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Some entire functions with multiply-connected wandering domains

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Abstract. A component U of the complement of the Julia set of an entire function f is a wandering domain if the sets $f^n(U)$ are mutually disjoint, where $n \in \mathbb{N}$ and f^n is the n-th iterate of f. Examples are given of entire f of order ρ, $0 \le \rho \le \infty$, which have multiply-connected wandering domains. An example is given where the connectivity is infinite.

1. *Introduction*

If f is a rational function of degree at least two or, alternatively, a non-linear entire function, denote by f^n, $n \in \mathbb{N}$, the nth iterate of f; further, by $N(f)$ the set

$$N(f) = \{z; (f^n) \text{ is normal in some neighbourhood of } z\},$$

and by $J(f)$, often called the Julia set of f, the complement of $N(f)$. The set $J(f)$ is non-empty and perfect. $J(f)$ is also completely invariant, meaning that $J(f)$ is mapped to itself both by $z \to f(z)$ and by $z \to f^{-1}(z)$. For proofs of these properties see e.g. [**5**], [**6**].

If U is a component of $N(f)$ then $f(U)$ lies in some component V of $N(f)$ and $f(U) = V$, except in the case when f is transcendental entire with a Picard-exceptional (omitted) value c such that $c \in V$, when $f(U) = V - \{c\}$. Suppose that $f^{n+k}(U) \cap f^n(U) \ne \varnothing$ for some non-negative integers n and k. Then $f^n(U)$ is a periodic component and the limiting behaviour of the sequence of iterates in this component can be classified completely. In the converse case, when all $f^n(U)$ are different components of $N(f)$, U is called a wandering domain of f. Rational functions have no wandering domains [**7**], but entire functions may do so [**2**], [**3**], [**4**], [**7**].

It was shown in [**3**, theorem 5.2] that for any ρ such that $1 \le \rho \le \infty$, there exists an entire function of order ρ which has wandering domains. The domains constructed in this proof are simply-connected. However, it is known [**2**] that multiply-connected wandering domains can occur. Indeed one has the stronger result, proved in § 2:

THEOREM 1. *For any ρ such that $0 \le \rho \le \infty$ there is an entire function of order ρ, which has multiply-connected wandering domains.*

The proof of theorem 1 involves the construction of an entire function g and concentric rings A_n, $n \in \mathbb{N}$, such that A_{n+1} lies outside A_n, $g(A_n) \subset A_{n+1}$ and $A_n \to \infty$ as $n \to \infty$. Further, each A_n lies in a wandering component U_n of $N(g)$. Although each U_n is clearly multiply-connected, the exact value of the connectivity does not seem to be clear. However, by modifying the construction one can obtain some cases where the connectivity of the wandering domain is known.

THEOREM 2. *There exists an entire function which has wandering domains of infinite connectivity.*

2. *Proof of theorem* 1

Let k_n, $n \in \mathbb{N}$, denote any sequence of positive integers and C a constant, such that

$$0 < C < \frac{1}{4e^2}. \tag{1}$$

Suppose further that n_0 is a positive integer and r_1 a number such that

$$r_1 > 1, \qquad 2^{n_0-1}C > 2r_1^{k_1}. \tag{2}$$

Denote by r_n, $n \in \mathbb{N}$, numbers such that

$$r_{n+1} > 2r_n, \qquad 1 \leq n < n_0, \tag{3}$$

and

$$r_{n+1} = C\left\{1+\left(\frac{r_n}{r_1}\right)^{k_1}\right\}\left\{1+\left(\frac{r_n}{r_2}\right)^{k_2}\right\}\cdots\left\{1+\left(\frac{r_n}{r_n}\right)^{k_n}\right\} \tag{4}$$

for $n \geq n_0$. By induction it follows from (1) to (4) that

$$r_{n+1} > 2r_n, \qquad n \geq n_0. \tag{5}$$

For we may take any $n \geq n_0$ and assume in the induction that $r_n > 2r_{n-1} > 2^2 r_{n-2}, \ldots,$ so that (4) gives

$$r_{n+1} > C\left(\frac{r_n}{r_1}\right)^{k_1} 2^{n-1} > 2r_n^{k_1} > 2r_n.$$

Define the entire function g by

$$g(z) = C\prod_{j=1}^{\infty}\left\{1+\left(\frac{z}{r_j}\right)^{k_j}\right\},$$

where the product converges uniformly in any compact region of the plane, since $r_j > r_1 \cdot 2^{j-1}$. Now

$$r_{n+1} < g(r_n) < er_{n+1}, \qquad n \geq n_0, \tag{6}$$

since

$$\frac{g(r_n)}{r_{n+1}} = \prod_{j=n+1}^{\infty}\left\{1+\left(\frac{r}{r_j}\right)^{k_n}\right\} < (1+\tfrac{1}{2})(1+\tfrac{1}{4})\cdots < e.$$

Note that for all $|z| \leq 1$ we have by (1), since $r_j > 2^{j-1}$, that

$$|g(z)| \leq g(1) \leq C\pi(1+r_j^{-1}) < e^2 C < \tfrac{1}{4}. \tag{7}$$

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LEMMA 1. *For $n > n_0$ we have:*

$$g(r_n^{1/2}) < r_{n+1}^{1/2}; \tag{8}$$

$$\tfrac{1}{4}g(r_n^2) > r_{n+1}^2. \tag{9}$$

Proof of the lemma. Since $g(r)$ is max $|g(z)|$ for $|z| = r$, it follows that $V(s) = \log g(e^s)$ is convex and for $s > 0$ we have

$$V(2s) - V(0) \geq 2(V(s) - V(0)).$$

Hence $V(2s) \geq 2V(s) - V(0)$, which gives, using (7),

$$g(r^2) \geq \frac{(g(r))^2}{g(1)} > 4(g(r))^2.$$

Putting $r = r_n$ and noting (6) gives (9); putting $r = r_n^{1/2}$ gives (8). □

We remark that there is an integer n_1 such that $r_{n+1} > r_n^4$ for $n > n_1$. Set

$$A_n = \{z; r_n^2 \leq |z| \leq r_{n+1}^{1/2}\}, \qquad n > n_1.$$

LEMMA 2. *There is an integer $n_2 > n_1$ such that for z in A_n*

$$|g(z)| > \tfrac{1}{4}g(|z|). \tag{10}$$

Further $g(A_n) \subset A_{n+1}$, $n > n_2$.

Proof of the lemma. For $z \in A_n$, putting $|z| = r$,

$$\frac{g(r)}{|g(z)|} \leq \prod_{j>n} \left(\frac{1+(r/r_j)}{1-(r/r_j)}\right) \cdot \prod_{j\leq n} \left(\frac{1+(r_1/r)}{1-(r_j/r)}\right).$$

For $n > n_2$ both $x = (r/r_j) \leq r/r_{n+1} < r_{n+1}^{-1/2}, j > n$, and $y = r_j/r < r_n^{-1}, j \leq n$, are so small that $\log\{(1+x)/(1-x)\} < 3x$ and $\log\{(1+y)/(1-y)\} < 3y$. Thus for $n > n_2$ we have

$$\log \frac{g(r)}{|g(z)|} \leq 3 \sum_{j>n} \frac{r}{r_j} + 3 \sum_{j\leq n} \frac{r_j}{r} < 6\left(\frac{r}{r_{n+1}} + \frac{r_n}{r}\right) < \log 4,$$

if n_2 is large enough. This proves (10).

For z in A_n the maximum modulus theorem and (8) give $|g(z)| \leq g(r_{n+1}^{1/2}) < r_{n+2}^{1/2}$, while the minimum modulus theorem and (9), (10) give

$$|g(z)| \geq \min\left(\tfrac{1}{4}g(r_n^2), \tfrac{1}{4}g(r_{n+1}^{1/2})\right) = \tfrac{1}{4}g(r_n^2) > r_{n+1}^2.$$

Hence $g(A_n) \subset A_{n+1}$. □

LEMMA 3. *For $n > n_3$ each A_n belongs to a multiply-connected wandering domain component of $N(g)$.*

Proof of the lemma. From lemma 2 it follows that $g^k(z) \to \infty$ uniformly in each A_n, $n > n_2$, as $k \to \infty$. Thus A_n belongs to $N(g)$. Since $J(g)$ is not empty, the bounded component of the complement of A_n meets $J(g)$ for all large n. Hence the component U_n of $N(g)$ which contains such an A_n is not simply-connected. It was shown in [1] that if g is entire transcendental and $N(g)$ has a multiply-connected component, then every component of $N(g)$ is bounded. Thus for $n > n_3$, say, U_n is bounded and this implies that U_n is disjoint from U_{n+1}. It follows that each U_n, $n > n_3$, is a wandering domain.

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To complete the proof of theorem 1 it remains to show that g can be made to have any prescribed order of growth. The maximum modulus function of g is $g(r)$ and we have

$$\log g(r)=\log C+\sum_{j\le n}\log\left\{1+\left(\frac{r}{r_j}\right)^{k_j}\right\}+\sum_{j>n}\log\left\{1+\left(\frac{r}{r_j}\right)^{k_j}\right\},$$

where n is chosen so that $r_n\le r<r_{n+1}$. But estimates like those of lemma 2 show that

$$\begin{aligned}\log g(r)&=\sum_{j\le n}\left[\log\left(\frac{r}{r_j}\right)^{k_j}+\log\left\{1+\left(\frac{r_j}{r}\right)^{k_j}\right\}\right]+O(1)\\&=\sum_{j\le n}k_j\log\left(\frac{r}{r_j}\right)+O(1)\\&=\int_0^r\frac{n(t)}{t}\,dt+O(1),\end{aligned}$$

where $n(t)$ is the number of zeros of $g(z)$ in $|z|\le t$. The term $O(1)$ is bounded as r (and hence n)$\to\infty$. We have $n(t)=k_1+\cdots+k_j$ in $r_j\le t<r_{j+1}$. In the construction r_n depends only on $r_1,\ldots,r_{n-1},k_1,\ldots,k_{n-1}$. Thus we can prescribe k_n as a function of r_n, e.g. $k_n=[r_n^\alpha]$, for a given positive constant α. This makes $k_1+\cdots+k_n=O(r_n^\alpha)$ and $n(t)=O(t^\alpha)$ as $t\to\infty$ and so $\log g(r)=O(r^\alpha)$ as $r\to\infty$. Since $\log g(2r_n)>k_n\log 2>[r_n^\alpha]\log 2$ we see that g is indeed exactly of order α. The cases $\alpha=0$ and ∞ are easily dealt with by similar arguments.

3. *Proof of theorem* 2

The exact connectivity of the wandering domains U_n in the preceding example was not determined. In this section the construction is modified in such a way that the corresponding domains U_n each contain exactly one critical point of the entire function. This is shown to ensure the infinite connectivity of U_n. The function constructed below is of very small growth, certainly of order 0.

Begin the construction by taking C, n_0, $r_1,\ldots,r_{n_0}$ to satisfy (1), (2), (3) as in the proof of theorem 1, with $k_1=1$, but define r_n, $n>n_0$, by

$$r_{n+1}=C^2\left(1+\frac{r_n}{r_1}\right)^2\cdots\left(1+\frac{r_n}{r_n}\right)^2. \tag{11}$$

By induction it follows from (1), (2), (3) and (11) that

$$r_{n+1}>2r_n,\qquad n\in\mathbb{N}, \tag{12}$$

and indeed

$$r_{n+1}>4r_n^2,\qquad n>n_0. \tag{13}$$

Define the entire function

$$f(z)=C^2\prod_{j=1}^{\infty}\left(1+\frac{z}{r_j}\right)^2. \tag{14}$$

Set

$$s_n=\left(\frac{n+1}{n+2}\right)r_{n+1}. \tag{15}$$

LEMMA 4. *The zeros of f' are at the points $-r_n$ and t_n, $n \in \mathbb{N}$, where $t_n \in (-r_{n+1}, -r_n)$. For large n the point t_n lies in $(-s_n, -r_n^2)$.*

Proof of the lemma. Since $h(z) = (f'/f)(z) = 2\sum_j \{1/(z+r_j)\}$, it is easily seen that all zeros of h are real and negative. Further h is decreasing, except for discontinuities at $-r_n$. The first statement of the lemma is now clear. The rest will follow if we show that $h(-s_n) > 0$, or equivalently, $h(-s_{n-1}) > 0$, and $h(-r_n^2) < 0$ for large n. Now

$$h(-s_{n-1}) > 2(n+1)r_n^{-1} + 2\sum_{j<n} \frac{1}{(r_j - s_{n-1})}. \tag{16}$$

But if $\alpha_j = 1/(r_j - s_{n-1})$, then for $j < n$,

$$-\alpha_j \le -\alpha_{n-1} = \frac{(n+1)}{[nr_n - (n+1)r_{n-1}]}.$$

By (13) we have for large n that

$$r_n > 4r_{n-1}^2 > 2^n r_{n-1} > (n+1)r_{n-1},$$

and so $-\alpha_{n-1} < (n+1)/\{(n-1)r_n\}$, whence $h(-s_{n-1}) > 0$, using (16).

Further, using (12), (13) we have for large n that

$$\begin{aligned} h(-r_n^2) &< 2\left(\frac{1}{r_n - r_n^2}\right) + 2\left(\frac{1}{r_{n+1}^2 - r_n^2} + \frac{1}{r_{n+2}^2 - r_n^2} + \cdots\right) \\ &< \frac{-2}{r_n^2} + \frac{2}{r_n^2}\left(\frac{1}{2^2-1} + \frac{1}{2^4-1} + \cdots\right) < 0. \end{aligned}$$

□

LEMMA 5. *Denote by B_n the annulus $B_n = \{z;\, r_n^2 < |z| < s_n\}$. Then for large n we have $f(B_n) \subset B_{n+1}$.*

Proof of the lemma. By the maximum and minimum modulus principles it is sufficient to show for large n

$$f(s_n) \le s_{n+1}, \tag{17}$$

$$f(-s_n) > r_{n+1}^2, \tag{18}$$

$$f(-r_n^2) > r_{n+1}^2. \tag{19}$$

Now

$$\begin{aligned} \frac{f(s_{n-1})}{s_n} &= \frac{n+2}{n+1} \cdot \prod_{j<n} \frac{(1+(s_{n-1}/r_j))^2}{(1+(r_n/r_j))^2} \cdot \frac{(2n+1)^2}{4(n+1)^2} \cdot P \\ &< \frac{(n+2)(2n+3)^2 P}{\{4(n+1)^3\}}, \end{aligned}$$

where

$$\begin{aligned} P = \prod_{j>n}\left(1 + \frac{n}{n+1}\cdot\frac{r_n}{r_j}\right)^2 &< \left(1 + \frac{1}{(4r_n)}\right)^2 \left(1 + \frac{1}{(8r_n)}\right)^2 \cdots \\ &< \exp\{(r_n)^{-1}\} = 1 + O(2^{-n}) \end{aligned}$$

as $n \to \infty$, by (13). Thus

$$\frac{f(s_{n-1})}{s_n} = 1 - \frac{3}{4n^2} + O\left(\frac{1}{n^3}\right) < 1$$

for large n, which proves (17).

Further $f(-s_{n-1}) = C^2 P_1 P_2/(n+1)^2$, where

$$P_1 = \prod_{j<n} \left(\frac{n}{n+1} \cdot \frac{r_n}{r_j} - 1\right)^2,$$

and

$$P_2 = \prod_{j>n} \left(1 - \frac{n}{n+1} \cdot \frac{r_n}{r_j}\right)^2 = 1 - O(2^{-n}) > \tfrac{1}{2},$$

for large n. Since, by (13) the bracket on the right-hand side of P_1 is at least $r_n/4r_j$ we have for large n that $P_1 > r_n^4/16r_1r_2 > 2r_n^3/C^2$, whence $f(-s_{n-1}) > r_n^3/(n+1)^2 > r_n^2$. Thus (18) holds.

To prove (19) note that we have from (1), (14) that

$$\frac{f(-r_n^2)}{r_{n+1}^2} = \frac{r_1^2 \cdots r_{n-1}^2 (r_n - 1)^2 P_1^2 P_2^2}{4C^2 P_3^4}$$

where

$$P_1 = \prod_{j<n} \left(1 - \frac{r_j}{r_n^2}\right), \quad P_2 = \prod_{j>n} \left(1 - \frac{r_n^2}{r_j}\right), \quad P_3 = \prod_{j<n} \left(1 + \frac{r_j}{r_n}\right).$$

Using (13) we see that P_1 and P_2 are at least as big as $\alpha = (1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots$, while $P_3 < (1 + \frac{1}{2})(1 + \frac{1}{4}) \cdots$. Hence $f(-r_n^2)/r_{n+1}^2 > 1$ for all sufficiently large n. □

LEMMA 6. *If h is a transcendental entire function, then no doubly-connected component of $N(h)$ contains a critical point of h.*

Proof of the lemma. Suppose that U is a doubly-connected component of $N(h)$. By [1] and [3] every component of $N(h)$ is bounded and U is a wandering domain for h. Denote by α and β the outer and inner boundary components of U. Write $U_1 = h(U)$, $\beta_1 = h(\beta)$, $\alpha_1 = h(\alpha)$. The complete invariance of $J(h)$ implies that $\partial U_1 = f(\partial U) = \alpha_1 \cup \beta_1$, which has at most two components. Suppose that ∂U_1 is connected. If $U_n = h^n(U)$, $n \in \mathbb{N}$, it then follows from the complete invariance of $J(h)$ that each ∂U_n is connected. For large n this conflicts with theorem 3.1 of [3], where it is shown that for such n the domain $f^n(U)$ contains a closed curve γ_n whose distance from 0 is large and whose winding number about 0 is not zero; that is, γ_n must separate some points of $J(h)$ and in particular boundary points of U_n.

Thus ∂U_1 has two distinct components α_1, β_1 and by the maximum principle α_1 is the outer and β_1 is the inner component. Denote by ψ and ψ_1, respectively, 1-1 conformal maps of the annuli $K = \{z; 1 < |z| < R\}$ and $K_1 = \{z; 1 < |z| < R_1\}$ to U and U_1. It is assumed that ψ (ψ_1) approaches α (α_1) or β (β_1), respectively, *according as $|z|$ approaches R (R_k) or 1. Then $F = \psi_1^{-1} f \psi$ maps K onto K_1, and as* $z \to \partial K$ so $F(z) \to \partial K_1$. Thus F extends analytically to $\bar{K}$ and $|F(z)| = 1$ on $|z| = 1$, $|F(z)| = R_1$ on $|z| = R$. Repeated application of the reflection principle shows that F can be continued to give an analytic map $\mathbb{C} \to \mathbb{C}$ such that the only solution of $F(z) = 0$ is $z = 0$. Further, for w in K_1 all solutions of $F(z) = w$ are in K. Hence F is a polynomial of the form cz^m, $|c| = 1$, in a positive integer. It follows that $F' = 0$ has no solution in K, whence $h' = 0$ has no solution in U. □

Conclusion of the proof of theorem 2. The preceding lemmas show, as in the proof of theorem 1, that for large n the annulus B_n belongs to a multiply-connected component U_n of $N(f)$, that $f^k(z) \to \infty$ locally uniformly in U_n as $k \to \infty$, and that U_n, U_{n+1} are disjoint, so that U_n is wandering.

Note that for the critical points $-r_n$ and t_n of f, described in lemma 4, we have $f(-r_n) = 0$, so that for large n, $f(-r_n)$ is not in U_{n+1} and so $-r_n$ is not in U_n, (or indeed in any U_k, k large). Thus U_n contains one critical point of f, namely t_n. By lemma 6 U_n is not doubly-connected.

Suppose that the connectivity of U_n is finite, say that U_n has d_n boundary components. It follows that $d_{n+1} \le d_n$ and since $d_k \ge 3$ for all k we may assume that all d_n have the same value, d, for $n \ge n_0$. Then we may denote the boundary components of U_n by α_n (outer), β_n (the boundary of that component of $\mathbb{C} \setminus U_n$ which contains 0), and γ_j^n, $1 \le j \le d-2$. Since f maps α_n to α_{n+1}, β_n to β_{n+1} it follows from the complete invariance of $J(f)$ that f maps each γ_j^n to a γ_k^{n+1} and we may number the components so that $f(\gamma_j^n) = \gamma_j^{n+1}$.

For a fixed n take a neighbourhood V of γ_1^n which meets no other boundary component of U_n. Since $\gamma_1^n \subset J(f)$ there is some $k > 0$ and $\xi \in V$ such that $f^k(\xi) \in U_n$. Then $\xi \in N(f)$ but $\xi \notin U_n$ since U_n is a wandering domain of f. Denote by W the component of $\mathbb{C} \setminus \gamma_1^n$ which contains ξ. Clearly W is bounded. Thus we have $\partial W \subset \gamma_1^n$ and $f^k(W)$ meets U_n. But $\partial f^k(W) \subset f^k(\partial W) \subset \gamma_1^{n+k}$. Now the domain U_n is in the unbounded component of the complement of γ_1^{n+k} and hence $f^k(W)$ must be unbounded. This contradicts the boundedness of W. We have shown that the connectivity of U_n is indeed infinite. □

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