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ON EXTENSION OF CHARACTERS FROM NORMAL SUBGROUPS

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1. Introduction

In what follows, character means irreducible complex character.

Let G be a finite group and let χ be a character of a normal subgroup N. If χ extends to a character of G then χ is stabilised by G, but the converse is false. The aim of this paper is to prove the following theorem which gives a sufficient condition for χ to be extended to a character of G.

Theorem. *Let the group G contain a subgroup B of order n such that $G=N\cdot B(N\Delta G)$ and let χ be a character of N which is stabilised by G. Then χ extends to a character of G if the following conditions hold:*

(1) $(m,n)=1$, $m=\chi(1)$;

(2) $N\cap B\leqq N'$.

The following well known results are corollaries of this theorem.

Corollary 1. (Gallagher, [**2**, Theorem 6]). *Let N be a normal Hall subgroup of G. Then each character of N which is stabilised by G extends to a character of G.*

Corollary 2. (Mackey, [**1**, p. 353]). *Suppose that $N\Delta G$. If N is abelian and complemented in G, then each character of N which is stabilised by G extends to a character of G.*

It follows from [**1**, Theorem 53.17] that the degrees of characters of a group divide the index of its abelian subnormal subgroup (and not only the index of abelian normal subgroup as is stated in [**1**, (53.18)]).

Hence we have a somewhat sharper result then the one given in Corollary 1.

Corollary 1′. *Suppose that $G=N\cdot B$ $(N\Delta G, N\cap B=1)$ and that the group N contains a subnormal abelian subgroup A such that $(m,n)=1$ where $m=(N:A)$ and n is the order of the group B. Then each character of N which is stabilised by G extends to a character of G.*

Finally note that if $O^p(G)$ is the subgroup of G generated by all the elements of order prime to p and if P is a Sylow p-subgroup of G then

$$G=O^p(G)\cdot P \quad \text{and} \quad O^p(G)\cap P\leqq O^p(G)'.$$

Thus we have the following result

Corollary 3. *Let χ be a character of $O^p(G)$ and suppose that χ is stabilised by G with $p\nmid\chi(1)$. Then χ extends to a character of G.*

Most of the notation we use is well known. In particular C^* denotes the set of all nonzero complex numbers, $o(g)$ is the order of g, and I denotes the $m\times m$ identity matrix.

2. Proof of the Theorem

Let Γ be a matrix representation of N which affords χ. Since χ is stabilised by G, any two representations

$$s\to\Gamma(s) \quad \text{and} \quad s\to\Gamma(g^{-1}sg)\ (s\in N)$$

of N are equivalent for all $g\in G$. Thus, if $g\in G$, there is a matrix $\psi(g)$ such that

$$\psi(g)^{-1}\Gamma(s)\psi(g)=\Gamma(g^{-1}sg) \qquad \text{for any } s\in N, \tag{1}$$

and so we may assume that

$$\Gamma(s)=\psi(s), \qquad \text{all } s\in N. \tag{2}$$

It is easy to see that the matrix $\psi(g_1)\psi(g_2)\psi(g_1g_2)^{-1}$ permutes with $\Gamma(s)$ for all $s\in N$, $g_1,g_2\in G$, and thus it follows from Schur's Lemma that

$$\psi(g_1)\psi(g_2)=\lambda(g_1,g_2)\psi(g_1g_2),\ \lambda(g_1g_2)\in C^*. \tag{3}$$

Therefore ψ is a projective representation of G. By replacing $\psi(g)$ by $\delta_g\psi(g)$ for a suitable $\delta_g\in C^*$ we may assume that, for any $g\in G-N$,

$$\psi(g)^{o(g)}=I. \tag{4}$$

If $g\in N\cap B$ then $\psi(g)=\Gamma(g)$ and so $N\cap B\leqq N'$ implies that $\det\psi(g)=1$. If $g\in B-N$ then it follows from (4) that $\det\psi(g)=\varepsilon$, $\varepsilon^k=1$, $k=o(g)$. The condition $(m,n)=1$ implies $(m,k)=1$ and hence there exists a natural number x such that $mx\equiv 1 \pmod k$. Thus $\det\varepsilon^{-x}\psi(g)=1$ and $[\varepsilon^{-x}\psi(g)]^{o(g)}=I$. We may therefore assume that

$$\det\psi(g)=1,\ \psi(g)^{o(g)}=I \text{ for all } g\in B. \tag{5}$$

Calculating the determinants in (3) and applying (5) we obtain

$$[\lambda(g_1,g_2)]^m=1 \text{ for all } g_1,g_2\in B. \tag{6}$$

Now consider the group $L=\langle\psi(g)\,|\,g\in B\rangle$. Then L contains the central subgroup $M=\langle\lambda(g_1,g_2)I\,|\,g_1,g_2\in B\rangle$.

It follows from (5), (6) and $(m,n)=1$ that the factor-group L/M has order prime to the order of M and hence by a theorem of Schur ([1], (7.5)) the group L is a direct product of M and another Hall subgroup. Since L is generated by the elements of orders prime to m it follows that $M=1$ and

$$\psi(g_1)\psi(g_2)=\psi(g_1g_2) \text{ for all } g_1, g_2 \in B. \tag{7}$$

Let R be a transversal to $N \cap B$ in B. Thus each g in G has a unique representation $g=rt$, $r \in R$, $t \in N$. If $g_1=r_1t_1$ is another element of G, write $r \cdot r_1 = r_2t_2$ with $r_2 \in R$, $t_2 \in N \cap B$. Define $\tilde{\psi}(g)=\psi(r)\psi(t)$. Using (1), (2) and (7) we get

$$\begin{aligned}\tilde{\psi}(g)\cdot\tilde{\psi}(g_1) &= \psi(r)\psi(t)\psi(r_1)\psi(t_1)=\psi(r)\psi(r_1)[\psi(r_1)^{-1}\psi(t)\psi(r_1)]\psi(t_1)\\ &= \psi(r\cdot r_1)[\psi(r_1)^{-1}\Gamma(t)\psi(r_1)]\psi(t_1)=\psi(r\cdot r_1)\Gamma(r_1^{-1}tr_1)\psi(t_1)\\ &= \psi(r\cdot r_1)\psi(r_1^{-1}tr_1)\psi(t_1)=\psi(r_2)\psi(t_2)\psi(r_1^{-1}tr_1)\psi(t_1)\\ &= \psi(r_2)\psi(t_2r_1^{-1}tr_1t_1)=\tilde{\psi}(gg_1).\end{aligned}$$

Thus, the character of $\tilde{\psi}$ extends χ.

REFERENCES

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