A NONEMBEDDING RESULT FOR COMPLEX GRASSMANN MANIFOLDS

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1. Introduction

A smooth map f: $M \to R^{n+k}$ of a differentiable n-manifold into Euclidean $(n+k)$-space is called an *immersion* if its Jacobian has rank n at each point of M. If f is also 1-1, it is called an *embedding*.

The embedding and immersion properties of projective spaces have been widely investigated (see for example (1), (7)). We generalise by considering the complex Grassmann manifold $G_2(C^n)$ of 2-planes in C^n. This is a compact differentiable manifold of dimension $d = 4n-8$, so by the classical results of Whitney (8, 9)

$$G_2(C^n) \subset R^{2d}, \quad G_2(C^n) \subseteq R^{2d-1},$$

where the first denotes an embedding and the second an immersion. We obtain a lower bound by proving

Main Theorem 1.1.

$$G_2(C^n) \not\subset R^{d+\frac{1}{2}d}, \quad G_2(C^n) \not\subseteq R^{d+\frac{1}{2}d-1}.$$

2. Notation and preliminaries

Henceforth we abbreviate $G_2(C^n)$ to G_n. Tangent and normal bundles will be denoted by τ and ν respectively. The *canonical bundle* γ has as fibre over each point p in G_n the points of the 2-plane p. Taking the orthogonal complement in C^n for each p gives the *complementary canonical bundle* λ, with $\gamma \oplus \lambda = \boldsymbol{n}$, the trivial bundle of dimension n.

If ξ is a real m-plane bundle, we can tensor each fibre with C over R to obtain a complex m-plane bundle called the *complexification* $c(\xi)$. In the reverse direction, the *decomplexification* $r(\eta)$ of a complex bundle η is realised by treating each fibre as a real space of twice the dimension. The *dual* of a real or complex bundle ξ is written ξ^*.

We denote the *total Chern class* of a complex m-plane bundle η by

$$C(\eta) = 1 + c_1(\eta) + c_2(\eta) + \ldots + c_m(\eta),$$

where the ith *Chern class* $c_i(\eta)$ is an element of the cohomology of the base space of η, in dimension $2i$.

There is the *Whitney formula* for two complex bundles

$$C(\eta \oplus \eta') = C(\eta) . C(\eta'). \tag{2.1}$$

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The ith *Pontrjagin class* of a real bundle ξ is defined as

$$p_i(\xi) = (-1)^i c_{2i}(c(\xi)).$$

We prove Theorem 1.1 by a calculation on the Chern classes of the complexified normal bundle of G_n, applied to

Proposition 2.2 (1, p. 132). *Let M be a differentiable m-manifold with normal bundle $\nu(M)$.* If $p_k(\nu(M)) \neq 0$ *then*

$$M \not\subset R^{m+2k}, \quad M \not\subseteq R^{m+2k-1}.$$

For an exposition of vector bundles and Chern classes, see (3) or (5).

3. Cohomology of G_n

Recall that γ is the canonical bundle over G_n, with complement λ. Write for convenience $x = -c_1(\gamma)$, $y = -c_2(\gamma)$, $v_i = c_i(\lambda)$. Then applying the Whitney formula (2.1) to $\gamma \oplus \lambda = \boldsymbol{n}$, we have

$$v_0 = 1, v_1 = x, \quad v_i = xv_{i-1} + yv_{i-2} \; (2 \leqq i \leqq n).$$

It follows from Borel (2) that $H^*(G_n)$ is generated as a ring with 1 by x and y, subject only to the relations $v_n = 0 = v_{n-1}$ expressed in terms of x and y. For our computation we require an additive basis, and this is provided for by a formally identical calculation in terms of K-theory ((4), 4.4, 4.6) which shows:

Proposition 3.1. *$H^*(G_n)$ is generated as a free abelian group by the monomials $x^r y^s$ of degree $\leqq n-2$. Monomials of degree $n-1$ are given in terms of the generators by*

$$v_r y^{n-1-r} = 0 \; (0 \leqq r \leqq n-1), \tag{3.2}$$

where

$$v_r = -\Sigma \binom{r-s}{s} x^{r-2s} y^s, \text{ summing over } 0 \leqq s \leqq \tfrac{1}{2}r; \tag{3.3}$$

$$v_0 = 1, \; v_1 = x, \; v_i = xv_{i-1} + yv_{i-2} \quad (2 \leqq i \leqq n). \tag{3.4}$$

Notice v_r has term of highest degree x^r. We need also the following result:

Lemma 3.5. *Let $z = x^2 + 2y$ and $r = 2p$ or $2p+1$, then*

$$z^r y^{n-2-r} = \binom{r}{p} y^{n-2}, \quad (0 \leqq r \leqq n-2).$$

Proof. First we claim that $z^r y^{n-2-r}$ is identically equal to

$$v_{2r} y^{n-2-r} + v_{2r-2} y^{n-r-1} + r(v_{2r-4} y^{n-r} + v_{2r-6} y^{n-r+1}) + \binom{r}{2} (v_{2r-8} y^{n-r+2} + v_{2r-10} y^{n-r+3}) + \ldots$$

the sum finishing with

$$\ldots + \binom{r}{p-1} (v_4 y^{n-4} + v_2 y^{n-3}) + \binom{r}{p} y^{n-2}$$

if $r = 2p$, and

$$\dots + \binom{r}{p}(v_2 y^{n-3} + y^{n-2}) \quad \text{if} \quad r = 2p+1.$$

For $r = 1$, we have

$$zy^{n-3} = (x^2+2y)y^{n-3} = (v_2+y)y^{n-3} = v_2 y^{n-3} + y^{n-2}.$$

Now we proceed by induction, showing that the result for case $r = 2p-1$, $p \geqq 1$, implies case $r = 2p$, which in turn implies case $r = 2p+1$. This all follows from (3.4). To complete the proof of the Lemma, apply relations (3.2).

4. Chern classes and the normal bundle

Our object in this section is to express $C(c(\nu))$ in terms of the generators of the cohomology of G_n, namely the Chern classes of the canonical bundle γ (3.1).

It is not hard to show that the tangent bundle of G_n as a complex manifold is (isomorphic to) Hom (γ, λ), the bundle whose fibre over a given point in G_n is the space of homomorphisms of the fibre of γ into that of λ. A similar result holds for Grassmannians in general (see (6), p. 411). Now since $\gamma \oplus \lambda = \boldsymbol{n}$ (see §2) and Hom $(\gamma, \lambda) = \gamma^* \otimes \lambda$ (3, p. 47), we have

$$\text{Hom}\,(\gamma, \lambda) \oplus (\gamma^* \otimes \gamma) = \gamma^* \otimes \boldsymbol{n} = n\gamma^*.$$

The tangent bundle τ itself is the decomplexification of Hom (γ, λ) (3, p. 67). Thus an elementary calculation, using " $cr = 1 + *$ " and other basic relations between bundles (3, p. 47 and 5, p. 242), shows

$$c(\tau) \oplus 2(\gamma^* \otimes \gamma) = n(\gamma^* \oplus \gamma).$$

The triviality of $\tau \oplus \nu$ implies that of $c(\tau) \oplus c(\nu)$, and we apply (2.1) to this and the formula above to obtain

$$C(c(\nu)) = (C(\gamma^* \otimes \gamma))^2 . (C(\gamma^* \oplus \gamma))^{-n}. \tag{4.1}$$

Next, using the method of (3), p. 64, we write formally

$$\sum_{i=0}^{2} c_i(\gamma) t^i = (1+at)(1+bt),$$

with t as indeterminate. Then

$$\sum_{i=0}^{2} c_i(\gamma^*) t^i = (1-at)(1-bt),$$

and with $x = -(a+b)$, $y = -ab$ (§3), we have

$$\sum_{i=0}^{4} c_i(\gamma^* \oplus \gamma) t^i = (1-a^2t^2)(1-b^2t^2) = 1-(x^2+2y)t^2 + y^2 t^4,$$

$$\sum_{i=0}^{4} c_i(\gamma^* \otimes \gamma) t^i = 1-(a-b)^2 t^2 = 1-(x^2+4y)t^2.$$

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Observe that no odd power of t appears in the two formulae above. In fact, substituting these in (4.1), replacing t^2 by the indeterminate T, and setting $z = x^2+2y$, we have finally

Lemma 4.2. *For the complexified normal bundle of G_n, the odd Chern classes c_{2i+1} are zero. The class $p_i(v)$ is the coefficient of T^i in*

$$(1+(z+2y)T)^2(1+zT+y^2T^2)^{-n}.$$

Remark. $c_{2i+1} = 0$ is also implied by the fact that in general the odd Chern classes of a complexified real bundle have order 2 (**5**, p. 243), whereas $H^*(G_n)$ is torsion-free (3.1). It means nothing is lost by the exclusion of these classes in Proposition 2.2.

5. Proof of the Main Theorem

According to Proposition 2.2, the Main Theorem 1.1 will follow if we can show the coefficient of T^{n-2} in Lemma 4.2 is non-zero. Define the *weight* of the monomial $x^r y^s \in H^{2r+4s}(G_n)$ to be $r+2s$, then all monomials of weight greater than $2n-4$ vanish, since $H^i(G_n) = 0$ for $i>4n-8$ (the dimension of G_n). Clearly T^r appears with coefficient of weight $2r$. Now the only generator of weight $2n-4$ is y^{n-2}, so that T^{n-2} has coefficient $D.y^{n-2}$ for some integer D. Therefore we must prove D is non-zero. Since higher powers of T vanish, Theorem 1.1 is the sharpest result obtainable by using Proposition 2.2.

Let $(1+zT+y^2T^2)^{-n} = \Sigma d_i T^i$ $(0 \leqq i \leqq n-2)$, then, from Lemma 4.2,

$$Dy^{n-2} = d_{n-2}+2(z+2y)d_{n-3}+(z+2y)^2 d_{n-4}.$$

The binomial expansions of the various powers of $zT+y^2T^2$ yield the d_i's required.

We evaluate D with the aid of Lemma 3.5. Set

$$S(L, m, s) = \sum_{k=0}^{t-L} (-1)^k \binom{4t-k-m-1}{2t-k-m}\binom{2t-k-m}{2t-2k-m}\binom{2t-2k-s}{t-k-[\frac{1}{2}s]},$$

where $[\frac{1}{2}s]$ means the integral part. Let S' be obtained by replacing $4t-k-m-1$ by $4t-k-m$. Then for $n = 2t$, we obtain

$$D = S(1, 2, 2)-2S(2, 3, 2)-4S(2, 3, 3)+S(2, 4, 2)+4S(2, 4, 3)+4S(2, 4, 4).$$

For $n = 2t+1$,

$$-D = S'(1, 1, 1)-2S'(1, 2, 1)-4S'(1, 2, 2)+S'(2, 3, 1) +4S'(2, 3, 2)+4S'(2, 3, 3).$$

After some calculation, we obtain

$$D = \binom{3t-3}{t-2}^2 \cdot \frac{t^2-8t+6}{(t-1)^2} \quad (n = 2t),$$

$$D = -3\binom{3t-1}{t-1}\binom{3t-2}{t-2}\frac{t^2-9t+6}{(t-1)(3t-2)} \quad (n = 2t+1).$$

Since the equations $t^2-8t+6=0$ and $t^2-9t+6=0$ have no integer solutions, D is non-zero for $t \geqq 2$, and the proof of the Main Theorem is complete.

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