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FINITE GENERATION OF THE EXTENSION ALGEBRA $\mathrm{Ext}^*_R(M, M)$

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Abstract

For a module M over an Artin algebra R, we discuss the question of whether the Yoneda extension algebra $\mathrm{Ext}^*_R(M, M)$ is finitely generated as an algebra. We give an answer for bounded modules M. (These are modules whose syzygies have direct summands of bounded lengths.)

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1. Introduction

Let R be an algebra over a commutative ring K, and let M be a finitely generated R-module. The group $\mathrm{Ext}^*_R(M, M) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_R(M, M)$, together with the multiplication induced by the Yoneda product, is a graded K-algebra called the extension algebra of M. It has been asked in various contexts whether this algebra is finitely generated.

An affirmative answer has been given by Evens [2] for modules over a group algebra KG of a finite group over a field K. For a local commutative noetherian ring R with residue field k, it has been conjectured that $\mathrm{Ext}^*_R(k, k)$ would always be a finitely generated R-algebra; see Gulliksen and Levin [4, p.115], and Levin [5, Conjecture II]. A counterexample to this conjecture has been given by Roos [6, p.315]. For a QF (= self-injective, artinian) algebra R, finiteness conditions on $\mathrm{Ext}^*_R(M, M)$ have been studied by the author [8]. Here, we will continue these studies and extend some of the results to arbitrary algebras R.

In Section 2, we will look at the extension algebra $\mathrm{Ext}^*_R(M, M)$ for a bounded module over an Artin algebra R over K. (M is called *bounded* if there is a number b with the following property: If

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$$\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

is a minimal projective resolution of M, then all the indecomposable direct summands of all syzygies $\mathrm{Ker}\, d_i$ have lengths bounded by b.) We consider two special cases of bounded modules, namely the non-repetitive ones (no indecomposable direct summand of M or of any $\mathrm{Ker}\, d_i$ is isomorphic to an indecomposable direct summand of $\mathrm{Ker}\, d_j$ with $j \neq i$) and the ultimately closed ones (there is some i with $\mathrm{Ker}\, d_i = \bigoplus A_k$, and every A_k is isomorphic to a direct summand of some syzygy with index less than i). For a bounded, non-repetitive module M, we will show that its extension algebra is semi-primary, and we will describe its Jacobson radical, and we will show that the extension algebra is a finitely generated K-algebra if and only if it is an Artin algebra over K (Theorem 2). For an ultimately closed module M, we can only give a weak result on finite generation of its extension algebra (Theorem 3).

In Section 3, we concentrate on QF algebras R. In this case, all syzygies $\mathrm{Ker}\, d_i$ are indecomposable whenever M is. Thus the properties of being non-repetitive and ultimately closed are complementary, and ultimately closed means periodic (that is, $K_i \cong M$, for some i). We will show that for periodic M the extension algebra is, roughly speaking, a factor algebra of a skew polynomial algebra $E[X; \sigma]$ with an Artin algebra E over K. As for M non-periodic, we will give two examples which show that the extension algebra of M may or may not be finitely generated, being an Artin algebra in the first case and semiprimary, non-artinian in the latter case. The example to the latter case will also be used to show that rationality of the Poincaré series does not imply finite generation of the extension algebra, for modules over QF algebras.

2. Bounded modules

In all what follows, let R be an Artin algebra over a commutative artinian ring K, and let M be a finitely generated right R-module. Let a minimal projective resolution

$$\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

of M be given, and let $K_i = \mathrm{Ker}\, d_i$ for $i \geq 0$, and let $K_{-1} = M$.

DEFINITION 1. (1) M is called *bounded* if there exists $b \in \mathbb{N}$ such that every indecomposable direct summand of $\bigoplus_{i=-1}^{\infty} K_i$ has length not greater than b.

(2) M is called *non-repetitive* if the indecomposable direct summands of K_i and K_j are pairwise non-isomorphic, whenever $i \neq j$.

(3) M is called *ultimately closed* (at index $n \geq 1$) if every indecomposable direct summand of K_{n-1} is isomorphic to an indecomposable direct summand of some K_i

with $i < n - 1$, where n is smallest possible.

Examples of ultimately closed modules are modules over radical square zero algebras, modules over algebras of finite representation type, and periodic modules. Examples of bounded, non-repetitive modules are harder to come by. Non-periodic modules over QF algebras have been constructed in [8] and by Gasharov and Peeva [3].

THEOREM 2. *Let M be a bounded, non-repetitive R-module. Then*

(1) *The Jacobson radical of* $\mathrm{Ext}^*_R(M, M)$ *is*

$$J = \mathrm{Rad}(\mathrm{End}(M_R)) \oplus \bigoplus_{i=1}^{\infty} \mathrm{Ext}^i_R(M, M).$$

J is nilpotent with index $\leq m2^b$, where m is the length of M and b is a bound in the sense of Definition 1(1). *Also,* $\mathrm{Ext}^*_R(M, M)$ *is a semiprimary ring.*

(2) *The following are equivalent:*

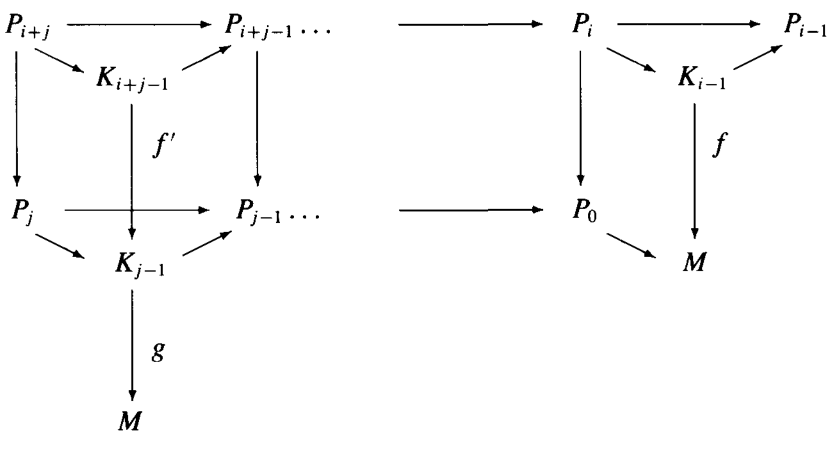
(i) $\mathrm{Ext}^*_R(M, M)$ *is a finitely generated K-algebra.*

(ii) *There is $n \in \mathbb{N}$ with* $\mathrm{Ext}^i_R(M, M) = 0$ *for all $i > n$.*

(iii) $\mathrm{Ext}^*_R(M, M)$ *is an Artin algebra over K.*

(iv) $\mathrm{Ext}^*_R(M, M)$ *is a noetherian ring.*

PROOF. (1) Recall that $\mathrm{Ext}^i_R(M, M)$ equals $\mathrm{Hom}_R(K_{i-1}, M)$ modulo the subgroup of maps which can be factorized over the inclusion $K_{i-1} \subset P_{i-1}$. Let $f \in \mathrm{Ext}^i_R(M, M)$ and $g \in \mathrm{Ext}^j_R(M, M)$ be given. Then the Yoneda product gf can be obtained from the diagram



where f' is a lift of f along the projective resolutions. Then the composite map gf' represents the extension $gf \in \mathrm{Ext}^{i+j}_R(M, M)$. Let now $k = m2^b$ (homogeneous)

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elements in J be given. Their product is a map

$$K_{i_0} \longrightarrow K_{i_1} \longrightarrow \cdots \longrightarrow K_{i_k}.$$

Writing every K_i as an indecomposable direct sum, it follows that the product map is a sum of chains $f_k \cdots f_2 f_1$, with each f_i starting and ending at an indecomposable module. If this chain contains m f_i's of degree 0 in sequence, then the product is zero since $\mathrm{Rad}(\mathrm{End}(M_R))$ has nilpotency index $\leq m$. Otherwise, the chain can be written as a product of at least 2^b non-isomorphisms between indecomposable modules of lengths bounded by b. Their product is 0 by the Lemma of Harada and Sai. It follows that $J^k = 0$. The factor ring $\mathrm{Ext}_R^*(M, M)/J$ is isomorphic to $\mathrm{End}(M_R)/\mathrm{Rad}(\mathrm{End}(M_R))$ which is semi-simple artinian. This shows that J is the radical of $\mathrm{Ext}_R^*(M, M)$, and the extension algebra has been shown to be semiprimary.

(2) We begin with (i) implies (ii). Let $\mathrm{Ext}_R^*(M, M)$, as a K-algebra, be generated by (homogeneous) elements $g_1, \dots, g_s$. Let c be the maximum of their degrees. Every element in $\bigoplus_{i=c2^b}^{\infty} \mathrm{Ext}_R^i(M, M)$ is then a sum of products of the g_i's having at least 2^b factors with positive degree. Again, by the Lemma of Harada and Sai, every such product is 0.

Now we show (ii) implies (iii). $\mathrm{Ext}_R^*(M, M) = \bigoplus_{i=0}^{n-1} \mathrm{Ext}_R^i(M, M)$ is obviously an Artin algebra over K.

That (iii) implies (iv) is trivial.

We prove that (iv) implies (ii). Since the ring is noetherian and semiprimary, it is artinian, and the descending chain of ideals

$$\bigoplus_{i=0}^{\infty} \mathrm{Ext}_R^i(M, M) \supset \bigoplus_{i=1}^{\infty} \mathrm{Ext}_R^i(M, M) \supset \cdots$$

must terminate.

Finally, (ii) implies (i) is trivial.

We now turn to ultimately closed modules. It has been shown by Wilson [9] that an ultimately closed module has a rational Poincaré series

$$\sum_{i=0}^{\infty} \dim_K(\mathrm{Ext}_R^i(M, M))T^i.$$

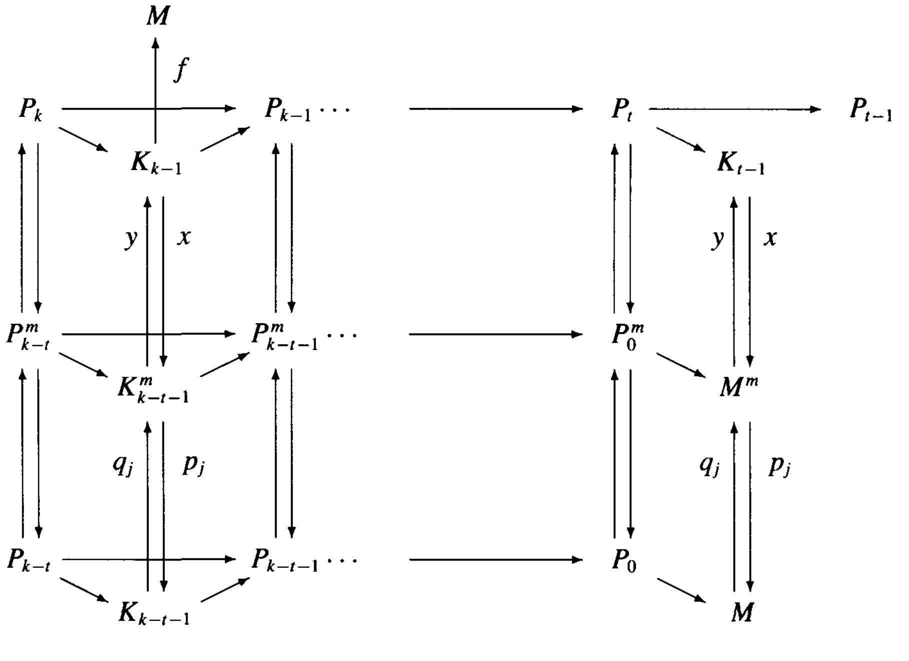
(Here, K is a field.) The problem of when an arbitrary module M has a rational Poincaré series is unsolved. The origin of this problem is a conjecture of Serre and Kaplansky suggesting that this would be true in case $M = k$, where k is the residue field of a local commutative noetherian ring R. A counterexample to this conjecture was given by Anick [1]. It was shown by Levin [5] that, in several special cases, rationality of the Poincaré series of k implies finite generation of the extension algebra $\mathrm{Ext}_R^*(M, M)$. We will show in Section 3 that this implication does not hold for bounded modules, in general. As for finite generation of the extension algebra, we get the following result for ultimately closed modules.

THEOREM 3. *Let the R-module M be ultimately closed, at index n. Then the extension algebra $\mathrm{Ext}_R^*(M', M')$, with $M' = M \oplus K_0 \oplus \cdots \oplus K_{n-2}$, is a finitely generated K-algebra.*

PROOF. Let K_0' be the kernel of a minimal projective cover of M'. Since M is ultimately closed at index n, K_0' embeds in M'^m, for some m. The result then follows from Lemma 4. With view to another application of Lemma 4 in Section 3, we are formulating it slightly more generally, and we are using the symbol M, instead of M'. Lemma 4 has already been proved in [8, Lemma 2.1(1)]. We include a proof here, for the convenience of the reader.

LEMMA 4. *Assume there exists a splitting monomorphism $x : K_{t-1} \longrightarrow M^m$, for some $t \geq 1$ and $m \in \mathbb{N}$. Then every element in $\mathrm{Ext}_R^*(M, M)$ is a sum of elements of the form gh (Yoneda product), where $g \in \bigoplus_{i=0}^{t-1} \mathrm{Ext}_R^i(M, M)$, and where $h = (p_{i_1}x)(p_{i_2}x) \cdots (p_{i_r}x)$ (Yoneda product), for some $r \in \mathbb{N}$ and some i_l between 1 and m, p_i denoting the canonical projection from M^m onto the ith summand M. In particular, $\mathrm{Ext}_R^*(M, M)$ is a finitely generated K-algebra.*

PROOF. Let $f \in \mathrm{Ext}_R^k(M, M)$ be given. Let y be a left inverse of x, and let $q_1, \ldots, q_m$ denote the canonical injections from M into M^m, with $\sum_j q_j p_j$ equal to the identity map of M^m. In the following diagram, let all the liftings of x, y, p_j, q_j from the right to the left along the projective resolutions be denoted by the same symbols, respectively.



From the diagram it follows that $f = fyx = fy(\sum_j q_j p_j)x = \sum_j (fyq_j)(p_j x) \in \sum_j \mathrm{Ext}_R^{k-t}(M, M)(p_j x)$. By induction over k we see that f has the form as stated. The algebra $\mathrm{Ext}^*_R(M, M)$ is then generated by the elements $p_1 x, \dots, p_m x$ together with a finite set of elements which generate $\bigoplus_{i=0}^{t-1} \mathrm{Ext}^i_R(M, M)$ as a K-module.

3. Bounded modules over QF algebras

In this section, let R be a QF algebra over K. It is well-known that if M is indecomposable, then all syzygies are indecomposable. An ultimately closed module M is then periodic. The smallest number t with $M \cong K_{t-1}$ is called the period of M. Also, M is non-repetitive if and only if M and its syzygies are pairwise non-isomorphic. Hence, for indecomposable modules over a QF algebra, the properties of being ultimately closed and non-repetitive are complementary.

THEOREM 5. *Let R be a* QF *Artin algebra over K, and let M be periodic with period t. Then the extension algebra* $\mathrm{Ext}^*_R(M, M)$ *is a finitely generated K-algebra and a noetherian ring.*

PROOF. Let E be the K-algebra generated by $\bigoplus_{i=0}^{t-1} \mathrm{Ext}^i_R(M, M)$. E carries the grading induced from the extension algebra $\mathrm{Ext}^*_R(M, M)$. Since M has period t, the modules $M, \dots, K_{t-2}$ are pairwise non-isomorphic. Elements in E of high degree are produced by long Yoneda products of elements in $\bigoplus_{i=1}^{t-1} \mathrm{Ext}^i_R(M, M)$. Following the arguments given in the proof of Theorem 2(1), such products are 0, by the Lemma of Harada and Sai. This means that E is already contained in $\bigoplus_{i=0}^{n} \mathrm{Ext}^i_R(M, M)$, for some n. Consequently, E is an Artin algebra over K, in particular it is a finitely generated K-algebra. Let now $x : K_{t-1} \longrightarrow M$ be an isomorphism. Then Lemma 4 (with $m = 1$) shows that the extension algebra of M equals $\sum_{i=0}^{\infty} Ex^i$. This algebra is generated by E and x; hence it is a finitely generated K-algebra, as well. We will show that $Ex = xE$. This will imply that $\mathrm{Ext}^*_R(M, M)$ is noetherian, by [8, Proposition 3.5.2]. It is clear from Lemma 4 that $xE \subset Ex$. To show the other inclusion, let $f = ex \in Ex$ be given. Without loss of generality, we may assume that f is homogeneous and $t \leq \deg f \leq 2t - 1$. In the following diagram, the rows are minimal projective resolutions, and, when being read from the left to the right, minimal injective resolutions, since R is QF. Hence, the map $x^{-1} f$ can be lifted, from the left to the right, to a map $g \in E$. Then f is equal to the Yoneda product xg, what was to be shown.

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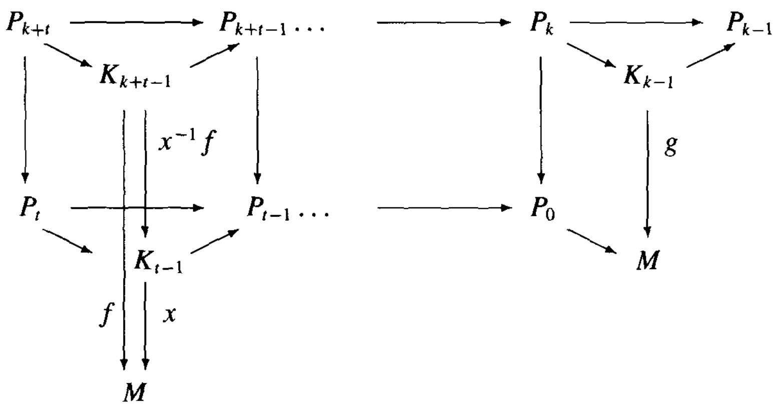


REMARK 6. Let us denote by $\underline{\mathrm{End}}(M_R)$ the endomorphism ring of M modulo the ideal of maps which factorize over a projective module. Let the extension algebras derived from $\mathrm{Ext}_R^*(M, M)$ and E, after replacing $\mathrm{Ext}_R^0(M, M) = \mathrm{End}(M_R)$ by $\underline{\mathrm{End}}(M_R)$, be denoted by $\underline{\mathrm{Ext}}_R^*(M, M)$ and $\underline{E}$, respectively.

It is well-known that the map which assigns ex to xg, in the proof of Theorem 5, induces a ring isomorphism $\sigma : \underline{E} \longrightarrow \underline{E}$, by taking $\sigma(e) = g$. Then Theorem 5 amounts to saying that $\underline{\mathrm{Ext}}_R^*(M, M)$ is a homomorphic image of the skew polynomial ring $\underline{E}[X; \sigma]$.

We now turn to non-periodic modules M over a QF algebra R. By Section 2, the extension algebra is either artinian (and hence a finitely generated K-algebra) or semiprimary and non-noetherian (and then not a finitely generated K-algebra). We will give two examples of modules over the same QF algebra R which show that both cases do occur.

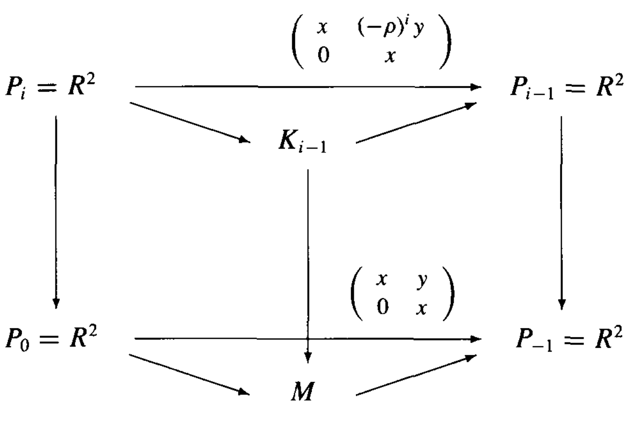
In what follows, let K be any field, and let $0 \neq \rho \in K$, ρ not a root of 1. Let $R = K[X, Y]/(X^2, Y^2)$, with the multiplication $YX = \rho XY$. Then R is a local Frobenius algebra with basis $1, x, y, xy$ over K (x, y denoting the residue classes of X, Y), and with $\mathrm{Rad}\, R = xR + yR$, $\mathrm{Rad}^2 R = \mathrm{Soc}\, R = xyR$ and $\mathrm{Rad}^3 R = 0$.

EXAMPLE 7. Let $M = (x + y)R$. Then M is an indecomposable R-module of length 2. The map $R \longrightarrow M$, $r \mapsto (x + y)r$, is a projective cover of M, whose kernel is the right annihilator of $x + y$ in R which is equal to $(x + (-\rho)y)R$. Hence, $K_0 = (x + (-\rho)y)R$. By induction over i, it follows that the syzygies of M are the modules $K_{i-1} = (x + (-\rho)^i y)R$ ($i \geq 1$) which are again indecomposable of length 2. M and its syzygies are pairwise non-isomorphic, since they have different annihilators in R. The group $\mathrm{Ext}_R^i(M, M)$ equals $\mathrm{Hom}_R(K_{i-1}, M)$ modulo the subgroup of maps which can be lifted over the inclusion $K_{i-1} \subset R$. Every $f \in \mathrm{Hom}_R(K_{i-1}, M)$ maps the top of K_{i-1} to the socle of M which means $f(x + (-\rho)^i y) = xys$, for some $s \in K$. Then f can be lifted to f' over the inclusion $K_{i-1} \subset M$ if and only if there is $t \in K$ with $f'(1) = (x + y)t$ and $(x + y)t(x + (-\rho)^i y) = xys$, or $t(\rho + (-\rho)^i) = s$. For $i \geq 2$, the factor $(\rho + (-\rho)^i)$ is non-zero due to the choice of ρ; hence this equation can be resolved by t for any s. This implies $\mathrm{Ext}_R^i(M, M) = 0$ for all $i \geq 2$. For $i = 1$, this equation cannot be satisfied for any non-zero s, and one obtains $\mathrm{Ext}_R^1(M, M) \cong K$. For the extension algebra, this means $\mathrm{Ext}_R^*(M, M) = \mathrm{End}(M_R) \oplus \mathrm{Ext}_R^1(M, M) \cong K[T]/T^2 \oplus K$.

EXAMPLE 8. Let the module M be given as the image of left multiplication by the matrix $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, on R^2. Then M is an indecomposable right R-module of length 4 with $M/\operatorname{Rad} M$ and $\operatorname{Rad} M = \operatorname{Soc} M$ both of length 2. We claim that the extension algebra of M is not finitely generated. To prove this, we consider the minimal projective resolution

$$\cdots P_i \xrightarrow{\begin{pmatrix} x & (-\rho)^i y \\ 0 & x \end{pmatrix}} P_{i-1} \cdots$$

of M, where $P_i = P_{i-1} = R^2$ for $i \geq 1$. The kernels are pairwise non-isomorphic since they have different annihilators in R. From the diagram



it follows that $\mathrm{Ext}_R^i(M, M)$ is K-isomorphic to the space of matrices

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} R^{2\times 2} \cap R^{2\times 2} \begin{pmatrix} x & (-\rho)^i y \\ 0 & x \end{pmatrix} \Big/ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} R^{2\times 2} \begin{pmatrix} x & (-\rho)^i y \\ 0 & x \end{pmatrix}.$$

A direct computation shows that the K-dimensions of these spaces are 4, for every $i \geq 1$. Then $\mathrm{Ext}_R^*(M, M)$ cannot be finitely generated, by Theorem 2(2).

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REMARK 9. In Example 8, the dimensions of all $\text{Ext}^i_R(M, M)$ $(i \geq 1)$ are 4. Then the Poincaré series of the extension algebra is rational. Hence, M is an example of a bounded module over a QF algebra whose extension algebra is not finitely generated but has a rational Poincaré series. Gasharov and Peeva [3, Example 3.2] have given a beautiful example of a non-periodic bounded module over a commutative QF algebra. This module can also be shown to have a non-finitely generated extension algebra with rational Poincaré series.

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