A THEOREM ON ISOMETRIES AND THE APPLICATION OF IT TO THE ISOMETRIES OF $H^p(S)$ FOR $2 < p < \infty$

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1. Introduction. 1.1. Let $X$ and $Y$ be sets, let $\lambda$ be a bounded positive measure on $X$, and let $\mu$ be a bounded positive measure on $Y$. Furthermore let $M$ be a subalgebra of $L^\infty(\lambda)$, let $p \in (0, \infty)$, and let $A$ be a linear transformation of $M$ into $L^p(\mu)$ such that

$$\int |Af|^p d\mu = \int |f|^p d\lambda$$

for all $f$ in $M$.

In § 2 of this paper we will prove the following theorem.

1.2. Theorem. If (a) $p > 2$, if (b) $(Af)(y) \neq 0$ for $\mu$-almost all $y$ in $Y$ whenever $f \in M$ and $f \neq 0$, and if (c) $A1 = 1$, then

$$A(fg) = AfAg$$

for all $f$ and $g$ in $M$ and

$$\int Afg d\mu = \int fgd\lambda$$

for all $f$ and $g$ in $M$.

1.3. If the hypotheses (b) and (c) of Theorem 1.2 hold and if instead of (a) we have $p < 2$, then we do not know if the conclusion of Theorem 1.2 holds. We will denote by $U$ the class of all $f$ in $M$ such that $ff = 1$. It was proved in [I] that if $M = C[U]$ and if the hypothesis (c) of Theorem 1.2 holds, then the conclusion of Theorem 1.2 holds for $p$ in $(0, \infty)$. Furthermore it was proved in [I] that if the hypothesis (c) of Theorem 1.2 holds and if instead of (a) we have $p \geq 4$, then the conclusion of Theorem 1.2 holds.

1.4. Let $V$ be a vector space over $\mathbb{C}$ of complex dimension $n$ with an inner product. If $x$ and $y$ are in $V$, then we will denote by $\langle x, y \rangle$ the inner product of $x$ and $y$. We will denote by $B$ the class of all $x$ in $V$ such that $\langle x, x \rangle < 1$, by $\overline{B}$ the class of all $x$ in $V$ such that $\langle x, x \rangle \leq 1$, and by $S$ the class of all $x$ in $V$ such that $\langle x, x \rangle = 1$. Thus $S$ may be regarded as the Euclidean sphere of real dimension $2n - 1$. We will denote by $\sigma$ the positive Radon measure on $S$ which assigns to each open subset of $S$ its Euclidean volume. We define $\alpha : \overline{B} \times B \rightarrow \mathbb{C}$ by

$$\alpha(x, y) = [\sqrt{(1 - \langle y, y \rangle)}]/(1 - \langle x, y \rangle)$$

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and we define \( \beta : \overline{B} \times B \to (0, \infty) \) by \( \beta = (a \bar{a})^n \). We recall that if \( \phi \) is a function which is defined on the Cartesian product \( E \times F \) of sets \( E \) and \( F \) and if \( (x, y) \in E \times F \), then \( \phi_x \) and \( \phi_y \) are the functions which are defined on \( F \) and \( E \) respectively by \( \phi_x(l) = \phi(x, l) \) and \( \phi_y(s) = \phi(s, y) \). If \( f \in L^1(\sigma) \), then we define \( f^\#: B \to C \) by

\[
f^\#(y) = (1/\sigma(S)) \int f \beta^d\sigma.
\]

We remark that if \( f \in L^1(\sigma) \), then \( f^\# \) is of differentiability class \( C^\infty \). If \( 1 \leq p \leq \infty \), then we will denote by \( H^p(S) \) the class of all \( f \) in \( L^p(\sigma) \) such that \( f^\# \) is holomorphic on \( B \). It follows that \( H^p(S) \) is a closed subspace of the Banach space \( L^p(\sigma) \), and hence that \( H^p(S) \) is a Banach space with respect to the norm of \( L^p(\sigma) \). The definition of \( H^p(S) \) is motivated by the change of variables formula with regard to holomorphic homeomorphisms of \( B \) that is expressed in Lemma 3.4. If \( n = 1 \), then \( H^p(S) \) is the familiar Hardy class \( H^p \) (if we regard \( S \) as the unit circle in the complex plane).

As an application of Theorem 1.2 we will prove the following theorem.

1.5. Theorem. If (a) \( T \) is a linear isometry of the Banach space \( H^p(S) \) onto itself and if (b) \( 2 < p < \infty \), then there is a holomorphic homeomorphism \( Z \) of \( B \) and a unimodular complex number \( \theta \) such that for every \( f \in H^p(S) \) we have

\[
Tf = \theta(\alpha^z)^{2n/p} f \circ Z
\]

where \( z \) in \( B \) is defined by \( Z(z) = 0 \).

1.6. The proof of Theorem 1.5 is in § 3. We remark that if \( Z \) is any holomorphic homeomorphism of \( B \) and if \( p \in [1, \infty) \), then the expression (1.1) defines a linear isometry of \( H^p(S) \) onto itself. (This follows from Lemma 3.4. The holomorphic homeomorphisms of \( B \) are described in Lemma 3.2.) If \( n \geq 2 \), if the hypothesis (a) of Theorem 1.5 holds, and if instead of (b) we have \( 1 \leq p < 2 \), then we do not know if the conclusion of Theorem 1.5 holds. Furthermore if \( n \geq 2 \), if \( p \in [1, \infty) \), and if \( p \neq 2 \), then it is not known if there are any linear isometries of \( H^p(S) \) into itself which are not onto.

2. The proof of Theorem 1.2. 2.1. If \( w \in C \) and if \( r \in (0, \infty) \), then we will denote by \( D(w, r) \) the open disc in \( C \) whose center is \( w \) and whose radius is \( r \). The proof of the following lemma is in [1].

2.2. Lemma. Let \( \rho \) be a bounded positive measure on \( X \), let \( \tau \) be a bounded positive measure on \( Y \), let \( s \in (0, \infty) \), let \( f \in L^s(\rho) \), and let \( g \in L^s(\tau) \). If for some \( r \) in \( (0, \infty) \) we have

\[
\int |1 + zf|^s d\rho = \int |1 + zg|^s d\tau
\]

for all \( z \) in \( D(0, r) \), then

\[
\int |f|^2 d\rho = \int |g|^2 d\tau.
\]
2.3. We will now prove Theorem 1.2. We will break the proof up into several statements.

2.3.1. If \( f \in M \) and \( f \neq 0 \), then

\[
(2.1) \quad \int |A(fg)|^2 |Af|^{p-2} d\mu = \int |g|^2 |f|^{p} d\lambda
\]

for all \( g \) in \( M \).

For the purpose of proving statement 2.3.1 we let \( d\rho = |f|^p d\lambda \) and \( d\tau = |Af|^p d\mu \). If \( g \in M \) and \( z \in \mathbb{C} \), then

\[
\int |1 + zg|^p d\rho = \int |f + zg|^p d\lambda
\]

\[
= \int |Af + zA(fg)|^p d\mu
\]

\[
= \int |1 + zA(fg)/Af|^p d\tau,
\]

and hence by Lemma 2.2 we have

\[
\int |g|^2 d\rho = \int |A(fg)/Af|^2 d\tau
\]

which completes the proof of statement 2.3.1.

We remark that the proof of statement 2.3.1 did not use either the fact that \( A1 = 1 \) or the fact that \( p > 2 \).

We will denote by \( M^{-1} \) the collection of all invertible elements of \( M \).

2.3.2. If \( f \in M^{-1} \), then

\[
(2.2) \quad \int |Af|^{p-2} |Ag|^2 d\mu = \int |f|^{p-2} |g|^2 d\lambda
\]

for all \( g \) in \( M \).

Statement 2.3.2 follows from statement 2.3.1 upon replacing \( g \) in the identity (2.1) by \( g/f \).

2.3.3. If \( f \in M \) and \( g \in M \), then

\[
\int |1 + zAf|^{p-2} |Ag|^2 d\mu = \int |1 + zf|^{p-2} |g|^2 d\lambda
\]

for all \( z \) in \( D(0, 1/||f||_\infty) \).

For the purpose of proving statement 2.3.3 we may assume that \( M \) is a closed subalgebra of \( L^\infty(\lambda) \). Since \( 1 + zf \in M^{-1} \) if \( z \in D(0, 1/||f||_\infty) \), statement 2.3.3 follows from statement 2.3.2 upon replacing \( f \) in the identity (2.2) by \( 1 + zf \).

We remark that the proof of statement 2.3.3 did not use the fact that \( p > 2 \).

2.3.4. If \( f \in M \) and \( g \in M \), then

\[
\int |Af|^2 |Ag|^2 d\mu = \int |f|^2 |g|^2 d\lambda.
\]
Statement 2.3.4 follows from statement 2.3.3 and Lemma 2.2 (with $d \rho = |g|^2 d \lambda$, $d \tau = |A g|^2 d \mu$, and $s = p - 2$).

It follows from statement 2.3.4 that if $f \in M$, then $Af \in L^4(\mu)$.

2.3.5. If $a$, $b$, $c$, and $d$ are in $M$, then

$$\int A a A b A c A d d \mu = \int a b c d d \lambda.$$ 

Statement 2.3.5 follows from statement 2.3.4 by the method of polarization. Statement 2.3.5 includes the second assertion of Theorem 1.2. Furthermore it follows from statement 2.3.5 that if $f \in M$ and $g \in M$, then

$$\int |A(f g) - A f g|^2 d \mu = 0,$$

which completes the proof of Theorem 1.2.

2.4. We will denote by $Z_+$ the class of all positive integers.

2.5. Corollary (of Theorem 1.2). If $f \in M$, then $||Af||_\infty = ||f||_\infty$.

Proof. If $k \in Z_+$, then

$$\left( \int |A f|^{2k} d \mu \right)^{1/2k} = \left( \int |A(f^k) A(f^k) d \mu \right)^{1/2k} = \left( \int |f|^{2k} d \lambda \right)^{1/2k},$$

from which the desired conclusion follows upon letting $k$ increase to $\infty$.

3. The proof of Theorem 1.5. 3.1. We will denote by $U(V)$ the class of all unitary transformations of $V$, and we will regard $SL(2, \mathbb{R})$ as the class of all $2 \times 2$ matrices $L$ of the form

$$L = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

where $a$ and $b$ are in $\mathbb{C}$ and $\det(L) = a\bar{a} - b\bar{b} = 1$. We define $\gamma : SL(2, \mathbb{R}) \times S \times \bar{B} \to \bar{B}$ by

$$\gamma(L, x, y) = [1/(b(y, x) + \bar{a})](y - \langle y, x \rangle x) + [(a(y, x) + b)/(b(y, x) + \bar{a})]x$$

and we define $\delta : U(V) \times SL(2, \mathbb{R}) \times S \times \bar{B} \to \bar{B}$ by

$$\delta(W, L, x, y) = W \gamma(L, x, y) = \gamma(L, Wx, Wy).$$

With regard to the definition of $\gamma$ we remark that if $x \in S$ and if $y \in V$, then $y - \langle y, x \rangle x$ is the orthogonal projection of $y$ into $V \ominus \mathbb{C}x$. Furthermore we
remark that $\delta_{(W, L, x)}$ is a holomorphic homeomorphism of $B$ for every triple $(W, L, x)$ in $U(V) \times \text{SL}(2, \mathbb{R}) \times S$. We recall the following fact of the theory of functions on $B$.

3.2. **Lemma.** If $Z$ is a holomorphic homeomorphism of $B$, then there is a triple $(W, L, x)$ in $U(V) \times \text{SL}(2, \mathbb{R}) \times S$ such that

$$Z(y) = \delta(W, L, x, y)$$

for all $y$ in $B$.

3.3. The following lemma (which is well-known) follows from Lemma 3.2.

3.4. **Lemma.** If $Z$ is a holomorphic homeomorphism of $B$, then

$$\int f \circ Z d\sigma = \int f \delta^{Z(\cdot)} d\sigma$$

for every $f$ in $L^1(\sigma)$.

3.5. The following lemma is due to R. Schneider [2] who stated it and proved it in terms of the Hardy spaces of torii. His proof applies as well to $H^p(S)$.

3.6. **Lemma.** If $p \in [1, \infty]$, if $g \in H^p(S)$ and $g \not\equiv 0$, if $h \in L^\infty(\sigma)$, and if $gh^k \in H^p(S)$ for all $k$ in $\mathbb{Z}_+$, then $h \in H^\infty(S)$.

3.7. We will now prove Theorem 1.5. For this purpose we recall that if $g \in H^p(S)$ and $g \not\equiv 0$, then $g(y) \not\equiv 0$ for $\sigma$ almost all $y$ in $S$. We let $a = T_1$, $d\mu = |a|^2 d\sigma$, and define $A : H^p(S) \to L^p(\mu)$ by $Af = T_f/a$. Since $H^\infty(S)$ is a subalgebra of $L^\infty(\sigma)$, it follows from Theorem 1.2 and Corollary 2.5 that if $f$ and $g$ are in $H^p(S)$, then $Af \in L^\infty(\sigma)$ and $A(fg) = AfAg$. It follows from this and Lemma 3.6 that if $f \in H^\infty(S)$, then $Af \in H^\infty(S)$ since $A(\bar{f}^k) = aA(f^k) = T(f^k)$ and $T(f^k) \in H^p(S)$ for all $k$ in $\mathbb{Z}_+$. Thus if $A$ is restricted to $H^\infty(S)$, then $A$ is an algebra homomorphism of $H^\infty(S)$ into $H^p(S)$. Furthermore we have $\|Af\|_\infty = \|f\|_\infty$ for all $f$ in $H^\infty(S)$.

We define $\chi : S \times V \to \mathbb{C}$ by $\chi(x, y) = \langle x, y \rangle$, we let $F$ be an orthonormal basis of $V$, and we define $Z : B \to V$ by

$$Z(x) = \sum_{\nu \in F} \langle A\chi^\nu(x) \rangle y.$$ 

It follows that if $(x, y) \in B \times V$, then $\langle Z(x), y \rangle = \langle A\chi^\nu \rangle \delta(x)$. Hence $Z$ (which is holomorphic) maps $B$ into itself, and $(A\chi^\nu)^\# = (\chi^\nu)^\# \circ Z$ for all $y$ in $V$. Thus if $g$ is in the ring $C[\chi^\nu : y \in V]$, then $(Tg)^\# = a^\# (Ag)^\# = a^\# g^\# \circ Z$, from which it follows that if $f \in H^p(S)$, then

$$\tag{3.1} (Tf)^\# = a^\# f^\# \circ Z$$

since $C[\chi^\nu : y \in V]$ is dense in $H^p(S)$.

We now consider $T^{-1}$. It follows that there is a function $b$ in $H^p(S)$ and a holomorphic transformation $W$ of $B$ into itself such that if $f \in H^p(S)$, then

$$\tag{3.2} (T^{-1}f)^\# = b^\# f^\# \circ W.$$
From (3.1) and (3.2) it follows that if \( f \in H^p(S) \), then \( f^t \circ W \circ Z = f^t = f^t \circ Z \circ W \), and hence \( Z \) is a holomorphic homeomorphism of \( B \) (whose inverse is \( W \)). Thus (by Lemma 3.2) \( Z \) is defined on \( B \) as well as on \( B \), \( Z \) maps \( S \) onto itself, and we have

\[
Tf = af \circ Z
\]

for all \( f \) in \( H^p(S) \).

We will now prove that for \( \sigma \)-almost all \( x \) in \( S \) we have

\[
|a(x)|^p = \beta(x, z)
\]

where \( z = W(0) \). If \( f \in H^p(S) \), then by (3.3) and Lemma 3.4 we have

\[
\int |f|^p |a|^p d\sigma = \int |f \circ W \circ Z|^p |a|^p d\sigma = \int |f \circ W|^p d\sigma = \int |f|^p \beta^p d\sigma.
\]

From this and Theorem 1.2 it follows that if \( f \) and \( g \) are in \( C[\chi : \chi \in V] \), then

\[
\int f g |a|^p d\sigma = \int f g \beta^p d\sigma,
\]

from which it follows by the Stone-Weierstrass theorem that (3.4) holds for \( \sigma \)-almost all \( x \) in \( S \). We will denote by \( A(S) \) the class of all \( f \) in \( C(S) \) such that \( f^t \) is holomorphic on \( B \). With regard to the proof of (3.4) we remark that if \( n \geq 2 \), then \( \{|f| : f \in A(S)\} \) is not dense in \( \{|f| : f \in C(S)\} \).

We let \( \theta = a / [(\alpha^t)^{2n/p}] \). Then \( \theta \tilde{\theta} = 1, \theta \in H^\infty(S) \), and if \( f \in H^p(S) \), then \( Tf = \theta(\alpha^t)^{2n/p} f \circ Z \). Thus if \( f = T^{-1} \), then \( f \in H^\infty(S) \) and \( \tilde{\theta} = (\alpha^t)^{2n/p} f \circ Z \), and hence \( \theta \) is a constant. This completes the proof of Theorem 1.5.

**References**


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