete.

Similarly, $(\overline{\mathbb{R}}, \varepsilon)$, where $\varepsilon = \{\epsilon : \epsilon \in \mathbb{R}^+\}$, is symmetric and upper complete but not lower complete; also $\big(\mathrm{Conv}(\mathbb{R}), \mathcal{V}\big)$ is only symmetric complete.

Remark 2.10 (i) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and let $a \in (\mathcal{P}, \mathcal{V})$. Every net in $\overline{a}$ converges to a with respect to the lower topology, *i.e.*, $\overline{a}$ is lower complete. Similarly, $\overline{\overline{a}}$ is upper complete. Also, every net in $\overline{a}^s$ converges to a with respect to each of the lower, upper, and symmetric topologies, so $\overline{a}^s$ is lower, upper, and symmetric complete.

In [1], the global preorder $\prec$ on $(\mathcal{P}, \mathcal{V})$ is defined by $a \prec b$, if $a \leq b + v$ for all $v \in \mathcal{V}$. If the global preorder $\prec$ coincides with the original one, then each of the subcones $\mathcal{P}^-, \mathcal{P}^+$ is lower and upper complete, respectively; in particular, the subcone $\mathcal{P}^- \cap \mathcal{P}^+$ is not only symmetric complete but also lower and upper complete, where $\mathcal{P}^- = \{a \in \mathcal{P} : a \leq 0\}$ and $\mathcal{P}^+ = \{a \in \mathcal{P} : a \geq 0\}$; indeed, applying the natural element $0 \in \mathcal{P}$, implies that $\mathcal{P}^-_\gamma = \overline{0}$, $\mathcal{P}^+_\gamma = \overline{\overline{0}}$ and $\mathcal{P}^-_\gamma \cap \mathcal{P}^+_\gamma = \overline{0}^s$.

(ii) A locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to have the *strict separation property*, in short (SP), if for all $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ with $a \not\leq b + \rho v$ for some $\rho > 1$, there is a $\mu \in v^\circ$ such that $\mu(a) > \mu(b) + 1$.

In a lower (upper) complete locally convex cone with (SP), we can find a base for the upper (respectively, lower) topology whose elements are lower (respectively, upper) complete. In particular, in a symmetric complete locally convex cone with (SP), the symmetric topology has a base whose elements are symmetric complete [2, 2.12, iv].

Proposition 2.11 *Let $(\mathcal{P}, \mathcal{V}) = \times_{\gamma \in \Gamma} (\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $x = (x_\gamma) \in \mathcal{P}$ and $(x_\alpha)_{\alpha \in \mathcal{I}}$ be a net in $\mathcal{P}$. Then*

(a) *$x_\alpha \downarrow x$ ($x_\alpha \uparrow x$ and $x_\alpha \to x$) if and only if $\varphi_\gamma(x_\alpha) \downarrow x_\gamma$ (respectively, $\varphi_\gamma(x_\alpha) \uparrow x_\gamma$ and $\varphi_\gamma(x_\alpha) \to x_\gamma$), $\gamma \in \Gamma$,*

(b) *$(x_\alpha)_{\alpha \in \mathcal{I}}$ is lower (upper and symmetric) Cauchy if and only if each $\varphi_\gamma(x_\alpha)$ is lower (respectively, upper and symmetric) Cauchy in $\mathcal{P}_\gamma$.*

Proof (a) If $x_\alpha \downarrow x$ in $\mathcal{P}$ and $\gamma \in \Gamma$, then for each $v_\gamma \in \mathcal{V}_\gamma$ we have $\Phi_\gamma^{-1}(\tilde{v}_\gamma) \in \widetilde{\mathcal{V}}$. So there is some $\alpha_{v_\gamma} \in \mathcal{I}$ such that for all $\alpha \geq \alpha_{v_\gamma}$, $(x, x_\alpha) \in \Phi_\gamma^{-1}(\tilde{v}_\gamma)$ or $x_{\alpha\gamma} \in (x_\gamma) v_\gamma$, that is, $x_{\alpha\gamma} \downarrow x_\gamma$. For the converse, let $\phi_\gamma(x_\alpha) \downarrow x_\gamma$ in $\mathcal{P}_\gamma$, $\gamma \in \Gamma$. Let $v \in \mathcal{V}$, $\tilde{v} = \bigcap_{i=1}^n \Phi_{\gamma_i}^{-1}(\tilde{v}_{\gamma_i})$, be arbitrary. Since $x_{\alpha\gamma_i} \downarrow x_{\gamma_i}$ $(i = 1, 2, \ldots, n)$, there is some $\alpha_0 \in \mathcal{I}$ such that

$$x_{\gamma_i} \leq x_{\alpha\gamma_i} + v_{\gamma_i} \quad (i = 1, 2, \ldots, n) \text{ for all } \alpha \geq \alpha_0.$$

This yields $(x, x_\alpha) \in \lambda\tilde{v}$ for all $\alpha \geq \alpha_0$, *i.e.*, $x_\alpha \downarrow x$. Part (b) is proved in a similar way. ∎

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Proposition 2.12 *For each $\gamma \in \Gamma$, let $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ be a locally convex cone and $A_\gamma \subseteq \mathcal{P}_\gamma$. Then $A = \times_{\gamma\in\Gamma} A_\gamma$ is a lower (upper and symmetric) complete subset of $(\mathcal{P}, \mathcal{V}) = \times_{\gamma\in\Gamma}(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ if and only if each A_γ is lower (respectively, upper and symmetric) complete.*

Proof Let A be lower complete and $(z_{\alpha\gamma})_{\alpha\in\mathcal{J}}$ be a lower Cauchy net in $\mathcal{P}_\gamma$. By Proposition 2.11 (b), there is a lower Cauchy net $(x_\alpha)_{\alpha\in\mathcal{J}}$ in $\mathcal{P}$ such that $(z_{\alpha\gamma})_{\alpha\in\mathcal{J}} = \phi_\gamma\big((x_\alpha)_{\alpha\in\mathcal{J}}\big)$. In fact, in each $\mathcal{P}_\delta$, $\delta \neq \gamma$, we choose a lower Cauchy net $(x_{\alpha\delta})_{\alpha\in\mathcal{J}}$ and put $x_\alpha = (x_{\alpha\lambda}) \in (\mathcal{P}, \mathcal{V})$, $\alpha \in \mathcal{J}$, where

$$x_{\alpha\lambda} = \begin{cases} z_{\alpha\gamma} & \text{for } \lambda = \gamma \\ x_{\alpha\delta} & \text{otherwise.} \end{cases}$$

Then there is $a \in A$ such that $x_\alpha \downarrow a$. So $z_{\alpha\gamma} \downarrow \phi_\gamma(a) \in A_\gamma$ by Proposition 2.11 (b), which shows that A_γ is lower complete. Conversely, let each A_γ be lower complete and let $(x_\alpha)_{\alpha\in\mathcal{J}}$ be a lower Cauchy net in $\mathcal{P}$. Each $\phi_\gamma\big((x_\alpha)_{\alpha\in\mathcal{J}}\big) = (x_{\alpha\gamma})_{\alpha\in\mathcal{J}}$ is a lower Cauchy net. Hence $x_{\alpha\gamma} \downarrow a_\gamma \in A_\gamma$, $a = (a_\gamma) \in A$ and $x_\alpha \downarrow a$. The proof of the others is similar. ■

In particular we have the following theorem.

Theorem 2.13 *The product locally convex cone $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$ is lower (upper and symmetric) complete if and only if each $\mathcal{P}_\gamma$ is lower (respectively, upper and symmetric) complete.*

3 Direct Sums

Let $\mathcal{P}_\gamma$, $\gamma \in \Gamma$ be cones and $\mathcal{P} = \times\mathcal{P}_\gamma$. The subcone of $\mathcal{P}$ spanned by $\bigcup \mathcal{P}_\gamma$ (more precisely, by $\bigcup j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma \colon \mathcal{P}_\gamma \to \mathcal{P}$ is the injection mapping) is called the *direct sum* of the cones $\mathcal{P}_\gamma$, $\gamma \in \Gamma$ and denoted by $\sum_{\gamma\in\Gamma}\mathcal{P}_\gamma$. It is worth remembering that here we only use positive scalars.

If each $\mathcal{P}_\gamma$, $\gamma \in \Gamma$ is a locally convex cone, the direct sum $\mathcal{Q} = \sum \mathcal{P}_\gamma$ can be endowed with the convex quasi-uniform structure induced by $\widetilde{\mathcal{V}}$, where $\mathcal{V}$ is the (abstract) 0-neighborhood system of $\mathcal{P}$. We call this the *product convex quasi-uniform structure* on $\sum\mathcal{P}_\gamma$. By Proposition 3.5, it induces the original convex quasi-uniform structure on each $\mathcal{P}_\gamma$. The finest such convex quasi-uniform structure on $\mathcal{Q}$ is obtained by regarding $\mathcal{Q}$ as the inductive limit of the cones $\mathcal{P}_\gamma$ by the injection mappings j_γ and denoted by $\widetilde{\mathcal{W}}$ [3, Theorem 3.1].

Theorem 3.1 *Let $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $\gamma \in \Gamma$ be locally convex cones, let $\mathbf{U}$ be the set of all convex sets $\mathtt{U}$ defined as*

$$\mathtt{U} = \bigcup\Big\{\sum_{\gamma\in\Delta}\lambda_\gamma J_\gamma(\tilde{v}_\gamma) : \sum_{\gamma\in\Delta}\lambda_\gamma = 1 \text{ and } \Delta \text{ is finite}\Big\}, \quad v_\gamma \in \mathcal{V}_\gamma, \gamma \in \Gamma.$$

Then $\mathbf{U}$ is a convex quasi-uniform structure on $\mathcal{Q}$ which is equivalent to $\widetilde{\mathcal{W}}$.

Note that in defining a $\mathtt{U} \in \mathbf{U}$ we take one v_γ from each $\mathcal{V}_\gamma$, $\gamma \in \Gamma$, and form all finite sums of these fixed v_γ, as above.

Proof For (u_2), let

$$\mathtt{U}_1 = \bigcup \Big\{ \sum_{\gamma_1 \in \Delta} \lambda_{\gamma_1} J_{\gamma_1}(\tilde{v}_{\gamma_1}) : \sum_{\gamma_1 \in \Delta} \lambda_{\gamma_1} = 1 \text{ and } \Delta \text{ is finite} \Big\}, \quad v_{\gamma_1} \in \mathcal{V}_\gamma \text{ and}$$

$$\mathtt{U}_2 = \bigcup \Big\{ \sum_{\gamma_2 \in \Delta} \lambda_{\gamma_2} J_{\gamma_2}(\tilde{v}_{\gamma_2}) : \sum_{\gamma_2 \in \Delta} \lambda_{\gamma_2} = 1 \text{ and } \Delta \text{ is finite} \Big\}, \quad v_{\gamma_2} \in \mathcal{V}_\gamma.$$

If we choose $v_\gamma \le v_{\gamma_1}, v_{\gamma_2}$ in $\mathcal{V}_\gamma$, $\gamma \in \Gamma$ and set

$$\mathtt{U} = \bigcup \Big\{ \sum_{\gamma \in \Delta} \lambda_{\gamma} J_{\gamma}(\tilde{v}_{\gamma}) : \sum_{\gamma \in \Delta} \lambda_{\gamma} = 1 \text{ and } \Delta \text{ is finite} \Big\},$$

we have $\mathtt{U} \subseteq \mathtt{U}_1 \cap \mathtt{U}_2$.

For (u_3), let $\mathtt{U} \in \mathbf{U}$, $\lambda, \mu > 0$. To show that $\lambda\mathtt{U} \circ \mu\mathtt{U} \subset (\lambda+\mu)\mathtt{U}$, let $(a,b) \in \lambda\mathtt{U} \circ \mu\mathtt{U}$. There is $z \in (\mathcal{Q}, \mathcal{W})$ such that $(a,z) \in \lambda\mathtt{U}$ and $(z,b) \in \mu\mathtt{U}$. There are finite subsets Δ, Θ of Γ such that

$$(a,z) = \lambda \sum_{\gamma \in \Delta} \lambda_\gamma \big(j_\gamma(a_\gamma), j_\gamma(z_\gamma) \big), (a_\gamma, z_\gamma) \in \tilde{v}_\gamma, \quad \text{with } \sum_{\gamma \in \Delta} \lambda_\gamma = 1,$$

and

$$(z,b) = \mu \sum_{\theta \in \Theta} \lambda_\theta \big(j_\theta(z_\theta), j_\theta(b_\theta) \big), (z_\theta, b_\theta) \in \tilde{v}_\theta, \quad \text{with } \sum_{\theta \in \Theta} \lambda_\theta = 1,$$

where v_γ, $\gamma \in \Delta$ and v_θ, $\theta \in \Theta$ are in defining $\mathtt{U}$.

For simplicity we write $a_\gamma = j_\gamma(a_\gamma)$, $\gamma \in \Gamma$, and so on. There are three possibilities.

(i) $\Delta = \Theta$. Without loss of generality let $\Delta = \Theta = \{1, 2, \dots, n\}$. Then

$$(a,z) = \lambda \sum_{i=1}^{n} \lambda_i (a_i, z_i), \quad (z,b) = \mu \sum_{i=1}^{n} \lambda_i' (z_i', b_i),$$

where $(a_i, z_i) \in J_i(\tilde{v}_i)$, $(z_i', b_i) \in J_i(\tilde{v}_i)$, $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \lambda_i' = 1$ and

$$z = \lambda \sum_{i=1}^{n} \lambda_i z_i = \mu \sum_{i=1}^{n} \lambda_i' z_i',$$

hence $\lambda\lambda_i z_i = \mu\lambda_i' z_i'$, $i = 1, 2, \dots, n$. Now we have

$$\begin{aligned}
(\lambda\lambda_i a_i, \mu\lambda_i' b_i) &= (\lambda\lambda_i a_i, \lambda\lambda_i z_i) \circ (\mu\lambda_i' z_i', \mu\lambda_i' b_i) \\
&= \lambda\lambda_i (a_i, z_i) \circ \mu\lambda_i' (z_i', b_i) \\
&\in \lambda\lambda_i \tilde{v}_i \circ \mu\lambda_i' \tilde{v}_i \subseteq (\lambda\lambda_i + \mu\lambda_i') J_i(\tilde{v}_i).
\end{aligned}$$

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Hence

$$(a,b)=\sum_{i=1}^{n}(\lambda\lambda_i a_i,\mu\lambda_i' b_i)\in(\lambda+\mu)\sum_{i=1}^{n}(\lambda\lambda_i+\mu\lambda_i')/\lambda+\mu J_i(\tilde{v}_i)$$

with $\sum_{i=1}^{n}(\lambda\lambda_i+\mu\lambda_i')/\lambda+\mu=1$, *i.e.*, $(a,b)\in(\lambda+\mu)\mathtt{U}$.

(ii) $\Delta\neq\Theta$ and $\Delta\cap\Theta=\varnothing$. Let $\Delta=\{1,2,\dots,n\}$, $\Theta=\{n{+}1,n{+}2,\dots,n{+}m\}$ say. Then for $1\leq i\leq n$ we have $a_i\neq 0$ but $b_i=0$ and $z_i=0$, also for $n<i\leq n+m$ we have $b_i\neq 0$ but $a_i=0$ and $z_i=0$. Hence for $1\leq i\leq n$, $(a,z)=\lambda\sum_{i=1}^{n}(\lambda_i a_i,0)$, $\sum_{i=1}^{n}\lambda_i=1$ and

$$\begin{aligned}\sum_{i=1}^{n}(\lambda\lambda_i a_i,\mu\lambda_i' b_i)&=\sum_{i=1}^{n}(\lambda\lambda_i a_i,\lambda\lambda_i z_i)\circ(\mu\lambda_i' z_i,\mu\lambda_i' b_i)\\&\in\sum_{i=1}^{n}(\lambda\lambda_i+\mu\lambda_i')J_i(\tilde{v}_i),\quad\text{where }\lambda_i'=0,1\leq i\leq n\end{aligned}$$

also for $n<i\leq n+m$, $(z,b)=\sum_{i=n+1}^{n+m}\lambda_i'(0,b_i)$, $\sum_{i=n+1}^{n+m}\lambda_i'=1$, and

$$\sum_{i=n+1}^{n+m}(\lambda\lambda_i a_i,\mu\lambda_i' b_i)\in\sum_{i=n+1}^{m+n}(\lambda\lambda_i+\mu\lambda_i')J_i(\tilde{v}_i),\quad\text{where }\lambda_i=0,n\leq i\leq n+m.$$

Hence

$$(a,b)=\sum_{i=1}^{n+m}(\lambda\lambda_i a_i,\mu\lambda_i' b_i)\in(\lambda+\mu)\sum_{i=1}^{n+m}(\lambda\lambda_i+\mu\lambda_i')/\lambda+\mu J_i(\tilde{v}_i)$$

with $\sum_{i=1}^{n+m}(\lambda\lambda_i+\mu\lambda_i')/\lambda+\mu=1$, *i.e.*, $(a,b)\in(\lambda+\mu)\mathtt{U}$.

(iii) $\Delta\neq\Theta$, $\Delta\cap\Theta\neq\varnothing$. Put $\Delta\cup\Theta=(\Delta-\Theta)\cup(\Delta\cap\Theta)\cup\Theta-\Delta$. For $\Delta\cap\Theta$ by (i) and for $\Delta-\Theta$ and $\Theta-\Delta$ by (ii) the requirements hold; combining these two we get the result for this case also.

The conditions (u_1) and (u_4) are trivial.

For (u_5), let $\mathtt{U}\in\mathbf{U}$ and $x=\sum_{\gamma\in\Delta'}j_\gamma(x_\gamma)\in\mathcal{Q}$. For each $\gamma\in\Delta'$, take $v_\gamma\in\mathcal{V}_\gamma$ as in defining U. There are $\mu_\gamma>0$, $\gamma\in\Delta'$ such that $\big(0,j_\gamma(x_\gamma)\big)\in\mu_\gamma J_\gamma(\tilde{v}_\gamma)$. Put $\mu=\sum_{\gamma\in\Delta'}\mu_\gamma$. Then

$$1/\mu(0,x)=1/\mu\Big(0,\sum_{\gamma\in\Delta'}j_\gamma(x_\gamma)\Big)\in\sum_{\gamma\in\Delta'}\mu_\gamma/\mu J_\gamma(\tilde{v}_\gamma)\subseteq\mathtt{U},$$

i.e., $(0,x)\in\mu\mathtt{U}$. Hence $\mathbf{U}$ is a convex quasi-uniform structure on $\mathcal{Q}$.

Now we show that $\mathbf{U}$ and $\widetilde{\mathcal{W}}$ are equivalent. If $\mathtt{U}\in\mathbf{U}$, then for each $\gamma\in\Gamma$, there is $v_\gamma\in\mathcal{V}_\gamma$ in defining $\mathtt{U}$. For this v_γ, we have $J_\gamma(\tilde{v}_\gamma)\subset\mathtt{U}$, hence j_γ is u-continuous. But $\widetilde{\mathcal{W}}$ is the finest convex quasi-uniform structure on $\mathcal{Q}$ that makes each j_γ u-continuous

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[3, Theorem 3.1]. Hence $\widetilde{\mathcal{W}}$ is finer than $\mathbf{U}$. Next, let $w \in \mathcal{W}$. For each $\gamma \in \Gamma$, $J_\gamma^{-1}(\tilde{w}) \in \widetilde{\mathcal{V}}_\gamma$, by definition. Put

$$\mathtt{U} = \bigcup \Big\{ \sum_{\gamma \in \Delta} \lambda_\gamma J_\gamma \circ J_\gamma^{-1}(\tilde{w}) : \sum_{\gamma \in \Delta} \lambda_\gamma = 1 \text{ and } \Delta \text{ is finite} \Big\}.$$

Since for each $\gamma \in \Delta$, $J_\gamma \circ J_\gamma^{-1}(\tilde{w}) \subseteq \tilde{w}$, we have

$$\sum_{\gamma \in \Delta} \lambda_\gamma J_\gamma \circ J_\gamma^{-1}(\tilde{w}) \subseteq \sum_{\gamma \in \Delta} \lambda_\gamma \tilde{w} = \tilde{w},$$

which shows that $\mathtt{U} \subseteq \tilde{w}$, hence $\mathbf{U}$ is also finer than $\widetilde{\mathcal{W}}$. ∎

If there is a one-to-one linear mapping t of $(\mathcal{P}, \mathcal{V})$ onto $(\mathcal{Q}, \mathcal{W})$ such that both t and its inverse t^{-1} are u-continuous, then these two locally convex cones are called *uniformly isomorphic (u-isomorphic)* and we say that t is a *u-isomorphism*.

Proposition 3.2 *Let* $(\mathcal{Q}, \mathcal{W}) = \sum_{\gamma \in \Gamma} (\mathcal{P}_\gamma, \mathcal{V}_\gamma)$. *Then*
(a) *for each* γ, $\phi_\gamma \colon (\mathcal{Q}, \mathcal{W}) \to (\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ *is a u-isomorphism,*
(b) *if* $A \subset (\mathcal{Q}, \mathcal{W})$, $A = \sum_{\gamma \in \Gamma} \phi_\gamma(A)$, *then*

$$\overline{A} = \sum_{\gamma \in \Gamma} \overline{\phi_\gamma(A)}, \quad \overline{\overline{A}} = \sum_{\gamma \in \Gamma} \overline{\overline{\phi_\gamma(A)}} \quad \textit{and} \quad \overline{A}^s = \sum_{\gamma \in \Gamma} \overline{\phi_\gamma(A)}^s.$$

Proof (a) Fix $\gamma \in \Gamma$ and let $v_{\gamma_\delta} \in \mathcal{V}_\gamma$, $\delta \in I_\gamma$. The neighborhoods v_λ, $\lambda \in \Gamma$, where $v_\lambda = v_{\gamma_\delta}$ for $\lambda = \gamma$ and $v_\lambda \in \mathcal{V}_\lambda$ otherwise, give some $\mathtt{U} \in \mathbf{U}$ in which $\Phi_\gamma(U) \subset \tilde{v}_{\gamma_\delta}$, *i.e.*, ϕ_γ is u-continuous. By [3, Theorem 3.1], $\phi_\gamma^{-1} = j_\gamma$ is also u-continuous. Therefore ϕ_γ is a u-isomorphism.

(b) We prove the first equality. Let $x \in \overline{A}$, $x = \sum_{\gamma \in \Delta} \phi_\gamma(x)$, where Δ is finite. If $\gamma \in \Delta$ and $v_\gamma \in \mathcal{V}_\gamma$, then part (a) gives some $\mathtt{U} \in \mathbf{U}$ such that $\Phi_\gamma(\mathtt{U}) \subseteq \tilde{v}_\gamma$. So $\big(\phi_\gamma(x), \phi_\gamma(a)\big) \in \tilde{v}_\gamma$ for some $a \in (x)\mathtt{U} \cap A$. Hence $\phi_\gamma(x) \in \overline{\phi_\gamma(A)}$. Conversely, let $x = \sum_{\gamma \in \Delta} \phi_\gamma(x)$, where Δ has n elements; say, and $\phi_\gamma(x) \in \overline{\phi_\gamma(A)}$, for each $\gamma \in \Delta$. Let $w \in \mathcal{W}$ and $\mathtt{U} \in \mathbf{U}$ with $\mathtt{U} \subset 1/n\tilde{w}$. We can find some $a \in A$ such that $\phi_\gamma(a) \in \big(\phi_\gamma(x)\big) v_\gamma \cap \phi_\gamma(A)$ for each $v_\gamma \in \mathcal{V}_\gamma$ in defining $\mathtt{U}$ and $\gamma \in \Delta$, which yields

$$(x, a) = \sum_{\gamma \in \Delta} \big(\phi_\gamma(x), \phi_\gamma(a)\big) \in \sum_{\gamma \in \Delta} \tilde{v}_\gamma \subset n\mathtt{U} \subset \tilde{w},$$

i.e., $a \in (x)\mathtt{U} \cap A$. So $x \in \overline{A}$. ∎

Corollary 3.3 *Let* $(\mathcal{Q}, \mathcal{W}) = \sum_{\gamma \in \Gamma} (\mathcal{P}_\gamma, \mathcal{V}_\gamma)$. *Then*
(a) *for each* $x \in \mathcal{Q}$, $x = \sum_{\gamma \in \Gamma} \phi_\gamma(x)$, *we have*

$$\overline{x} = \sum_{\gamma \in \Gamma} \overline{\phi_\gamma(x)}, \quad \overline{\overline{x}} = \sum_{\gamma \in \Gamma} \overline{\overline{\phi_\gamma(x)}} \quad \textit{and} \quad \overline{x}^s = \sum_{\gamma \in \Gamma} \overline{\phi_\gamma(x)}^s,$$

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(b) $(\mathcal{Q},\mathcal{W})$ *is separated if and only if each* $(\mathcal{P}_\gamma,\mathcal{V}_\gamma)$, $\gamma\in\Gamma$ *is separated,*
(c) *if each* $\mathcal{P}_\gamma$ *is separated, then* $j_\gamma(\mathcal{P}_\gamma)$ *is closed with respect to the symmetric topology in* $(\mathcal{Q},\mathcal{W})$.

Remark 3.4 (i) $\big(\overline{\mathrm{DConv}}(\mathcal{Q}),\overline{\mathcal{W}}\big)$ is the direct sum of $\big(\overline{\mathrm{DConv}}(\mathcal{P}_\gamma),\overline{\mathcal{V}}_\gamma\big)$, $\gamma\in\Gamma$; in particular $(\overline{\mathcal{Q}},\overline{\mathcal{W}})$ is the direct sum of the locally convex cones $(\overline{\mathcal{P}}_\gamma,\overline{\mathcal{V}}_\gamma)$.

(ii) For each $\gamma\in\Gamma$ and $a_\gamma\in\mathcal{P}_\gamma$, $j_\gamma(\overline{a}_\gamma)$ and $j_\gamma(\overline{\overline{a}}_\gamma)$ is lower and upper complete in both of $(\mathcal{Q},\mathcal{W})$ and $(\mathcal{P},\mathcal{V})$, respectively. Also $j_\gamma(\overline{a}^s_\gamma)$ is lower, upper, and symmetric complete; in particular if the global preorders coincide with the original ones, for each $\gamma\in\Gamma$, $j_\gamma(\mathcal{P}^-_\gamma)$ and $j_\gamma(\mathcal{P}^+_\gamma)$ is lower and upper complete, respectively, and the subcone $j_\gamma(\mathcal{P}^-_\gamma\cap\mathcal{P}^+_\gamma)$ is lower, upper, and symmetric complete.

Proposition 3.5 *Let* $(\mathcal{P}_\gamma,\mathcal{V}_\gamma)$, $\gamma\in\Gamma$ *be locally convex cones. The direct sum convex quasi-uniform structure* $\widetilde{\mathcal{W}}$ *is finer than the product convex quasi-uniform structure* $\widetilde{\mathcal{V}}$. *For every finite subset* Δ *of* Γ, *these two coincide on* $\sum_{\gamma\in\Delta}\mathcal{P}_\gamma$. *The direct sum convex quasi-uniform structure* $\widetilde{\mathcal{W}}$ *induces the original convex quasi-uniform structure* $\widetilde{\mathcal{V}}_\gamma$ *on each* $\mathcal{P}_\gamma$, $\gamma\in\Gamma$.

Proof If $v\in\mathcal{V}$, $v=\bigcap_{i=1}^n\phi_{\gamma_i}^{-1}(\tilde{v}_{\gamma_i\delta_i})$, then $\mathtt{U}\subset\tilde{v}$, where $\mathtt{U}$ is defined by neighborhoods $v_\gamma=v_{\gamma_i\delta_i}$ for $\gamma=\gamma_i$ $(i=1,2,\dots,n)$ and $v_\gamma\in\mathcal{V}_\gamma$ otherwise. So $\widetilde{\mathcal{W}}$ is finer than $\widetilde{\mathcal{V}}$.

Let $w\in\mathcal{W}$ and $v_\gamma\in\mathcal{V}_\gamma$, $\tilde{v}_\gamma=J_\gamma^{-1}(\tilde{w})$ $(\gamma\in\Delta)$. If we put $\tilde{v}=\bigcap_{\gamma\in\Delta}\Phi_\gamma^{-1}(\tilde{v}_\gamma)$, then $1/n\tilde{v}\subset\tilde{w}$. Therefore $\widetilde{\mathcal{V}}$ is also finer than $\widetilde{\mathcal{W}}$. In the special case $\Delta=\{\gamma\}$, Proposition 3.2 (a) gives the last part. ∎

Lemma 3.6 *Let* $(\mathcal{P},\mathcal{V})$ *and* $(\mathcal{Q},\mathcal{W})$ *be the product and direct sum locally convex cones; respectively. Then*
(a) *the upper topology of* $(\mathcal{Q},\mathcal{W})$ *has a base whose members are closed with respect to the lower topology of* $(\mathcal{P},\mathcal{V})$,
(b) *the lower topology of* $(\mathcal{Q},\mathcal{W})$ *has a base whose members are closed with respect to the upper topology of* $(\mathcal{P},\mathcal{V})$,
(c) *the symmetric topology of* $(\mathcal{Q},\mathcal{W})$ *has a base whose members are closed with respect to the symmetric topology of* $(\mathcal{P},\mathcal{V})$.

Proof (a) We show that for each $\mathtt{U}\in\mathbf{U}$ and $a\in\mathcal{Q}$, $\overline{\mathtt{U}(a)}\subseteq 4\mathtt{U}(a)$, where $\overline{\mathtt{U}(a)}$ is closure of $\mathtt{U}(a)$ with respect to lower topology of $(\mathcal{P},\mathcal{V})$. Let $x\in\overline{\mathtt{U}(a)}$. There is a finite subset Δ of Γ, containing say n elements, such that $\phi_\gamma(a)=0$, for each $\gamma\notin\Delta$. Let $v\in\mathcal{V}$, $\tilde{v}=1/n\bigcap_{\gamma\in\Delta}\Phi_\gamma^{-1}(\mathtt{U}\cap\mathcal{P}_\gamma^2)$. There is some $y\in(x)v\cap\mathtt{U}(a)$, such that

$$\Big(\sum_{\gamma\in\Delta}\phi_\gamma(x),\sum_{\gamma\in\Delta}\phi_\gamma(y)\Big)\in\mathtt{U}\quad\text{and}\quad(y,a)\in\mathtt{U}.\tag{3.1}$$

Put $\Delta'=\{\gamma\in\Gamma:\phi_\gamma(x)\neq 0,\gamma\notin\Delta\}$ and denote by κ the number of the elements in Δ'. Fix $\gamma\in\Delta'$ and let $v_\gamma\in\mathcal{V}_\gamma$ be the corresponding 0-neighborhood in defining $\mathtt{U}$. By Proposition 2.1, $\phi_\gamma(x)\in\overline{\phi_\gamma\big(\mathtt{U}(a)\big)}$, which yields $\big(\phi_\gamma(x),\phi_\gamma(z)\big)\in\frac{1}{\kappa}\tilde{v}_\gamma$ for

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some $z \in \mathtt{U}(a)$ and by $(z, a) \in \mathtt{U}$, we choose some λ_γ with $0 \leq \lambda_\gamma \leq 1$ such that $\big(\phi_\gamma(z), \phi_\gamma(a)\big) \in \lambda_\gamma \tilde{v}_\gamma$. Thus

$$\begin{aligned}\big(\phi_\gamma(x), \phi_\gamma(a)\big) &= \big(\phi_\gamma(x), \phi_\gamma(z)\big) \circ \big(\phi_\gamma(z), \phi_\gamma(a)\big) \\ &\in \tfrac{1}{\kappa}\tilde{v}_\gamma \circ \lambda_\gamma \tilde{v}_\gamma \subseteq (\tfrac{1}{\kappa} + \lambda_\gamma)\tilde{v}_\gamma.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{\gamma\in\Delta'} \big(\phi_\gamma(x), \phi_\gamma(a)\big) &\in \frac{1}{\kappa} \sum_{\gamma\in\Delta'} \tilde{v}_\gamma + \sum_{\gamma\in\Delta'} \lambda_\gamma \tilde{v}_\gamma \\ &\subseteq \mathtt{U} + \mathtt{U} = 2\mathtt{U},\end{aligned}$$

which by (3.1) implies that

$$\begin{aligned}(x, a) &= \Big(\sum_{\gamma\in\Delta} \phi_\gamma(x), \sum_{\gamma\in\Delta} \phi_\gamma(a)\Big) + \Big(\sum_{\gamma\in\Delta'} \phi_\gamma(x), \sum_{\gamma\in\Delta'} \phi_\gamma(a)\Big) \\ &\in \mathtt{U} + \mathtt{U} + 2\mathtt{U} = 4\mathtt{U},\end{aligned}$$

i.e., $x \in 4\mathtt{U}(a)$. In a similar way we prove parts (b) and (c). ∎

Theorem 3.7 *Let each $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ be separated. Then*
(a) *if each $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is lower (upper) complete, then $(\mathcal{Q}, \mathcal{W})$ is lower (respectively, upper) complete,*
(b) *$(\mathcal{Q}, \mathcal{W})$ is symmetric complete if and only if each $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is symmetric complete.*

Proof (a) Let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a lower Cauchy net in $\mathcal{Q}$. Then $(x_\alpha)_{\alpha\in\mathcal{I}}$ is also lower Cauchy in $(\mathcal{P}, \mathcal{V})$, and since $(\mathcal{P}, \mathcal{V})$ is lower complete (Theorem 2.13), there is some $x \in \mathcal{P}$ such that $x_\alpha \downarrow x$ in $(\mathcal{P}, \mathcal{V})$. We show that $x \in \mathcal{Q}$ and $x_\alpha \downarrow x$ in $(\mathcal{Q}, \mathcal{W})$.

If $x = 0$, then $x \in \mathcal{Q}$. For $x \neq 0$, let Δ be the subset of Γ such that for each $\gamma \in \Delta$, $\phi_\gamma(x) \neq 0$. Since the upper topology of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $\gamma \in \Gamma$ is $\mathtt{T}_0$ [1, 3.9], for each $\gamma \in \Delta$ there is $v_\gamma \in \mathcal{V}_\gamma$ such that $\phi_\gamma(x) \notin v_\gamma(0_{\mathcal{P}_\gamma})$. Then for $v \in \mathcal{V}$, $\tilde{v} = \frac{1}{2}\bigcap_{\gamma\in\Delta} \Phi_\gamma^{-1}(\tilde{v}_\gamma)$, there is some $w \in \mathcal{W}$ with $\tilde{w} \subset \tilde{v}$ (Proposition 3.5). Take some $\alpha_w \in \mathcal{I}$ such that

$$x_\beta \leq x_\alpha + w \quad \text{for all } \alpha, \beta \in \mathcal{I} \quad \text{with} \quad \beta \geq \alpha \geq \alpha_w.$$

Suppose that $\alpha \geq \alpha_w$ is arbitrary. If Δ is infinite, there is $\delta \in \Delta$ with $\varphi_\delta(x_\alpha) = 0$, because $x_\alpha \in \mathcal{Q}$ and only finite number of $x_{\alpha\gamma}$ are nonzero. Put $\tilde{v} = 1/2\phi_\delta^{-1}(\tilde{v}_\delta)$, $v \in \mathcal{V}$. Since $x_\alpha \downarrow x$ in $(\mathcal{P}, \mathcal{V})$, we can find some $\eta \in \mathcal{I}$ with $\eta \geq \alpha$ such that $x_\eta \in (x)v$. Then

$$\begin{aligned}\big(\phi_\delta(x), 0\big) &= \big(\phi_\delta(x), \phi_\delta(x_\eta)\big) \circ \big(\phi_\delta(x_\eta), \phi_\delta(x_\alpha)\big) \\ &\in 1/2\tilde{v}_\delta \circ 1/2\tilde{v}_\delta \subseteq \tilde{v}_\delta,\end{aligned}$$

i.e., $\phi_\delta(x) \in v_\delta(0_{\mathcal{P}_\gamma})$, which is a contradiction. Hence Δ is finite and $x \in \mathcal{Q}$.

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Now we show that $x_\alpha \downarrow x$ in $\mathcal{Q}$. Let $w \in \mathcal{W}$ and $\mathtt{U} \subseteq \frac{1}{4}\tilde{w}$, $\mathtt{U} \in \mathbf{U}$. Since $(x_\alpha)_{\alpha\in\mathcal{I}}$ is lower Cauchy, there is $\alpha_{\mathtt{U}} \in \mathcal{I}$ such that

$$(x_\beta, x_\alpha) \in \mathtt{U} \quad \text{for all } \alpha, \beta \in \mathcal{I} \quad \text{with} \quad \beta \geq \alpha \geq \alpha_{\mathtt{U}}. \tag{3.2}$$

Fix $\alpha \geq \alpha_{\mathtt{U}}$ and let $v \in \mathcal{V}$. Since $x_\alpha \downarrow x$ in $(\mathcal{P}, \mathcal{V})$, there is some $\beta \in \mathcal{I}$ with $\beta \geq \alpha$ such that $x_\beta \in (x)v$ and so (3.2) yields $x_\beta \in (x)v \cap \mathtt{U}(x_\alpha)$. Thus $x \in \overline{\mathtt{U}(x_\alpha)}$. According to Lemma 3.6 (a), $\overline{\mathtt{U}(x_\alpha)} \subseteq 4\mathtt{U}(x_\alpha)$, which yields $x \in 4\mathtt{U}(x_\alpha) \subseteq w(x_\alpha)$ or $x_\alpha \in (x)w$. Therefore $x_\alpha \downarrow x$ in $(\mathcal{Q}, \mathcal{W})$.

(b) If $(\mathcal{Q}, \mathcal{W})$ is symmetric complete, then by Corollary 3.3 (c) and Proposition 2.8 (a), each $\mathcal{P}_\gamma$ is symmetric complete. The proof of the converse is similar to part (a). ∎

In [1], dual pair and X-topology are defined as follows.

Definition 3.8 A *dual pair* $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping

$$(a, x) \longmapsto \langle a, x\rangle : \mathcal{P} \times \mathcal{Q} \longrightarrow \overline{\mathbb{R}}.$$

Definition 3.9 Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair and X be a collection of subsets of $\mathcal{Q}$ such that:

(P_0) $\inf\{\langle a, x\rangle : x \in A\} > -\infty$ for all $a \in \mathcal{P}$ and $A \in X$;
(P_1) $\lambda A \in X$ for all $A \in X$ and $\lambda > 0$;
(P_2) for all $A, B \in X$ there is some $C \in X$ such that $A \cup B \subseteq C$.

For each $A \in X$ we define

$$U_A = \{(a, b) \in \mathcal{P} \times \mathcal{P} : \langle a, x\rangle \leq \langle b, x\rangle + 1 \text{ for all } x \in A\}.$$

The set of all U_A, $A \in X$ is a convex quasi-uniform structure with property (u_5) and defines a locally convex structure on $\mathcal{P}$. This is called the *X-topology* on $\mathcal{P}$. For each $A \in X$ we denote by v_A the (abstract) 0-neighborhood induced on $\mathcal{P}$ by U_A. Therefore $(a, b) \in U_A$ if and only if $a \leq b + v_A$. Obviously an X-topology on $\mathcal{P}$ defines at the same time upper, lower, and symmetric topologies on $\mathcal{P}$.

Theorem 3.10 *Let $(\mathcal{Q}, \mathcal{W})$ be direct sum of the locally convex cones $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$. Then*
(a) *if $v_\gamma \in \mathcal{V}_\gamma$, $\gamma \in \Gamma$ and $\mathtt{U} \in \mathbf{U}$, which is defined by these neighborhoods, then $w^\circ_{\mathtt{U}} = \times_{\gamma\in\Gamma} v^\circ_\gamma$; in particular $\mathcal{Q}^* = \times_{\gamma\in\Gamma}\mathcal{P}^*_\gamma$,*
(b) *if each $\mathcal{P}^*_\gamma$, $\gamma \in \Gamma$ has the X_γ-topology, then the product $\times_{\gamma\in\Gamma}\mathcal{P}^*_\gamma$ has the X-topology, where X is the set of all finite unions of the sets $j_\gamma(X_\gamma)$.*

Proof (a) Let $\mathtt{U} \in \mathbf{U}$ and $\mu \in w^\circ_{\mathtt{U}}$. Fix $\gamma \in \Gamma$. Given $(a_{\gamma_\delta}, b_{\gamma_\delta}) \in \tilde{v}_\gamma$, $\delta \in \mathcal{I}_\gamma$ yields $j_\gamma(a_{\gamma_\delta}) \leq j_\gamma(b_{\gamma_\delta}) + w_{\mathtt{U}}$, hence $\mu \circ j_\gamma(a_{\gamma_\delta}) \leq \mu \circ j_\gamma(b_{\gamma_\delta}) + 1$, *i.e.*, $\mu_\gamma(a_{\gamma_\delta}) \leq \mu_\gamma(b_{\gamma_\delta}) + 1$ or $\mu_\gamma \in v^\circ_\gamma$. Conversely, let $\mu \in \times_{\gamma\in\Gamma} v^\circ_\gamma$ and $(a, b) \in \mathtt{U}$, then

$$(a, b) = \sum_{\gamma\in\Delta} \lambda_\gamma\big(j_\gamma(a_\gamma), j_\gamma(b_\gamma)\big), \quad \text{where } (a_\gamma, b_\gamma) \in \tilde{v}_\gamma \text{ and } \sum_{\gamma\in\Delta} \lambda_\gamma = 1,$$

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which yields $\mu_\gamma(a_\gamma) \le \mu_\gamma(b_\gamma) + 1$ or $\mu \circ j_\gamma(a_\gamma) \le \mu \circ j_\gamma(b_\gamma) + 1$. Therefore

$$\sum_{\gamma\in\Delta} \lambda_\gamma \mu \circ j_\gamma(a_\gamma) \le \sum_{\gamma\in\Delta} \lambda_\gamma \mu \circ j_\gamma(a_\gamma) + \sum_{\gamma\in\Delta} \lambda_\gamma,$$

i.e., $\mu(a) \le \mu(b) + 1$.

Part (b) follows from [2, Theorem 3.15]. Indeed the adjoint operator of each j_γ is the projection mapping ϕ'_γ of $\times_{\gamma\in\Gamma}\mathcal{P}^*_\gamma$ onto $\mathcal{P}^*_\gamma$. ■

Theorem 3.11 *Let $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $\gamma \in \Gamma$, be locally convex cones.*

(a) *If $v_{\gamma_i} \in \mathcal{V}_{\gamma_i}$, $i = 1, 2, \dots, n$, then $v^\circ \subseteq \sum_{i=1}^n v^\circ_{\gamma_i} \subseteq nv^\circ$, where $\tilde{v} = \bigcap_{i=1}^n \Phi^{-1}_{\gamma_i}(\tilde{v}_{\gamma_i})$; in particular $\mathcal{P}^* = \sum_{\gamma\in\Gamma} \mathcal{P}^*_\gamma$.*

(b) *If each $\mathcal{P}^*_\gamma$ has the X_γ-topology, then the direct sum locally convex cone $\sum_{\gamma\in\Gamma} \mathcal{P}^*_\gamma$ has the X-topology, where X is the set of all products $\times_{\gamma\in\Gamma} A_\gamma$ with $A_\gamma \in X_\gamma$ for each $\gamma \in \Gamma$.*

Proof (a) If $\mu \in v^\circ$, then μ is bounded on $v(0)v$. We claim that $\mu = \sum_{i=1}^n \mu_{\gamma_i}$ and $\mu_{\gamma_i} \in v^\circ_{\gamma_i}$ for $i = 1, 2, \dots, n$. The mapping μ vanishes on $\mathcal{P}_\gamma$ for each γ ($\gamma \ne \gamma_i$, $i = 1, 2, \dots, n$). Let $x_{\gamma_\delta} \in \mathcal{P}_\gamma$, $\delta \in I_\gamma$. Since $(\phi_{\gamma_i} \circ j_\gamma)(x_{\gamma_\delta}) = 0$, we have

$$\big(j_\gamma(x_{\gamma_\delta}), 0\big) \in \Phi^{-1}_{\gamma_i}(\tilde{v}_{\gamma_i}), \quad \big(0, j_\gamma(x_{\gamma_\delta})\big) \in \Phi^{-1}_{\gamma_i}(\tilde{v}_{\gamma_i}) \quad (i = 1, 2, \dots, n),$$

so $j_\gamma(x_{\gamma_\delta}) \in v(0)v$, and since $\mu_\gamma(x_{\gamma_\delta}) = (\mu \circ j_\gamma)(x_{\gamma_\delta})$, by boundedness of $\mu\big(v(0)v\big)$ we conclude that $\mu_\gamma(x_{\gamma_\delta}) = 0$. Now, let $a_{\gamma_i}, b_{\gamma_i} \in \mathcal{P}_{\gamma_i}$ with $a_{\gamma_i} \le b_{\gamma_i} + v_{\gamma_i}$. Then $j_{\gamma_i}(a_{\gamma_i}) \le j_{\gamma_i}(b_{\gamma_i}) + v$, which implies that $\mu \circ j_{\gamma_i}(a_{\gamma_i}) \le \mu \circ j_{\gamma_i}(b_{\gamma_i}) + 1$ or $\mu_{\gamma_i}(a_{\gamma_i}) \le \mu_{\gamma_i}(b_{\gamma_i}) + 1$. So $\mu_{\gamma_i} \in v^\circ_{\gamma_i}$.

Suppose that $\mu \in \sum_{i=1}^n \mu_{\gamma_i}$, $\mu_{\gamma_i} \in v^\circ_{\gamma_i}$ and $a \le b + v$. Then $a_{\gamma_i} \le b_{\gamma_i} + v_{\gamma_i}$ for $i = 1, 2, \dots, n$, which yields $\mu_{\gamma_i}(a_{\gamma_i}) \le \mu_{\gamma_i}(b_{\gamma_i}) + 1$ or $\mu(a) \le \mu(b) + n$. Hence $\mu \in nv^\circ$.

(b) Let us denote by $\widetilde{\mathcal{W}}_X = \{\widetilde{w}_A : A \in X\}$ the convex quasi-uniform structure on $\sum_{\gamma\in\Gamma} \mathcal{P}^*_\gamma$ induced by the X-topology. By Theorem 3.1, the direct sum convex quasi-uniform structure on $\sum_{\gamma\in\Gamma} \mathcal{P}^*_\gamma$ induced by $\widetilde{\mathcal{V}}_{X_\gamma}$, $\gamma \in \Gamma$, where $\widetilde{\mathcal{V}}_{X_\gamma} = \{\tilde{v}_{A_\gamma} : A_\gamma \in X_\gamma\}$, is equivalent to a convex quasi-uniform structure $\mathbf{U}$, where each $\mathrm{U} \in \mathbf{U}$ is defined as

$$\mathrm{U} = \bigcup\Big\{\sum_{\gamma\in\Delta} \lambda_\gamma J_\gamma(\tilde{v}_{A_\gamma}) : \sum_{\gamma\in\Delta} \lambda_\gamma = 1 \text{ and } \Delta \text{ is finite}\Big\}, \quad A_\gamma \in X_\gamma, \gamma \in \Gamma.$$

The proof will therefore be complete if we show that $\mathrm{U} \subseteq \widetilde{w}_A \subseteq n\mathrm{U}$ for each $A \in X$, $A = \times_{\gamma\in\Gamma} A_\gamma$ and some $n \in \mathbb{N}$, where U is defined by A.

For $\mathrm{U} \subseteq \widetilde{w}_A$, let $(\tau, k) \in \mathrm{U}$. Then for a finite subset Δ of Γ we have

$$(\tau, k) = \sum_{\gamma\in\Delta} \lambda_\gamma (j_\gamma \circ \tau_\gamma, j_\gamma \circ k_\gamma), \quad \text{where } (\tau_\gamma, k_\gamma) \in \tilde{v}_{A_\gamma}, \text{ and } \sum_{\gamma\in\Delta} \lambda_\gamma = 1.$$

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Hence $x = (x_\gamma) \in A$ yields

$$\begin{aligned}\langle \tau, x\rangle &= \Big\langle \sum_{\gamma\in\Delta} \lambda_\gamma j_\gamma \circ \tau_\gamma, x\Big\rangle \\ &\leq \Big\langle \sum_{\gamma\in\Delta} \lambda_\gamma j_\gamma \circ k_\gamma, x\Big\rangle + \sum_{\gamma\in\Delta} \lambda_\gamma = \langle k, x\rangle + 1,\end{aligned}$$

i.e., $(\tau, k) \in \widetilde{w}_A$. For the second inclusion, if $(\tau, k) \in \tilde{w}_A$ then $(\tau, k) = \sum_{\gamma\in\Delta}(j_\gamma \circ \tau_\gamma, j_\gamma \circ k_\gamma)$, where $(\tau_\gamma, k_\gamma) \in \tilde{v}_{A_\gamma}$, hence

$$(\tau, k) \in \sum_{\gamma\in\Delta} J_\gamma(\tilde{v}_{A_\gamma}) \subseteq n\mathrm{U}, \quad \text{for some } n \in \mathbb{N}. \qquad \blacksquare$$

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