Canad. Math. Bull. Vol. **22** (4), 1979

ON REGULAR-CLOSED AND MINIMAL REGULAR SPACES

BY

JAMES E. JOSEPH

ABSTRACT. This paper contains several characterizations of regular-closed and minimal regular topological spaces along with some related properties.

1. Introduction. In [2], characterizations of regular-closed and minimal regular spaces have been given in terms of arbitrary filterbases and a type of convergence called s-convergence. In the present article, we employ these characterizations 1) to obtain characterizations of regular-closed spaces in terms of projections and in terms of graphs of functions into the spaces and 2) to obtain characterizations of minimal regular spaces in terms of graphs of functions into the spaces; both of these—projections and graphs—are utilized in conjunction with a class $\mathcal{S}$ of spaces containing as a subclass the Hausdorff completely normal fully normal spaces to effect the characterizations. See [1] for a survey of results on regular-closed and minimal regular spaces.

2. Preliminaries. The closure of a subset K of a space will be denoted by $\mathrm{cl}(K)$; $\mathrm{ad}\,\Omega$ will represent the adherence of a filterbase Ω on a space. If $\psi, \lambda : X \to Y$ are functions, $E(\psi, \lambda, X, Y)$ will represent $\{x \in X : \psi(x) = \lambda(x)\}$. We let $G(\psi)$ denote the graph of ψ. We assume that regular spaces are T_1. A family of open subsets, Σ, of a space X is called a *strinkable family of open subsets about a point* x of X if for each $V \in \Sigma$, there is a $W \in \Sigma$ satisfying $x \in V \subset \mathrm{cl}(V) \subset W$ and a point x in a space is in *the* s*-adherence of a filterbase* Ω *on the space* ($x \in s-\mathrm{ad}\,\Omega$) if for each strinkable family of open subsets, Σ, about x there is a $V \in \Sigma$ satisfying $V \cap F \neq \emptyset$ for each $F \in \Omega$ [2]. We recall that a point x in a space is in the θ*-closure of a subset* K ($x \in \theta - \mathrm{cl}(K)$) of the space if each V open about x satisfies $K \cap \mathrm{cl}(V) \neq \emptyset$; and a point x in a space is in the θ*-adherence of a filterbase* Ω *on the space* ($x \in \theta - \mathrm{ad}\,\Omega$) if $x \in \theta - \mathrm{cl}(F)$ for each $F \in \Omega$ [8]. K is θ-closed if $\theta - \mathrm{cl}(K) = K$. It is obvious that $\mathrm{ad}\,\Omega \subset \theta - \mathrm{ad}\,\Omega \subset s - \mathrm{ad}\,\Omega$ for each filterbase Ω on a space. A function $\psi : X \to Y$ is *weakly-continuous* if for each $x \in X$ and U open about $\psi(x)$, there is a V open about x

Received by the editors January 25, 1977 and, in revised form, April 25, 1978.

AMS (MOS) subject classifications (1970). Primary 54D25, 54D30; Secondary 54C10.

Key words and phrases. Regular closed spaces, minimal regular spaces, functions with closed graphs.

https://doi.org/10.4153/CMB-1979-064-8 Published online by Cambridge University Press

with $\psi(V)\subset \mathrm{cl}(U)$ [6]. It is known that a function $\psi: X\to Y$ is weakly-continuous if and only if $\psi(\mathrm{ad}\,\Omega)\subset \theta-\mathrm{ad}\,\psi(\Omega)$ for each filterbase Ω on X. We define a function $\psi: X\to Y$ to be *s-weakly-continuous* if $\psi(\mathrm{ad}\,\Omega)\subset s-\mathrm{ad}\,\psi(\Omega)$ for each filterbase Ω on X. It is clear that a weakly-continuous function is s-weakly-continuous. In [3] a function $\psi: X\to Y$ is said to have a *strongly-closed graph* if for each $(x, y)\in(X\times Y)-G(\psi)$ there is a V open about x and U open about y satisfying $(V\times \mathrm{cl}(U))\cap G(\psi)=\emptyset$. It is known that a function $\psi: X\to Y$ has a strongly-closed graph if and only if for each $x\in X$, $\{\psi(x)\}$ is θ-closed in Y and $\theta-\mathrm{ad}\,\psi(\Omega)\subset\{\psi(x)\}$ for each filterbase Ω on $X-\{x\}$ with $\Omega\to x$. We say that $\psi: X\to Y$ has an *s-strongly-subclosed graph* if $s-\mathrm{ad}\,\psi(\Omega)\subset\{\psi(x)\}$ for each $x\in X$ and filterbase Ω on $X-\{x\}$ with $\Omega\to x$. Let X be a set, let $x_0\in X$ and let Ω be a filterbase on X; $\{A\subset X: x_0\in X-A$ or $F\cup\{x_0\}\subset A$ for some $F\in\Omega\}$ is a topology on X and will be called *the topology on X associated with x_0 and Ω*. X equipped with this topology will be called *the space associated with x_0 and Ω*. We will denote this space by $X(x_0,\Omega)$. The following readily established theorem is used frequently in the sequel.

THEOREM. *Let X be a set, let $x_0\in X$ and let Ω be a filterbase on X which has empty intersection on $X-\{x_0\}$. Then $X(x_0,\Omega)$ is in class $\mathcal{S}$.*

3. Characterizations of regular-closed spaces via graphs. In [3], it is proved that a Hausdorff space Y is H-closed if and only if all functions with strongly-closed graphs from a space in class $\mathcal{S}$ to Y are weakly-continuous. In this section, we present-via graphs-some characterizations of spaces satisfying condition $\mathcal{R}(i)$ (below), which has been observed by Herrington [2] to be equivalent to regular-closedness for regular spaces. This condition is also equivalent to the $\mathcal{R}(i)$ condition of Scarborough and Stone [7].

$\mathcal{R}(i)$ *Each filterbase on the space has a nonempty s-adherence.*

We now use graphs to give several characterizations of $\mathcal{R}(i)$ spaces which are not necessarily regular.

3.1. THEOREM. *The following statements are equivalent for a space Y:*

(a) *Y is $\mathcal{R}(i)$.*

(b) *For each space H, each function $\psi: X\to Y$ with an s-strongly-subclosed graph is s-weakly-continuous.*

(c) *For each space X, and any functions $\psi, \lambda: X\to Y$ with s-strongly-subclosed graphs, $E(\psi, \lambda, X, Y)$ is closed in X.*

(d) *For each space X and any functions $\psi, \lambda: X\to Y$ with s-strongly-subclosed graphs, $E(\psi, \lambda, X, Y)=X$ whenever $E(\psi, \lambda, X, Y)$ is dense in X.*

Proof that (a) implies (b). Let Y be an $\mathcal{R}(i)$ space, X be a space, and $\psi: X\to Y$ be a function with an s-strongly-subclosed graph; let Ω be a filterbase on X, choose $y\in\psi(\mathrm{ad}\,\Omega)$ and $x\in\mathrm{ad}\,\Omega$ with $\psi(x)=y$. If $\Omega_1=\{(V\cap F)-\{x\}: V$ open about x, $F\in\Omega\}$ is not a filterbase on $X-\{x\}$, then $x\in F$

for each $F\in\Omega$, so $y=\psi(x)\in\psi(F)$ for each $F\in\Omega$. If Ω_1 is a filterbase on $X-\{x\}$, then $\Omega_1\to x$ and, since Y is $\mathcal{R}(i)$ and $G(\psi)$ is s-strongly-subclosed, we have $\emptyset\neq s-\text{ad }\psi(\Omega_1)\subset\{\psi(x)\}$. This gives $y\in s-\text{ad }\psi(\Omega_1)\subset s-\text{ad }\psi(\Omega)$.

Proof that (*b*) implies (*c*). Let Y be $\mathcal{R}(i)$, X be a space, and $\psi,\lambda:X\to Y$ be functions with s-strongly-subclosed graphs. Let $x\in\text{cl}(E(\psi,\lambda,X,Y))-E(\psi,\lambda,X,Y)$. There is a filterbase Ω on $E(\psi,\lambda,X,Y)$ with $\Omega\to x$. Since λ has an s-strongly-subclosed graph, and ψ is s-weakly-continuous from (b), we get $\{\psi(x)\}\subset\psi(\text{ad }\Omega)\subset s-\text{ad }\psi(\Omega)=s-\text{ad }\lambda(\Omega)\subset\{\lambda(x)\}$. This is a contradiction.

Proof that (c) implies (*d*). Let Y be an $\mathcal{R}(i)$ space, X be a space and $\psi,\lambda:X\to Y$ be functions with s-strongly-subclosed graphs. From (c), $E(\psi,\lambda,X,Y)$ is closed in X; so $E(\psi,\lambda,X,Y)=X$ if $E(\psi,\lambda,X,Y)$ is dense in X.

Proof that (*d*) implies (*a*). Suppose Ω is a filterbase on the space Y with $s-\text{ad }\Omega=\emptyset$. Choose x_0, $y_0\in Y$ with $x_0\neq y_0$ and let $\Omega_1=\{F-\{x_0,y_0\}:F\in\Omega\}$. Then Ω_1 is a filterbase on Y. Let $\psi:Y(x_0,\Omega_1)\to Y$ be the identity function and define $\lambda:Y(x_0,\Omega_1)\to Y$ by $\lambda(x_0)=y_0$, and $\lambda(x)=x$ otherwise. Then $E(\psi,\lambda,Y(x_0,\Omega),Y)=Y-\{x_0\}$ which is dense in $Y(x_0,\Omega_1)$. We will reach a contradiction of (d) by establishing that ψ and λ have s-strongly-subclosed graphs.

(a) $G(\psi)$ *is* s*-strongly-subclosed.* Let $x\in Y$ and let Ω_2 be a filterbase on $Y-\{x\}$ with $\Omega_2\to x$ in $Y(x_0,\Omega_1)$. Then $x=x_0$ and Ω_2 is stronger than Ω; so, by Theorem 2.3 (b) of [2], we have $s-\text{ad }\psi(\Omega_2)=s-\text{ad }\Omega_2\subset s-\text{ad }\Omega\subset\{x_0\}$. Hence $G(\psi)$ is s-strongly-subclosed.

(b) $G(\lambda)$ *is* s*-strongly-subclosed.* Let $x\in Y$ and let Ω_2 be a filterbase on $Y-\{x\}$ with $\Omega_2\to x$ in $Y(x_0,\Omega_1)$. Again, $x=x_0$ and Ω_2 is stronger than Ω_1; so we have $s-\text{ad }\lambda(\Omega_2)=s-\text{ad }\Omega_2\subset s-\text{ad }\Omega=\emptyset\subset\{\lambda(x)\}$. Thus $G(\lambda)$ is s-strongly-subclosed.

The proof is complete.

4. Characterizations of regular-closed spaces via projections. In [5], the author has proved that a Hausdorff space is H-closed if and only if the projection $\pi_y:X\times Y\to Y$ maps θ-closed subsets onto θ-closed subsets for every space Y. In this section, we give a similar characterization of regular-closed spaces. A point (x,y) in a product space $X\times Y$ is in the *first-coordinate* s*-closure of* $K\subset X\times Y((x,y)\in(1)s-\text{cl}(K))$ if for each strinkable family of open sets, Σ, about x, there is a $V\in\Sigma$ satisfying $(V\times W)\cap K=\emptyset$ for each W open about y. $K\subset X\times Y$ is *first-coordinate* s*-closed* (K *is* $(1)s$*-closed*) if $(1)s-\text{cl}(K)=K$.

4.1. Theorem. *A space* X *is* $\mathcal{R}(i)$ *if and only if* $\pi_y:X\times Y\to Y$ *satisfies* $\text{cl}(\pi_y(K))\subset\pi_y((1)s-\text{cl}(K))$ *for every* Y *in class* $\mathcal{S}$ *and* $K\subset X\times Y$.

https://doi.org/10.4153/CMB-1979-064-8 Published online by Cambridge University Press

Proof. *Strong necessity*. Let X be $\mathscr{R}(i)$, Y be any space, and $K \subset X \times Y$. Let $y \in \text{cl}(\pi_y(K))$; then $\Omega = \{\pi_x(K \cap (X \times W)) : W$ is open about $y\}$ is a filterbase on X. Let $x \in s-\text{ad}(\Omega)$; let Σ be a strinkable family of open sets about x. There is a $V \in \Sigma$ satisfying $V \cap \pi_x(K \cap (X \times W)) \neq \emptyset$ for each W open about y. So $(V \times W) \cap K \neq \emptyset$ is true for each W open about y. Thus $(x, y) \in (1)s-\text{cl}(K)$ and $y \in \pi_y((1)s-\text{cl}(K))$.

Sufficiency. Let X be a space satisfying the condition and let Ω be a filterbase on X. Choose $y_0 \notin X$, let $K = \{(x, x) : x \in X\}$ and $Y = X \cup \{y_0\}$. Then $K \subset X \times Y(y_0, \Omega)$ and $y_0 \in \text{cl}(\pi_y(K))$. So, by hypothesis, we have $y_0 \in \pi_y((1)s-\text{cl}(K))$. Let $x \in X$ satisfy $(x, y_0) \in (1)s-\text{cl}(K)$ and let Σ be a strinkable family of open sets about x. There is a $V \in \Sigma$ satisfying $(V \times (F \cup \{y_0\})) \cap K \neq \emptyset$ for each $F \in \Omega$. Thus $V \cap F \neq \emptyset$ for each $F \in \Omega$, and $x \in s-\text{ad}\,\Omega$.

The proof is complete.

4.2. THEOREM. *A regular space X is regular-closed if and only if $\pi_y : X \times Y \to Y$ maps* (1) *s-closed subsets onto closed subsets of Y for each space Y in class $\mathscr{S}$.*

Proof. *Strong necessity*. This is immediate from the strong necessity of Theorem 4.1.

Sufficiency. Let X be regular satisfying the condition and let Ω be a filterbase on X with $s-\text{ad}\,\Omega = \emptyset$. Choose $y_0 \notin X$, let $K = \{(x, x) : x \in X\}$ and $Y = X \cup \{y_0\}$. Then $K \subset X \times Y(y_0, \Omega)$ and $y_0 \in \text{cl}(\pi_y(K)) - \pi_y(K)$. We show that K is (1)s-closed to obtain a contradiction. Let $(x, y) \in (X \times Y(y_0, \Omega)) - K$. Then $x \neq y_0$ and $x \neq y$. There is a strinkable family of open sets, Σ_1, about x such that for each $V \in \Sigma_1$, some $F(V) \in \Omega$ satisfies $F(V) \cap V = \emptyset$. Since X is regular, $\Sigma_2 = \{X - \text{cl}(V) : V$ is open about y and $x \notin \text{cl}(V)\}$ is a strinkable family of open sets about x. Now, $\Sigma_3 = \{V \cap W : V \in \Sigma_1, W \in \Sigma_2\}$ is a strinkable family of open sets about x satisfying $((V \cap W) \times (F(V) \cup \{y, y_0\})) \cap K = \emptyset$ for each $V \cap W \in \Sigma_3$.

The proof is complete.

5. Characterizations of minimal regular spaces via graphs. In [4], it is proved that a Hausdorff space Y is minimal Hausdorff if and only if all functions with strongly-closed graphs from a space in class $\mathscr{S}$ to Y are continuous. In this section, we present-via graphs-some characterizations of spaces satisfying condition $\mathscr{R}(ii)$ (below) which has been observed by Herrington [2] to be equivalent to minimal regularness for regular spaces.

$\mathscr{R}(ii)$ *Each filterbase on the space with at most one point in its s-adherence converges.*

We use graphs to give several characterizations of $\mathscr{R}(ii)$ spaces which are not necessarily regular. We say that a function $\psi : X \to Y$ has a *subclosed graph* if $\text{ad}\,\psi(\Omega) \subset \{\psi(x)\}$ for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$. We recall that a function $\psi : X \to Y$ is continuous if and only if $\psi(\text{ad}\,\Omega) \subset \text{ad}\,\psi(\Omega)$ for each filterbase Ω on X.

5.1. THEOREM. *The following statements are equivalent for a space Y:*
(*a*) *Y is $\mathcal{R}(ii)$.*
(*b*) *For each space X, each function $\psi: X \to Y$ with an s-strongly-subclosed graph is continuous.*
(*c*) *For each space X and any functions $\psi, \lambda: X \to Y$ with s-strongly-subclosed and subclosed graph, respectively, $E(\psi, \lambda, X, Y)$ is closed in $\mathcal{X}$.*
(*d*) *For each space X and any functions $\psi, \lambda: X \to Y$ with s-strongly-subclosed graph and subclosed graph, respectively, $E(\psi, \lambda, X, Y) = X$ whenever $E(\psi, \lambda, X, Y)$ is dense in X.*

Proof that (*a*) implies (*b*). Let Y be an $\mathcal{R}(ii)$ space, let X be a space, let $\psi: X \to Y$ be a function with an s-strongly-subclosed graph and let $x \in X$. If $\{x\}$ is open in X, ψ is continuous at x. If not, then $\Omega = \{V - \{x\}: V \text{ is open about } x\}$ is a filterbase on $X - \{x\}$ and $\Omega \to x$. So $s - \text{ad}\ \psi(\Omega) \subset \{\psi(x)\}$ since ψ has an s-strongly-subclosed graph. Therefore $\psi(\Omega) \to \psi(x)$ and for each W open about $\psi(x)$, there is a V open about x with $\psi(V) \subset W$.

Proof that (*b*) implies (*c*). Let Y be $\mathcal{R}(ii)$, let X be a space and let $\psi, \lambda: X \to Y$ be functions with s-strongly-subclosed graph and subclosed graph, respectively. Then ψ is continuous from (*b*). Let $x \in \text{cl}(E(\psi, \lambda, X, Y)) - E(\psi, \lambda, X, Y)$ and let Ω be a filterbase on $E(\psi, \lambda, X, Y)$ with $\Omega \to x$. Then $\{\psi(x)\} \subset \psi(\text{ad}\ \Omega) \subset \text{ad}\ \psi(\Omega) = \text{ad}\ \lambda(\Omega) \subset \{\lambda(x)\}$. This is a contradiction, so $E(\psi, \lambda, X, Y)$ is closed.

The proofs that (*c*) implies (*d*) and that (*d*) implies (*a*) parallel the corresponding proofs that (*c*) implies (*d*), and (*d*) implies (*a*) in Theorem 3.1.

The proof is complete.

5.2. REMARK. If replacements of phrases are made in Theorems 3.1 or 5.1 as indicated below, the statement resulting from such replacements is valid.

i) *In* (*b*), (*c*), *or* (*d*) *replace the phrase "for each space X" by the phrase "for each space X in class $\mathcal{S}$".*
ii) *In* (*b*) *or* (*c*), *replace "function*(*s*)" *by "bijection*(*s*)".
iii) *In* (*d*), *replace "functions" by "functions* (*one a bijection*)".

6. First countable regular spaces. See [1] for definitions and results used but not given here. Noting that $X(x_0, \Omega)$ is metrizable when $\Omega = \{F_n\}$ is countable with empty intersection on $X - \{x_0\}$ ($X(x_0, \Omega)$ is regular and $\{\mathcal{V}(n)\}$ defined by $\mathcal{V}(n) = \{F_n \cup \{x_0\}\} \cup \{\{x\}: x \in X - (F_n \cup \{x_0\})\}$ is a σ-locally finite base), we may establish the following theorems by appropriate use of the first countability and arguments similar to those in the last section.

6.1. THEOREM. *The following statements are equivalent for a first countable regular space Y. $\mathcal{A}$ may represent the class of first countable spaces, the class of first countable spaces in class $\mathcal{S}$, or the class of metric spaces.*

https://doi.org/10.4153/CMB-1979-064-8 Published online by Cambridge University Press

(a) *Each countable filterbase on Y has nonvoid s-adherence.*
(b) *Each sequence in Y s-accumulates to some point in Y* [2].
(c) *For each $X \in \mathcal{A}$, each function (bijection) $\psi: X \to Y$ with an s-strongly-subclosed graph is s-weakly-continuous.*
(d) *For each $X \in \mathcal{A}$ and any two functions (bijections) $\psi, \lambda: X \to Y$ with s-strongly-subclosed graphs, $E(\psi, \lambda, X, Y)$ is closed in X.*
(e) *For each $X \in \mathcal{A}$ and any two functions (one a bijection) $\psi, \lambda: X \to Y$ with s-strongly-subclosed graphs, $E(\psi, \lambda, X, Y) = X$ when $E(\psi, \lambda, X, Y)$ is dense in X.*

6.2. Theorem. *The following statements are equivalent for a first countable regular space Y. $\mathcal{A}$ is as in Theorem* 6.1.
(a) *Each countable filterbase on Y with at most one s-adherent point is convergent.*
(b) *Each sequence in Y with at most one s-accumulation point converges.*
(c) *For each $X \in \mathcal{A}$, each function (bijection) $\psi: X \to Y$ with an s-strongly-subclosed graph is continuous.*
(d) *For each $X \in \mathcal{A}$ and any two functions (bijections) $\psi, \lambda: X \to Y$ with s-strongly-subclosed graph and subclosed graph, respectively, $E(\psi, \lambda, X, Y)$ is closed in X.*
(e) *For each $X \in \mathcal{A}$ and any two functions (one a bijection) $\psi, \lambda: X \to Y$ with s-strongly-subclosed graph and subclosed graph, respectively, $E(\psi, \lambda, X, Y) = X$ whenever $E(\psi, \lambda, X, Y)$ is dense in X.*

6.3. Corollary. *A first countable regular space Y is first countable and regular-closed (minimal regular) if Y satisfies any of the equivalent statements of Theorem* 6.1 (6.2).

Proof. Follows from Theorem 4.2 (4.4) of [2].

Acknowledgement. The author wishes to thank Professor Larry L. Herrington for so graciously providing the author with a preprint of his paper [2].

References

1. M. P. Berri, J. R. Porter, and R. M. Stephenson, Jr., *A survey of minimal topological spaces*, General Topology and its Relations to Modern Analysis and Algebra, **III** (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, 93–114.

2. L. L. Herrington, *Characterizations of regular-closed spaces*, Math. Chronicle **5** (1977) 168–178.

3. L. L. Herrington and P. E. Long, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc. **48** (1975) 469–475.

4. L. L. Herrington and P. E. Long, *A characterization of minimal Hausdorff spaces*, Proc. Amer. Math. Soc. **57** (1976) 373–374.

5. J. E. Joseph, *On H-closed spaces*, Proc. Amer. Math. Soc. **55** (1976) 223–226.

6. Norman Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961) 44–46.

7. C. T. Scarborough and A. H. Stone, *Products of nearly compact spaces*, Trans. Amer. Math. Soc. **124** (1966), 131–147.

8. N. V. Velichko, *H-closed topological spaces*, Mat. Sb. **70** (112) (1966), 98–112; Amer. Math. Soc. Transl. **78** (Series 2) (1969), 103–118.

Department of Mathematics
Howard University
Washington, D.C. 20059

https://doi.org/10.4153/CMB-1979-064-8 Published online by Cambridge University Press