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Metabelian groups and varieties

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Here we announce results which will appear in full in due course elsewhere. The aim is a study of varieties of metabelian groups. The technique is to study varieties of certain universal algebras, which are very similar to groups, called bigroups. For certain varieties of bigroups all the non-nilpotent join-irreducible subvarieties are determined, and this is used to reduce the same problem for varieties of metabelian groups to the case of prime-power exponent. Questions of distributivity of the lattice of metabelian varieties are also discussed.

In this paper we announce some results with applications to the theory of varieties of metabelian groups; some of them are to be found in [4]. A fuller account will appear in due course [5].

1. Introduction

Let G be a non-nilpotent metabelian critical group (see Hanna Neumann's book [8] for definitions), with Fitting subgroup F. Then it may be shown that F is a p-group complemented in G by a p'-cycle (say of order t) and that G', the derived group of G, is contained in F. Is it true that, if G' has exponent p^{α} and its factor group exponent n,

(*) $$\mathrm{var}\, G = (\mathrm{var}\, F)\underline{\underline{A}}_t \wedge \underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_n \text{ ?}$$

A positive answer would lead to a classification of metabelian varieties of finite exponent in terms of prime-power exponent subvarieties. Unfortunately examples exist which show that (*) is in general false,

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though it is true if either $p^2 \nmid n$ or the class of F is at most p, and there is a sense in which (*) does hold generally. To explain these statements one needs the concepts of bigroup and bivariety introduced in §2 below. Having developed the concept it seems natural to ask familiar varietal questions about bivarieties: answers to several such questions are presented in §3 and provide some light on similar questions for varieties of groups that have not yet been answered - see §4.

2. Definitions

A *bigroup* is an ordered triple $\mathsf{G} = (G, A, B)$, where G is a group, A a normal subgroup of G and B a subgroup of G such that $G = AB$, $A \cap B = 1$. G is the *carrier* of G. We shall often write $A_1(\mathsf{G}) = A$, $A_2(\mathsf{G}) = B$. A sub-bigroup of G is a bigroup (G^*, A^*, B^*) where G^* is a subgroup of G and $A^* = G^* \cap A$, $B^* = G^* \cap B$; G^* is *normal* in G if G^* is normal in G. A *morphism* $\mu : \mathsf{G}_1 \to \mathsf{G}_2$ is a group-homomorphism $\mu : G_1 \to G_2$ such that $A_i(\mathsf{G}_1)\mu \leqq A_i(\mathsf{G}_2)$ $(i = 1, 2)$. If N is a normal sub-bigroup of G then the *quotient bigroup* G/N is $(G/N, AN/N, BN/N)$. A sub-bigroup of G is *fully invariant* in G if it is invariant under all self-morphisms of G; note that full invariance does not imply normality. The *cartesian product* of a collection $\{\mathsf{G}_j : j \in J\}$ of bigroups is the bigroup carried by $G = \prod\{G_j : j \in J\}$ such that $A_i(\mathsf{G}) = \prod\{A_i(\mathsf{G}_j) : j \in J\}$ $(i = 1, 2)$ in the natural way.

(Notice that, since a given splitting of a group G determines, and is determined by, a unique idempotent endomorphism of G, a bigroup may be thought of as a universal algebra, and that the definitions just given and those about to be given, accord with this.)

Let Y, Z be free groups of ranks m, n respectively on the free generating sets $\{y_i : i \in I\}$, $\{z_j : j \in J\}$. $\mathsf{Q}(m,n)$ is carried by the group $Y * Z$ with $A_1(\mathsf{Q}(m,n))$ equal to the normal closure of Y in $Y * Z$, and $A_2(\mathsf{Q}(m,n))$ equal to Z. $\mathsf{Q}(m,n)$ is the *absolutely free bigroup of rank* (m,n) on the free generating set $\{y_i : i \in I\} \cup \{z_j : j \in J\}$. Put $\mathsf{Q} = \mathsf{Q}(\aleph_0, \aleph_0)$. A *biword* is an element of Q. A biword q is a *bilaw* in a bigroup G if $q\alpha = 1$ for all $\alpha : \mathsf{Q} \to \mathsf{G}$. Let S be a subset of Q; then the *variety of bigroups*

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(bivariety, for short*)* determined by S is the class of bigroups for which the elements of S are bilaws. Alternatively, a bivariety is a class of bigroups closed under taking sub-bigroups, quotient bigroups and cartesian products. Standard varietal concepts such as verbal subgroup, free groups of a variety, critical group, go over to varieties of bigroups, but we will not here set down the definitions: with a little care one merely copies those in [8].

Products of bivarieties, however, need a mention. We imitate the definition in [8]: if $\mathcal{B}$, $\mathcal{C}$ are bivarieties, $\mathcal{B}\,\mathcal{C}$ is the class of all bigroups G with a normal sub-bigroup in $\mathcal{B}$ whose quotient is in $\mathcal{C}$. If $\underline{\underline{U}}$ is a variety of groups denote by $\underline{\underline{U}} \circ \underline{\underline{E}}$, $\underline{\underline{E}} \circ \underline{\underline{U}}$ the varieties of bigroups $(G,G,1)$, $(G,1,G)$ $(G \in \underline{\underline{U}})$ respectively: and put

$$\underline{\underline{U}} \circ \underline{\underline{V}} = (\underline{\underline{U}} \circ \underline{\underline{E}})(\underline{\underline{E}} \circ \underline{\underline{V}})$$

for varieties of groups $\underline{\underline{U}}$, $\underline{\underline{V}}$. Define the mapping σ from the lattice $\Lambda(\underline{\underline{U}}\underline{\underline{V}})$ of subvarieties of $\underline{\underline{U}}\,\underline{\underline{V}}$ to the lattice $\Lambda(\underline{\underline{U}} \circ \underline{\underline{V}})$ of sub-bivarieties of $\underline{\underline{U}} \circ \underline{\underline{V}}$ by

$$\underline{\underline{W}}\sigma = \{(G,A,B) \;:\; G \in \underline{\underline{W}}\;,\; A \in \underline{\underline{U}}\;,\; B \in \underline{\underline{V}}\}\;,\quad \underline{\underline{W}} \leq \underline{\underline{U}}\,\underline{\underline{V}}\;.$$

(Note that this definition depends very much on $\underline{\underline{U}}$ and $\underline{\underline{V}}$.) Unfortunately σ is, in general, practically devoid of desirable properties: it is a meet -, but not necessarily a join -, homomorphism, and need be neither one-to-one nor onto.

Notation. $\underline{\underline{A}}$, $\underline{\underline{A}}_m$ denote the varieties of all abelian groups, and all abelian groups of exponent dividing m respectively; N_c is the variety of all bigroups whose carriers have class at most c. If $\mathcal{B}$ is a bivariety write

$$\Phi(\mathcal{B}) = \{\mathcal{U} \in \Lambda(\mathcal{B}) : \mathcal{U} \text{ generated by critical bigroups } \mathsf{G} \text{ in } \mathcal{B} \text{ with } A_1(\mathsf{G}) \neq 1\} \cup \underline{\underline{E}} \circ \underline{\underline{E}}\;.$$

If S is a subset of (the carrier of) G then $\mathrm{cl}S$ denotes the normal, fully invariant closure of S in G .

3. The lattice $\Lambda(\underline{\underline{A}} \circ \underline{\underline{A}})$

The following theorems show how a description (but not a complete classification) of subvarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}$ can be reduced to the finite

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exponent, and then to the prime-power exponent, case. If we content ourselves with a list of join-irreducibles, a further reduction to the nilpotent case can be had, but here, with minor exceptions, we are stuck. Call a bivariety *torsion-free* if it is generated by bigroups G such that the groups $A_1(G)$ are torsion-free.

THEOREM 1. *(a) Let B be a proper subvariety of $\underline{\underline{A}} \circ \underline{\underline{A}}$ which is not contained in $\underline{\underline{A}}_m \circ \underline{\underline{A}} \vee \underline{\underline{A}} \circ \underline{\underline{A}}_n$ for any natural numbers m, n. There exists a unique torsion-free subvariety T of B and a unique natural number u such that*

$$B = T \vee \underline{\underline{A}}_u \circ \underline{\underline{A}} \vee P$$

where P has finite exponent.

(b) Let B be a subvariety of $\underline{\underline{A}}_m \circ \underline{\underline{A}} \vee \underline{\underline{A}} \circ \underline{\underline{A}}_n$. Then there exists a unique (minimal) subset Δ of the divisors of n, and a unique divisor u of m such that

$$B = \bigvee\{\underline{\underline{A}} \circ \underline{\underline{A}}_\delta : \delta \in \Delta\} \vee \underline{\underline{A}}_u \circ \underline{\underline{A}} \vee P$$

where P has finite exponent.

In neither case is the bivariety P unique in general, even when it is minimized.

THEOREM 2. *The torsion-free join-irreducible subvarieties of $A \circ A$ comprise the following list:*

$$\underline{\underline{E}} \circ \underline{\underline{A}}, \quad \underline{\underline{E}} \circ \underline{\underline{A}}_{p^\beta}, \quad \underline{\underline{A}} \circ \underline{\underline{A}}_s, \quad N_c(\underline{\underline{E}} \circ \underline{\underline{A}}_s) \wedge \underline{\underline{A}} \circ \underline{\underline{A}}, \quad \underline{\underline{A}} \circ \underline{\underline{A}}$$

where p is prime, $\beta \geq 0$, $s \geq 1$, $c \geq 2$. Moreover, every torsion-free subvariety T of $\underline{\underline{A}} \circ \underline{\underline{A}}$ is uniquely expressible as an irredundant join of torsion-free join-irreducibles and this is the only decomposition of T as an irredundant join of join-irreducibles.

Next comes the finite exponent case. To this end let m, n be natural numbers greater than 1, and write $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, $n = p_i^{\beta_i} n_i$ $(1 \leq i \leq r)$ where $p_1, \ldots p_r$ are distinct primes, and $p_i \nmid n_i$. Put $B_i = \underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_{p_i^{\beta_i}}$.

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THEOREM 3. *Let* $\underline{\underline{E}} \circ \underline{\underline{A}}_n \leq B \leq \underline{\underline{A}}_m \circ \underline{\underline{A}}_n$. *To each* i *in* $\{1,\ldots,r\}$ *and each* t *dividing* n_i *there exists a unique* B_{it} *with* B_{i1} *in* $\Lambda(B_i)$, B_{it} *in* $\Phi(B_i)$ $(t > 1)$ *such that*

$$\text{i)} \qquad t|u|n_i \Longrightarrow B_{iu} \leq B_{it} , \qquad i \in \{1,\ldots,r\} ,$$

$$\text{ii)} \qquad B = V_i \, V_{t|n_i} \{B_{it}(\underline{\underline{E}} \circ \underline{\underline{A}}_t) \wedge \underline{\underline{A}} \circ \underline{\underline{A}}\} .$$

Then, for the prime-power exponent subvarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}$ we have

THEOREM 4. *If* $\underline{\underline{E}} \circ \underline{\underline{A}}_{p^\beta} \leq U \leq \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$ *(p prime,* $\beta \geq 1$*) then there exists a unique subvariety* L *of* $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta-1}}$ *containing* $\underline{\underline{E}} \circ \underline{\underline{A}}_{p^{\beta-1}}$ *such that*

$$U = (L(\underline{\underline{E}} \circ \underline{\underline{A}}_p) \wedge \underline{\underline{A}} \circ \underline{\underline{A}}) \vee N$$

where N *is nilpotent (and, in general, not unique).*

Combining all these theorems it can be shown that

THEOREM 5. *A proper non-nilpotent subvariety of* $\underline{\underline{A}} \circ \underline{\underline{A}}$ *is join-irreducible if and only if it takes one of the following forms:*

$L(\underline{\underline{E}} \circ \underline{\underline{A}}_p) \wedge \underline{\underline{A}} \circ \underline{\underline{A}}$, p *prime,* L *of p-power exponent and join irreducible,* $L \nleq \underline{\underline{E}} \circ \underline{\underline{A}}$;

$U(\underline{\underline{E}} \circ \underline{\underline{A}}_t) \wedge \underline{\underline{A}} \circ \underline{\underline{A}}$, U *join-irreducible of p-power exponent,* $p \nmid t$, $U \nleq \underline{\underline{E}} \circ \underline{\underline{A}}$, $t > 1$;

$\underline{\underline{A}} \circ \underline{\underline{A}}_s$, $N_c(\underline{\underline{E}} \circ \underline{\underline{A}}_s) \wedge \underline{\underline{A}} \circ \underline{\underline{A}}$, $\underline{\underline{A}}_s \circ \underline{\underline{A}}$, $s > 1$, $c > 1$.

For the bivarieties $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$, $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge N_p$ it is possible, though very tedious to write down a complete description of their subvarieties. In view of Theorems 5, 8 perhaps the most interesting things that emerge from the descriptions are the next two theorems.

THEOREM 6. *(a) The join-irreducible subvarieties of* $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$ *are*

$$\underline{\underline{E}} \circ \underline{\underline{A}}_p , \quad \underline{\underline{A}}_{p^\sigma} \circ \underline{\underline{E}} , \quad \underline{\underline{A}}_{p^\tau} \circ \underline{\underline{A}}_p , \quad \underline{\underline{A}}_{p^\tau} \circ \underline{\underline{A}}_p \wedge N_c$$

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where $0 \leq \sigma \leq \alpha$, $1 \leq \tau \leq \alpha$, $(\tau - 1)(p - 1) + 2 \leq c$.

(b) (cf. [2], [10]). *The join-irreducible subvarieties of* $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge N_p$ *are*

$$\underline{\underline{E}} \circ \underline{\underline{A}}_{p^\sigma} , \quad \underline{\underline{A}}_{p^\sigma} \circ \underline{\underline{E}} , \quad \underline{\underline{A}}_{p^\beta} \circ \underline{\underline{A}}_{p^\beta} \wedge N_c ,$$

where $0 \leq \sigma \leq \alpha$, $1 \leq \beta \leq \alpha$, $2 \leq c \leq p$.

THEOREM 7. *The lattices* $\Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p)$, $\Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge N_p)$ *are distributive. By contrast* $\Lambda(\underline{\underline{A}}_{p^2} \circ \underline{\underline{A}}_{p^2} \wedge N_{p+1})$ *is not distributive.*

From the foregoing theorems, and some unpublished lattice theory of L.G. Kovács we have the next theorem, which gives point to Theorem 7.

THEOREM 8. *Let* B *be a subvariety of* $\underline{\underline{A}} \circ \underline{\underline{A}}$. *Then* $\Lambda(B)$ *is distributive if and only if* $\Lambda(B \wedge N_c)$ *is distributive for all natural numbers* c .

The proof of this chain of theorems is rather intricate, and does not follow the order in which they appear; part (b) of Theorem 1 (whose proof is a special case of that of part (a)) is needed to prove minimum condition on $\Lambda(\underline{\underline{A}} \circ \underline{\underline{A}})$ which in turn is needed to prove part (a) of Theorem 1. The proofs of Theorems 1, 3, 4, 5 rely rather heavily on commutator calculations; and Theorem 3 depends on some knowledge of the structure of non-nilpotent critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$. Several crucial lemmas will be cited in order to give the flavour of the proofs.

LEMMA 1. *If* $G = (G,A,B)$ *in* $\underline{\underline{A}} \circ \underline{\underline{A}}$ *is critical and not nilpotent, then* A *is a p-group for some prime* p , *it is self-centralizing in* G *and is equal to* G' . *If* $B = H \times K$ *where* H *is the Sylow* p-*subgroup of* B , *then* K *is a* p'-*cycle each non-trivial element of which acts fixed-point-free on* A .

LEMMA 2. *Let* q *be a commutator biword and* t *a natural number. There exist biwords* $q_1, \ldots, q_v$ *depending on* q *and on* t *such that, if* q *is a bilaw in a critical bigroup* $(G,A,H \times K)$ *in* $\underline{\underline{A}} \circ \underline{\underline{A}}$ *where* $|K| = t$ *then* $q_1, \ldots, q_v$ *are bilaws in* (AH,A,H) . *Conversely, if* $(G_1,A_1,H_1 \times K_1)$ *is in* $\underline{\underline{A}} \circ \underline{\underline{A}}$ *with* $\exp K_1 | t$, *and* $q_1, \ldots, q_v$ *are bilaws in*

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(A_1H_1, A_1, H_1) , *then* q *is a bilaw in* G_1 .

Theorem 3 is almost entirely covered by these two lemmas. In dealing with Theorem 4 the following lemma is crucial.

LEMMA 3. *Let* (G,A,B) *belong to* $\underline{\underline{A}} \circ \underline{\underline{A}}_{p^\nu}$. *If for fixed* $a_1, \ldots, a_m$ $(m \geqq p)$ *in* A , *and all* b *in* B

$$\prod_{i=1}^{m} [a_i , b^i] = 1 ,$$

then

$$\prod_{i=1}^{[(m-j)/p]} [a_{ip+j} , b^{ip}] \in Z_{m-1}(G) , \qquad j \in \{0,\ldots,p-1\} ,$$

and

$$\prod_{i=0}^{[(m-j)/p]} a_{ip+j} \in Z_{m+p-2}(G) , \qquad j \in \{1,\ldots,p-1\} .$$

Let W_β be the free bigroup of rank $(\aleph_0, \aleph_0)$ in $\underline{\underline{A}} \circ \underline{\underline{A}}_{p^\beta}$ $(\beta \geqq 0)$. Define, for some $\nu \geqq 1$, $\xi_\nu : W_{\nu-1} \to W_\nu$, a one-to-one morphism by

$$\hat{a}_i\xi_\nu = a_i , \quad \hat{b}_j\xi_\nu = b_j^p , \qquad i,j \in \{1,2,\ldots\}$$

where $\{\hat{a}_i : i = 1,2,\ldots\} \cup \{b_j : j = 1,2,\ldots\}$ and $\{a_i : i = 1,2,\ldots\} \cup \{b_j : j = 1,2,\ldots\}$ freely generate $W_{\nu-1}$ and W_ν respectively. If Λ_β denotes the lattice (dually isomorphic to $\Lambda(\underline{\underline{A}} \circ \underline{\underline{A}}_{p^\beta})$) of normal fully invarient sub-bigroups of W_β denote by $\lambda_\nu : \Lambda_{\nu-1} \to \Lambda_\nu$ the mapping defined by

$$L\lambda_\nu = \mathrm{cl}\{l\xi_\nu : l \in L\} , \qquad L \in \Lambda_{\nu-1} .$$

One then shows that

LEMMA 4. *If* $L \leqq A_1(W_{\nu-1})$ *then (the carrier of)* $W_\nu/L\lambda_\nu$ *has trivial centre.*

From Lemma 3 we get

LEMMA 5. *If* q *is in* $A_1(W_\nu)$ *then there exist* $l_1, \ldots, l_d$ *in* $A_1(W_{\nu-1})$ *and an integer* v *such that*

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$$q \in \operatorname{nmcl}\{l_i\xi_\nu : 1 \leq i \leq d\}$$

and

$$[l_i\xi_\nu , vW_\nu] \leq \operatorname{cl} q , \qquad 1 \leq i \leq d ,$$

(where, as usual, $[l_i\xi_\nu , vW_\nu]$ *denotes the subgroup generated by all elements* $[l_i\xi_\nu , w_1,\ldots,w_v]$, $w_j \in W_\nu$*)* .

Then from Lemmas 3, 4, 5 one can deduce

LEMMA 6. *The mapping* λ_ν *is a one-to-one lattice homomorphism. To each* U *in* Λ_ν *, with* U *contained in* $A_1(W_\nu)$ *, there exists a unique* L *in* $\Lambda_{\nu-1}$ *, with* L *contained in* $A_1(W_{\nu-1})$ *, and an integer* v *such that*

$$[L\lambda_\nu , vW_\nu] \leq U \leq L\lambda_\nu .$$

A little extra effort gives us Theorem 4 from Lemma 6.

From the proof of Theorem 1 we quote one interesting lemma, a trivial adaptation of a result of L.G. Kovács and M.F. Newman [*6*].

LEMMA 7. *If* B *is a subvariety of* $\underline{\underline{A}} \circ \underline{\underline{A}}$ *which does not contain* $\underline{\underline{A}}_m \circ \underline{\underline{A}}$ *, then there exist natural numbers* r , s , t *with* t *not divisible by* m *such that* B *has a bilaw* $[y_1 , rz_1^s]^t$.

4. Applications to varieties of metabelian groups

Let G be a non-nilpotent metabelian critical group. It can be proved using a result of Schenkman [9] that G' is complemented in G , and that all such complements are conjugate. Hence G carries a critical bigroup in $\underline{\underline{A}} \circ \underline{\underline{A}}$, unique up to isomorphism. Conversely it can be shown that the carrier of a non-nilpotent critical bigroup in $\underline{\underline{A}} \circ \underline{\underline{A}}$ is a critical group. Referring back to (*) in §*1* , it is the bigroup - (rather than the group -) varietal properties of F which are important: equation (*) is true if we interpret it in bivarietal language (this is just Theorem 3), but fails as stated, essentially because the mapping $\sigma : \Lambda(\underline{\underline{A}}_{p^\alpha} \underline{\underline{A}}_{p^\beta}) \to \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})$ need not be onto. However the close connexion between non-nilpotent critical groups and bigroups that we have just observed enables some results to be obtained: indeed using Lemma 2 we can write down a rather complicated theorem, similar to Theorem 3, classifying subvarieties of $\underline{\underline{A}}_m \underline{\underline{A}}_n$ in terms of varieties and bivarieties of prime-power exponent. A consequence of this is easier to state and is interesting.

THEOREM 9. *In the notation of Theorem 3 let* $\sigma_i : \Lambda(\underline{\underline{A}}_{p_i^{\alpha_i}} \underline{\underline{A}}_n) \to \Lambda(\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n)$ *be defined as in* §2. *Then if* $\underline{\underline{V}}$ *is a subvariety of* $\underline{\underline{A}}_m \underline{\underline{A}}_n$, $\Lambda(\underline{\underline{V}})$ *is distributive if and only if* $\Lambda(\underline{\underline{V}} \wedge \underline{\underline{A}}_{p_i^{\alpha_i}} \underline{\underline{A}}_{p_i^{\beta_i}})$ *and* $\Lambda((\underline{\underline{V}} \wedge \underline{\underline{A}}_{p_i^{\alpha_i}} \underline{\underline{A}}_n)\sigma_i)_{i1})$ *are distributive for* $1 \leq i \leq r$.

It follows from Theorem 7 and results of Brisley [1,2] and Weichsel [10] that

THEOREM 10. *Let* $\underline{\underline{V}}$ *be a variety of metabelian groups of bounded exponent in which* p*-groups have class at most* p . *Then* $\Lambda(\underline{\underline{V}})$ *is distributive. On the other hand if* $\underline{\underline{W}}$ *is the subvariety of* $\underline{\underline{A}}_{p^2} \underline{\underline{A}}_{p^2 N}$ $(p \nmid N , N \neq 1)$ *which consists of groups whose Sylow* p*-subgroups have class at most* $p + 1$, *then* $\Lambda(\underline{\underline{W}})$ *is not distributive.*

Finally we note that $\sigma : \Lambda(\underline{\underline{A}}_{p^\alpha} \underline{\underline{A}}_p) \to \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p)$ is onto - this follows from Theorem 6(a) - and that a theorem analogous to Theorem 3 is possible. Use the notation of Theorem 3 with $\underline{\underline{V}}_i = \underline{\underline{A}}_{p_i^{\alpha_i}} \underline{\underline{A}}_{p_i^{\beta_i}}$ $(1 \leq i \leq r)$.

THEOREM 11. *Let* $\underline{\underline{A}}_n \leq \underline{\underline{V}} \leq \underline{\underline{A}}_m \underline{\underline{A}}_n$ *where* m *is nearly prime to* n *(that is* $p^2 | n \Longrightarrow p \nmid m$*)* . *To each* i *in* $\{1, \ldots r\}$ *and each* $t | n_i$ *there exists a subvariety* $\underline{\underline{V}}_{it}$ *of* $\underline{\underline{V}}_i$, *with* $\underline{\underline{V}}_{it}\sigma_i$ *in* $\Phi(\underline{\underline{V}}_i \sigma_i)$ *for* $t > 1$, *and a natural number* $\alpha(i , t) \leq \min(\alpha_i , \exp \underline{\underline{V}}_{it})$ *such that*

i) $$\underline{\underline{V}} = V_i V_{t|n_i} \{\underline{\underline{V}}_{it} \underline{\underline{A}}_t \wedge \underline{\underline{A}}_{p_i^{\alpha(i,t)}} \underline{\underline{A}}_n\} ;$$

ii) $t_0 | t | n_i \Longrightarrow \underline{\underline{V}}_{it} \leq \underline{\underline{V}}_{it_0}$, $\alpha(i,t) \leq \alpha(i,t_0)$ $(1 \leq i \leq r)$;

iii) $\underline{\underline{V}}_{i1}$, $\underline{\underline{V}}_{it}\sigma_i$, $\alpha(i,t)$ *are unique,* $1 \neq t | N$ $(1 \leq i \leq r)$.

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In the case when m is nearly prime to n, therefore, Theorems 11 and 6(a), and the classification [7] by Kovács and Newman of $\Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_p)$, allow a complete classification of $\Lambda(\underline{\underline{A}}_m\underline{\underline{A}}_n)$. In a similar way the description of $\Lambda(\underline{\underline{A}}^2 \wedge \underline{\underline{B}}_{p^k} \wedge \underline{\underline{N}}_p)$ [2] and Theorem 6(b) can be generalized to cover metabelian varieties of finite exponent in which p-groups have class at most p: as the statement of this is almost identical to that of Theorem 11 we omit it. Theorem 11 yields

THEOREM 12. *When m is nearly prime to n, $\Lambda(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is distributive.*

5. Conclusion

The proof of Theorem 1 is modelled on the proof, by L.G. Kovács and M.F. Newman, of an analogous result [6] for subvarieties of $\underline{\underline{A}}^2$. Their result in turn depends on our Theorem 11; much of the proof will appear in Appendix I of [5]. However Theorem 4 has, as yet, no analogue except for the case of subvarieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^2}$ (M.S. Brooks [3]): thus only for restricted m, n (namely such that $p^2|n \Longrightarrow p^2 \nmid m$ and $p^3|n \Longrightarrow p \nmid m$) do our Theorems 3, 4, 6 and the results of Brooks [3] and Kovács and Newman [7] enable us to describe all non-nilpotent join irreducibles in $(\underline{\underline{A}}_m\underline{\underline{A}}_n)$.

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