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An elementary approach to the abelianization of the Hitchin system for arbitrary reductive groups

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Abstract. We consider the moduli space $\mathcal{M}$ of stable principal G-bundles over a compact Riemann surface C of genus $g \geqslant 2$, G being any reductive algebraic group and give an explicit description of the generic fibre of the Hitchin map $\mathcal{H}\colon T^*\mathcal{M} \to \mathcal{K}$.

If $T \subset G$ is a fixed maximal torus with Weyl group W, for each given generic element $\phi \in \mathcal{K}$ one may construct a W-Galois covering $\widetilde{C}$ of C and consider the generalized Prym variety $\mathcal{P} = \mathrm{Hom}_W(X(T), J(\widetilde{C}))$, where $X(T)$ denotes the group of characters of T and $J(\widetilde{C})$ the Jacobian. The connected component $\mathcal{P}_0 \subset \mathcal{P}$ which contains the trivial element is an abelian variety. In the present paper we use the classical theory of representations of finite groups to compute $\dim \mathcal{P} = \dim \mathcal{M}$. Next, by means of mostly elementary techniques, we explicitly construct a finite map $\mathcal{F}$ from each connected component $\mathcal{H}^{-1}(\phi)_c$ of the Hitchin fibre to $\mathcal{P}_0$. In case $G = \mathrm{PGl}(2)$ one has that the generic fibre of $\mathcal{F}\colon \mathcal{H}^{-1}(\phi)_c \to \mathcal{P}_0$ is a principal homogeneous space with respect to a product of $(2d-2)$ copies of $\mathbf{Z}/2\mathbf{Z}$ where d is the degree of the canonical bundle over C. However if the Dynkin diagram of G does not contain components of type B_l, $l \geqslant 1$ or when the commutator subgroup (G, G) is simply connected the map $\mathcal{F}$ is injective.

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Key words: principal G-bundles, generalized Prym varieties.

Introduction

We consider here the moduli space $\mathcal{M}$ of stable principal G-bundles over a compact Riemann surface C, with G an algebraic complex group. We denote by K the canonical bundle over C. In [Hi] N. Hitchin defined an analytic map $\mathcal{H}$ from the cotangent bundle $T^*\mathcal{M}$ to the 'characteristic space' $\mathcal{K}$ by associating to each G-bundle P and section $s \in H^0(C, \mathrm{ad}\, P \otimes K)$ the spectral invariants of s. Hitchin showed for $G = \mathrm{Gl}(n), \mathrm{SO}(n), \mathrm{Sp}(n)$ that the generic fibre of $\mathcal{H}$ is an open set in an abelian variety $\mathcal{A}$. In fact, he considers in each case a nonsingular spectral curve S covering C : for $G = \mathrm{Gl}(n)$, $\mathcal{A}$ is identified with the Jacobian $J(S)$; in the other cases, there is a naturally defined involution on S and $\mathcal{A}$ is the associated Prym variety. More recently, Faltings extended these results and described an abelianization procedure for the moduli space of Higgs G-bundles, with G any reductive group (see [F]). If $T \subset G$ is a fixed maximal torus with Weyl group W, one may construct for each given generic element $\phi \in \mathcal{K}$ a ramified covering $\widetilde{C}$ of

C having $|W|$ sheets. The combined action of W on $\widetilde{C}$ and on the group of one parameter subgroups of T induces an action on the space of all principal T-bundles τ over $\widetilde{C}$ and we may consider the subvariety $\widehat{\mathcal{P}}$ of those τ which are W-invariant in this sense. The connected component $\mathcal{P}_0$ of $\widehat{\mathcal{P}}$ which contains the trivial T-bundle is an abelian variety. In [F] it is shown that the generic fibre of the Hitchin map is a principal homogeneous space with respect to a group (namely the first étale cohomology group of C with coefficients in a suitably defined group scheme) which is isogenous to $\widehat{\mathcal{P}}$. In the present paper, by means of mostly elementary techniques, we explicitly construct a map $\mathcal{F}$ from each connected component $\mathcal{H}^{-1}(\phi)_c$ of $\mathcal{H}^{-1}(\phi)$ to $\mathcal{P}_0$ and show that $\mathcal{F}$ has finite fibres. We use the classical theory of representations of finite groups to compute $\dim \mathcal{P}_0 = \dim \mathcal{M}$ and conclude that the image under $\mathcal{F}$ of $\mathcal{H}^{-1}(\phi)_c$ contains a Zariski open set in $\mathcal{P}_0$.

In case $G = \mathrm{PGl}(2)$ one can check directly that the generic fibre of $\mathcal{F}: \mathcal{H}^{-1}(\phi)_c \to \mathcal{P}_0$ is a principal homogeneous space with respect to a product of $(2 \cdot \deg K - 2)$ copies of $\mathbf{Z}/2\mathbf{Z}$. However in case the Dynkin diagram of G does not contain components of type B_l, $l \geqslant 1$ or when the commutator subgroup (G, G) is simply connected the map $\mathcal{F}$ is injective.

Such results were announced in our previous paper [Sc], in which we showed that $\mathcal{P}_0$ is isogenous to a 'spectral' Prym–Tjurin variety P_λ for each given dominant weight λ. Results concerning the description of the Hitchin fibre in terms of generalized Prym varieties were also announced in R. Donagi, *Spectral covers*, preprint, alg-geom/9505009 (1995).

1. The Hitchin map for any reductive group

We denote by C a compact Riemann surface of genus $g \geqslant 2$ and by G a reductive algebraic group over the field of complex numbers. We also write **g** as the Lie algebra of G. The moduli space of stable principal G-bundles over C is a quasi-projective complex variety $\mathcal{M}$ with $\dim \mathcal{M} = (g-1)\dim G + \dim Z(G)$, $Z(G)$ being the center of G. Note here that semistability for a principal G-bundle P corresponds to semistability for the holomorphic vector bundle $\mathrm{ad}\, P$ associated to the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{gl}(\mathbf{g})$ ([A-B], [R]).

We denote by K the canonical line bundle over C. By deformation theory and Serre duality, a point in the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$ is a pair (P, s) with P a stable principal G-bundle over C and s a section of the vector bundle $\mathrm{ad}\, P \otimes K$. The ring of polynomials on **g** which are invariant with respect to the adjoint action is freely generated by homogeneous polynomials $h_1, \ldots, h_k$. Each h_i induces a map $\mathcal{H}_i\colon \mathrm{ad}\, P \otimes K \to K^{d_i}$ where $d_i = \deg h_i$, and the Hitchin map

$$\mathcal{H}\colon T^*\mathcal{M} \to \mathcal{K} = \oplus_{i=1}^{k} H^0(C, K^{d_i})$$

takes (P, s) to the element in $\mathcal{K}$ whose ith component is the composition of $\mathcal{H}_i$ with s ([Hi]). It is a remarkable fact that the dimension of $\mathcal{K}$ is equal to the dimension of

$\mathcal{M}$. Moreover the map $\mathcal{H}$ is surjective. This fact can be deduced from the existence of very stable G-bundles (see [L], [BR], [KP] Lemma 1.4).

We fix once and for all a maximal torus $T \subset G$ with associated root system $R = R(G,T)$ and Weyl group $W = N_G(T)/T$. We also fix a subset $R^+ \subset R$ of positive roots (or equivalently a Borel subgroup $B \supset T$). If $\mathbf{t}$ denotes the Lie algebra of T, the differential of each root $\alpha \in R$ induces a map $d\alpha\colon \mathbf{t} \otimes K \to K$ and the homogeneous W-invariant polynomials $\sigma_1, \ldots, \sigma_k$ on $\mathbf{t}$ obtained by restriction of $h_1, \ldots, h_k$ define a Galois covering

$$\underline{\sigma} = (\sigma_1, \ldots, \sigma_k)\colon \mathbf{t} \otimes K \to \oplus_{i=1}^k K^{d_i}$$

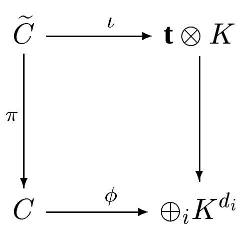
whose discriminant Ξ is given by the zeroes of the W-invariant function $\prod_{\alpha \in R} d\alpha$. For generic $\phi \in \mathcal{K} = H^0(C, \oplus_i K^{d_i})$, we consider the curve $\widetilde{C} := \phi^*(\mathbf{t} \otimes K)$. This is a ramified covering of C having $m = |W|$ sheets, whose branch locus Ram satisfies by construction

$$\mathcal{O}(Ram) \cong K^{|R|} \equiv K^{(\dim G - \operatorname{rank} G)}. \tag{1.1}$$

If we indicate by $\iota\colon \widetilde{C} \to \mathbf{t} \otimes K$ the natural inclusion map, we have by definition, for each $w \in W$,

$$\iota(w\eta) = \operatorname{Ad}(n_w)\iota(\eta), \tag{1.2}$$

where $n_w \in N_G(T)$ is any representative of w. Note also that, if $\pi\colon \widetilde{C} \to C$ denotes the projection map, $d\alpha \circ \iota$ is a holomorphic section of $\pi^* K$.



As a consequence of our genericity hypothesis, $\widetilde{C}$ has the following properties:

(a) it is smooth and irreducible.
(b) each ramification point $p \in \pi^{-1}(Ram)$ has index 1; i.e. is a simple zero for the section $\prod_{\alpha \in R^+}(d\alpha \circ \iota)\colon \widetilde{C} \to \pi^* K^{|R|/2}$.

This may be checked as follows. Let us denote by $\pi_i\colon K^{d_i} \to C$, $i = 1, \ldots, k$ and $q\colon \mathbf{t} \otimes K \to C$ the projections. Moreover for every $i = 1, \ldots, k$ let us denote by $\gamma_i\colon K^{d_i} \to \pi_i^* K^{d_i}$ the tautological section. For each i we consider those sections of $q^* K^{d_i}$ that have the form $s = c \cdot \sigma_i^* \gamma_i + q^* a_i$ for some $c \in \mathbf{C}$ and $a_i \in H^0(C, K^{d_i})$. As c varies in $\mathbf{C}$ and a_i in $H^0(C, K^{d_i})$ the zero divisor of s forms a linear system δ_i of divisors in $\mathbf{t} \otimes K$ that has no base points since the linear system $|K^{d_i}|$ on C has

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no base points. For $\phi = (a_1, \dots, a_k) \in \mathcal{K}$, the curve $\widetilde{C}$ is defined by the equations $\sigma_i^*\gamma_i = q^*a_i$, $i = 1, \dots, k$. One immediately checks that the map

$$\begin{aligned} K^{d_i} &\longrightarrow \mathbf{P}^{\dim H^0(C, K^{d_i})} \\ x &\longmapsto [\gamma_i(x), \pi_i^* a_{i,1}(x), \dots, \pi_i^* a_{i,m_i}(x)], \end{aligned}$$

where the $a_{i,j}$'s form a basis of $H^0(C, K^{d_i})$ has image of dimension 2 and that $\sigma_1 : \mathbf{t} \otimes K \to K^{d_1}$ is dominant. By Bertini's theorem (see [J], Theorem 6.3) the divisor $X_1 \in \delta_1$ of the section $\sigma_1^*(\gamma_1 - \pi_1^* a_1) = \sigma_1^*\gamma_1 - q^* a_1$ with a_i generic in $H^0(C, K^{d_i})$ is smooth and irreducible. If $k \geqslant 2$, we next consider the linear system on X_1 given by the restriction of δ_2. Since the polynomial σ_2 is algebraically independent from σ_1 the map $\sigma_2 \mid_{X_1}: X_1 \to K^{d_2}$ is dominant. We use the same argument as above and from Bertini's theorem we obtain that the divisor $X_2 \subset X_1$ of the section $\sigma_2^*\gamma_2 - q^*a_2 \mid_{X_1}$ with generic a_2 is smooth and irreducible. We can repeat the same argument for the linear system $\delta_i \mid_{X_{i-1}}$ for every $i \leqslant k$ (since the map $\sigma_i \mid_{X_{i-1}}: X_{i-1} \to K^{d_i}$ is dominant) and thus prove (a). As for the statement (b) one may consider the restriction of the linear systems above both to the discriminant locus Ξ and to the locus $\mathcal{Z} \subset \Xi$ where $\prod_{\alpha \in R^+} d\alpha$ vanishes with multiplicity $\geqslant 2$ ($\mathcal{Z} = \operatorname{Sing} \Xi$). Again from Bertini's theorem one obtains that $\widetilde{C}$ does not intersect $\mathcal{Z}$ and intersects $\Xi \setminus \mathcal{Z}$ transversely.

Remark 1.1. For each $\alpha \in R^+$, let $s_\alpha \in W$ denote the corresponding reflection. As a consequence of condition (b) above we may consider the ramification locus in $\widetilde{C}$ as a disjoint union: $\mathcal{D} = \coprod_{\alpha \in R^+} \mathcal{D}_\alpha$, with $\mathcal{D}_\alpha = \{\text{zeroes of } d\alpha \circ \iota\} = \{\eta \in \widetilde{C} \mid s_\alpha \eta = \eta\}$. By our previous considerations $\mathcal{D}_\alpha$ belongs to the linear system $|\pi^* K|$. In case G is simple and simply laced, i.e. W acts transitively on the set of roots R, we may write for each $y \in Ram$

$$\pi^{-1}(y) = \coprod_{\alpha \in R^+} \mathcal{D}_\alpha^y,$$

where $\mathcal{D}_\alpha^y := \mathcal{D}_\alpha \cap \pi^{-1}(y)$ is nonempty for every $\alpha \in R^+$.

If G is not simply laced and has connected Dynkin diagram, R is the union of two W-orbits R_1, R_2, each one consisting of roots having the same length. Then we have

$$\pi^{-1}(y) = \coprod_{\alpha \in R_1 \cap R^+} \mathcal{D}_\alpha^y \quad \text{or} \quad \pi^{-1}(y) = \coprod_{\alpha \in R_2 \cap R^+} \mathcal{D}_\alpha^y \tag{1.3}$$

depending on whether y corresponds to a short or a long root.

More generally, if the Dynkin diagram of G has more than one connected component, we have as many different 'kinds' of fibers

$$\pi^{-1}(y) = \coprod_{\alpha \in R_j \cap R^+} \mathcal{D}_\alpha^y$$

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as are the W-orbits $R_j \subset R$. Since for each $\alpha \in R^+$ we have $|\mathcal{D}_\alpha| = |W| \cdot \deg K$ and each fibre over a branch point consists of $|W|/2$ points, the number of fibres which correspond to the same orbit R_j is equal to

$$\begin{aligned} n_j &= |R_j^+| \cdot |W| \cdot \deg K / \tfrac{1}{2}|W| \\ &= |R_j| \cdot \deg K. \end{aligned} \tag{1.4}$$

Let now $X(T)$ be the group of characters of T and consider the group $H^1(\widetilde{C}, T)$ of isomorphism classes of holomorphic principal T-bundles over $\widetilde{C}$. Each pair (τ, μ) with τ a principal T-bundle, $\mu \in X(T)$, defines a line bundle $\tau_\mu \equiv \tau \times_\mu \mathbf{C}$ and this way $H^1(\widetilde{C}, T)$ is identified with

$$\mathrm{Pic}(\widetilde{C}) \otimes X(T)^*,$$

$X(T)^* \equiv \mathrm{Hom}(X(T), \mathbf{Z})$ being the dual group. For the same reason, the group of isomorphism classes of topologically trivial principal T-bundles is a tensor product

$$J(\widetilde{C}) \otimes X(T)^*$$

(here, as usual, $J(\widetilde{C})$ denotes the group of divisors with zero degree modulo linear equivalence). Now, the action of W on the sheets of $\widetilde{C}$ induces an action on $J(\widetilde{C})$. On the other hand, W acts by conjugation on T, hence on $X(T)^*$. If $\tau = D_1 \otimes \chi_1 + \cdots + D_l \otimes \chi_l$ is a principal T-bundle over $\widetilde{C}$ and $w \in W$ an element of the Weyl group, we set

$${}^w\tau = wD_1 \otimes {}^w\chi_1 + \cdots + wD_l \otimes {}^w\chi_l.$$

DEFINITION 1.1. The generalized Prym variety $\mathcal{P} = [J(\widetilde{C}) \otimes X(T)^*]^W$ consists of those isomorphism classes of topologically trivial T-bundles τ which satisfy ${}^w\tau \cong \tau$ for each $w \in W$.

Note that $\mathcal{P}$ is an algebraic group whose connected component of the identity $\mathcal{P}_0$ is an abelian variety.

2. Computing the dimension of $\mathcal{P}$

The following can be deduced from the above mentioned Faltings' result describing the generic Hitchin fibre as isogenous to $\widehat{\mathcal{P}} = [\mathrm{Pic}(\widetilde{C}) \otimes X(T)^*]^W$ ([F], Theorem III.2) and the fact (due to G. Laumon and proved in [F], Theorem II.5) that all Hitchin fibers have the same dimension:

PROPOSITION 2.1. *The dimension of $\mathcal{P}$ is equal to the dimension of $\mathcal{M}$.*

In this section we give a direct proof of such statement. If we set $\mathcal{S} \equiv X(T) \otimes_{\mathbf{Z}} \mathbf{C}$ and denote by H^1 the first cohomology W-representation $H^1(\widetilde{C}, \mathbf{C})$, by Doulbault theorem we have

$$\dim \mathcal{P} = \tfrac{1}{2}\dim[H^1 \otimes \mathcal{S}^*]^W = \tfrac{1}{2}\dim \mathrm{Hom}_W(\mathcal{S}, H^1).$$

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We will compute $M \equiv \dim \mathrm{Hom}_W(\mathcal{S}, H^1)$ by use of the classical theory of representations of finite groups and associated characters (for more details about this subject, see for example [Se]).

For any W-representation V considered here, we denote by $\chi_V : W \to \mathbf{C}$ its character (for $\rho\colon W \to \mathrm{Gl}(V)$ the homomorphism defining the representation, we have by definition $\chi_V(w) = \mathrm{trace}(\rho(w)),\ \forall w \in W$). By the theory of characters of finite groups we have

$$M = \langle \chi_{\mathcal{S}}, \chi_{H^1} \rangle, \tag{2.1}$$

where $\langle\ ,\ \rangle$ is the usual scalar product between characters. If N is the number of connected components of the Dynkin diagram Π of G and $h = \dim Z(G)$ we have a decomposition

$$\mathcal{S} = \underbrace{\mathcal{B} \oplus \cdots \oplus \mathcal{B}}_{h} \oplus \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_N,$$

where $\mathcal{B}$ is the 1-dimensional trivial representation and $\mathcal{S}_i$ the irreducible reflection representation corresponding to the ith component of Π, $i = 1, \ldots, N$. Then we may rewrite (2.1) as

$$M = h\langle \chi_{\mathcal{B}}, \chi_{H^1} \rangle + \sum_{i=1}^{N} \langle \chi_{\mathcal{S}_i}, \chi_{H^1} \rangle. \tag{2.2}$$

We observe that W acts trivially on the cohomology groups $H^0(\widetilde{C}, \mathbf{C}) \cong H^2(\widetilde{C}, \mathbf{C}) \cong \mathbf{C}$. Hence the Lefschetz character $\chi_L \equiv \chi_{H^0} - \chi_{H^1} + \chi_{H^2}$ satisfies $\chi_L = 2\chi_{\mathcal{B}} - \chi_{H^1}$ and we have

$$\langle \chi_{\mathcal{B}}, \chi_{H^1} \rangle \ = \ 2 - \langle \chi_{\mathcal{B}}, \chi_L \rangle, \tag{2.3}$$

$$\langle \chi_{\mathcal{S}_i}, \chi_{H^1} \rangle \ = \ -\langle \chi_{\mathcal{S}_i}, \chi_L \rangle. \tag{2.4}$$

On the other hand, it is well known (Hopf trace formula, see e.g.[CR]) that the Lefschetz character satisfies

$$\chi_L = \chi_{\widetilde{C}^0} - \chi_{\widetilde{C}^1} + \chi_{\widetilde{C}^2},$$

$\widetilde{C}^n$ being the free $\mathbf{C}$-module generated by the n-cells of some cellular decomposition of $\widetilde{C}$ ($\widetilde{C}^n \cong H_n(K^n, K^{n-1}; \mathbf{C})$, with K^j the jth skeleton of $\widetilde{C}$, $j = n, n-1$).

We choose one finite triangulation Δ of C whose set of vertices contains all branch points. We denote by C^n the free module generated by the n-cells of Δ for $n = 1, 2$, and by C_0^0 and D_j the free modules whose generators are respectively all vertices not lying in the branch locus Ram and all branch points corresponding to the same W-orbit $R_j \subset R$ (see Remark 1.1.). Let N' be the number of W-orbits in R, and for each $j = 1, \ldots, N'$ let us fix one positive root $\alpha_j \in R_j^+$ and set $H_j = \{1, s_{\alpha_j}\} \subset W$. We denote by $\mathrm{Ind}_{H_j}^W(B_j)$ the W-representation induced by the 1-dimensional trivial representation B_j of H_j (by definition, $\mathrm{Ind}_{H_j}^W(B_j) = \oplus_{[w]\in W/H_j}\mathbf{C}v_{[w]}$ with W acting by $u\circ v_{[w]} = v_{[uw]}$). We have the following isomorphisms of W-modules:

$$\begin{aligned}
\widetilde{C}^2 &\cong \mathbf{C}[W]\otimes C^2,\\
\widetilde{C}^1 &\cong \mathbf{C}[W]\otimes C^1,\\
\widetilde{C}^0 &\cong \mathbf{C}[W]\otimes C^0_0 \oplus \bigoplus_{j=1}^{N'} \mathrm{Ind}_{H_j}^W(B_j)\otimes D_j\\
&\equiv \mathbf{C}[W]\otimes C^0_0 \oplus \bigoplus_{j=1}^{N'} (\mathrm{Ind}_{H_j}^W(B_j))^{n_j},
\end{aligned}$$

where $\mathbf{C}[W]$ denotes as usual the regular representation and the n_j's satisfy (1.4).

By Frobenius reciprocity formula we have

$$\langle \chi_{\mathcal{B}}, \chi_{\mathrm{Ind}_{H_j}^W(B_j)}\rangle = \langle \chi_{B_j}, \chi_{B_j}\rangle = 1;$$

and since from the general theory each irreducible W-representation occurs as a subrepresentation of $\mathbf{C}[W]$ as many times as is its dimension, we obtain

$$\langle \chi_{\mathcal{B}}, \chi_L\rangle = \mathrm{rk}\, C^2 - \mathrm{rk}\, C^1 + \mathrm{rk}\, C^0_0 + \mid Ram \mid = (2-2g). \tag{2.5}$$

Analogously, we have

$$\langle \chi_{\mathcal{S}_i}, \chi_L\rangle = (\mathrm{rk}\, C^2 - \mathrm{rk}\, C^1 + \mathrm{rk}\, C^0_0)\dim \mathcal{S}_i + \sum_{j=1}^{N'} n_j \langle \chi_{B_j}, \chi_{\mathrm{res}_j\, \mathcal{S}_i}\rangle,$$

where $\mathrm{res}_j\mathcal{S}_i$ denotes the representation obtained by restriction to H_j.

Now, given some positive root $\alpha \in R^+$, the corresponding reflection $s_\alpha \in W$ acts trivially on $\mathcal{S}_i$ whenever $\alpha \notin \mathcal{S}_i$, otherwise it acts trivially on one subspace of codimension 1 in $\mathcal{S}_i$. Thus we get

$$\begin{aligned}
\langle \chi_{\mathcal{S}_i}, \chi_L\rangle &= (\mathrm{rk}\, C^2 - \mathrm{rk}\, C^1 + \mathrm{rk}\, C^0_0)\dim \mathcal{S}_i + \sum_{R_j\subset \mathcal{S}_i} n_j(\dim \mathcal{S}_i - 1) +\\
&\qquad + \sum_{R_j\not\subset \mathcal{S}_i} n_j\cdot \dim \mathcal{S}_i\\
&= (2-2g)\dim \mathcal{S}_i - \sum_{R_j\subset \mathcal{S}_i} n_j.
\end{aligned} \tag{2.6}$$

By substituting (2.5) and (2.6) respectively in (2.3) and (2.4) and then (2.3) and (2.4) in (2.2), we finally obtain

$$\begin{aligned}
M &= 2h + (2g-2)\left(h + \sum_{i=1}^{N} \dim \mathcal{S}_i\right) + \sum_{j=1}^{N'} n_j\\
&= 2h + (2g-2)\dim T + \mid Ram \mid .
\end{aligned}$$

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Since $\dim T+ \mid R \mid = \dim G$, by (1.1) we get

$$\dim \mathcal{P} \equiv \tfrac{1}{2}M = (g-1)\dim G + h.$$

3. The main results

In this section we will define a map $\mathcal{F}$ from each connected component of the generic Hitchin fibre to the abelian variety $\mathcal{P}_0$ and study its properties. We first show how one can associate to each given pair $(P, s) \in \mathcal{H}^{-1}(\phi)$ a T-bundle $\mathcal{T} = \mathcal{T}(P, s)$ which satisfies ${}^w\mathcal{T} \cong \mathcal{T}\ \forall w \in W$.

For $\phi \in \mathcal{K}$ generic, let then P be a principal G-bundle and $s \in H^0(C, \mathrm{ad}\, P \otimes K)$ such that $(P, s) \in \mathcal{H}^{-1}(\phi)$. We first consider the restriction P_0 of P to the open set C_0. Since for every $\xi \in C_0$, $s(\xi) \in \mathbf{g}$ is regular semisimple (for an analysis of the regular elements in **g**, see for example [K]), we have a morphism of vector bundles

$$[s,\ ]\colon \mathrm{ad}\, P_0 \to \mathrm{ad}\, P_0 \otimes K$$

whose kernel $\mathcal{N}$ is a bundle of Cartan subalgebras in **g**. We thus have a section

$$\gamma\colon C_0 \to P/N_G(T) \equiv P \times_G G/N_G(T)$$

locally defined by $\gamma(\xi) = \nu(\xi)N_G(T)$ where $\nu(\xi) \in G$ satisfies $\mathrm{Ad}\,\nu(\xi)\mathbf{t} = \mathcal{N}_\xi \equiv c\mathbf{g}(s(\xi))$. If we pull back P_0 over $\widetilde{C}_0$ we actually have a section

$$\varphi\colon \widetilde{C}_0 \to \pi^*P_0/T \tag{3.1}$$

locally defined by $\varphi(\eta) = \mu(\eta)T$ where $\mu(\eta) \in G$ satisfies

$$\mathrm{Ad}\,\mu(\eta)(\iota(\eta)) = s(\pi(\eta)). \tag{3.2}$$

Thus over $\widetilde{C}_0$ the bundle π^*P has a reduction of its structure group to T. Moreover, from (1.2) we have for each $w \in W$

$$\varphi(w\eta) = \mu(\eta)n_w^{-1}T \tag{3.3}$$

which implies that such T-reduction $\tau_0 = \varphi^*(\pi^*P_0)$ is W-invariant with respect to the action previously defined. Now if we consider a Borel subgroup $B \subset G$ containing T, the inclusion map $T \hookrightarrow B$ and φ define a section: $\widetilde{C}_0 \to \pi^*P \times_G G/B$. Since G/B is a complete variety, by the valuative criterion of properness this section can be extended to the whole curve $\widetilde{C}$ and we thus obtain (uniquely up to isomorphisms) a B-reduction P_B of the G-bundle π^*P such that $P_B\mid_{\widetilde{C}_0}$ is the B-extension of τ_0.

If $(\ ,\ )$ denotes a W-invariant scalar product on $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\beta \in R$, we define as usual the one parameter subgroup $\beta' \in \mathrm{Hom}(X(T), \mathbf{Z})$ by

$$\beta'(\lambda) = \langle \lambda, \beta \rangle \equiv \frac{2(\lambda, \beta)}{(\beta, \beta)} \quad \forall \lambda \in X(T). \tag{3.4}$$

We want to prove the following:

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THEOREM 3.1. *Let* $\tau_B = \tau(P, s)$ *be the* T*-bundle associated to* P_B *via the natural projection* $B \to T$*. Let us fix one theta characteristic* $\frac{1}{2}K$ *and consider the* T*-bundle* $K_\rho = \frac{1}{2}\pi^*K \otimes \sum_{\beta\in R^+}\beta'$*, where* $R^+ \subset R$ *is the subset of positive roots that corresponds to* B*. Then* $\mathcal{T}(P, s) := \tau_B + K_\rho$ *is* W*-invariant.*

The proof will be organized in a few lemmas. We first observe that since W is generated by the simple reflections it suffices to show

$$^{s_\alpha}\tau_B \cong \tau_B + \pi^*K \otimes \alpha' \tag{3.5}$$

for every simple root α. In fact we have $\sum_{\beta\in R^+} s_\alpha(\beta') = \sum_{\substack{\beta\in R^+\\ \beta\neq\alpha}} \beta' - \alpha'$, so, if relation (3.5) holds, one has $^{s_\alpha}(\tau_B + K_\rho) \cong \tau_B + K_\rho$. In terms of line bundles associated to characters on T, relation (3.5) can be rewritten as

$$(^{s_\alpha}\tau_B - \tau_B) \times_\lambda \mathbf{C} \cong \langle\lambda, \alpha\rangle \pi^*K \quad \forall\lambda \in X(T). \tag{3.6}$$

Given a simple root α, let us denote by $s_\alpha(B)$ the Borel subgroup $n_\alpha B n_\alpha^{-1}$, where $n_\alpha \in N_G(T)$ represents s_α. One analogously obtains another T-bundle $\tau_{s_\alpha(B)}$ such that $\tau_{s_\alpha(B)}\,|_{\widetilde{C}_0} \cong \tau_0$ from the completion of τ_0 to an $s_\alpha(B)$-reduction $P_{s_\alpha(B)}$. The first lemma treats the relationship between τ_B and $\tau_{s_\alpha(B)}$.

LEMMA 3.2. *We have* $\tau_{s_\alpha(B)} \cong {}^{s_\alpha}\tau_B$.

Proof. We consider an open covering $\{V_h\}_{h\in H}$ of C over which P and the canonical bundle K can be trivialized and with the property that each V_h contains at most one branch point. We choose a Čech covering $\mathcal{U} = \{U_h\}_{h\in H}$ of $\widetilde{C}$ to be given by all open sets $U_h = \pi^{-1}(V_h)$ (by definition each U_h is stable with respect to the action of W). For $h \in H$ we choose frames $e_1^h, \dots, e_q^h$ for the vector bundle ad $P \otimes K$ over $V_h \subset C$, q being equal to the dimension of **g**. With respect to this choice the section $s: C \to adP \otimes K$ is locally given by 'coordinates' $s_h: V_h \to \mathbf{g}$ satisfying

$$s_h = \mathrm{Ad}\, g_{hl} \cdot k_{hl} s_l \quad \text{for } V_h \cap V_l \neq \emptyset, \tag{3.7}$$

g_{hl} and k_{hl} being transition functions for P, K respectively. Let $\iota_h : U_h \to \mathbf{t}$ be coordinates for $\iota: \widetilde{C} \to \mathbf{t} \otimes K$. We define $J \subset H$ to be the subset of those indices j such that V_j contains a branch point and set $I = H \setminus J$. For each $h \in H$ we fix maps $\mu_h: U_h \to G$ such that, for each $i \in I$, μ_i satisfies

$$\mathrm{Ad}\, \mu_i(\eta)(\iota_i(\eta)) = s_i(\pi(\eta)) \tag{3.8}$$

(compare with (3.2)) and the 0-chain $\{\mu_h(\eta)B\}_{h\in H}$ defines the section $\widehat{\varphi}_B: \widetilde{C} \to \pi^*P/B$ completing φ in (3.1). By definition, the B-bundle P_B is represented by the cocycle $\{b_{hl}\} \in \mathcal{Z}^1(\mathcal{U}, B)$ where $b_{hl}(\eta) \equiv \mu_h(\eta)^{-1} g_{hl}(\pi(\eta))\mu_l(\eta)$. Define $\{b'_{hl}\} \in \mathcal{Z}^1(\mathcal{U}, s_\alpha(B))$ by $b'_{hl}(\eta) = n_\alpha b_{hl}(s_\alpha\eta)n_\alpha^{-1}$ $\forall\eta \in U_h \cap U_l$. We have $b'_{hl}(\eta) \equiv n_\alpha\mu_h(s_\alpha\eta)^{-1} g_{hl}(\pi(\eta))\mu_l(s_\alpha\eta)n_\alpha^{-1}$, hence $\{b'_{hl}\}$ represents an $s_\alpha(B)$-reduction of π^*P. On the other hand, from (3.3) we have $\{\mu_i(s_\alpha\eta)n_\alpha^{-1}T\}_{i\in I} =$

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$\{\mu_i(\eta)T\}_{i\in I}$ hence $\{b'_{hl}\}$ represents $P_{s_\alpha(B)}$. Now, if we denote by $p\colon B \to T, p'\colon s_\alpha(B) \to T$ the natural projections we have $p' \circ b'_{hl}(\eta) = n_\alpha(p \circ b_{hl}(s_\alpha\eta))n_\alpha^{-1}$ (since every Borel subgroup is a semidirect product of its maximal torus and its maximal unipotent subgroup). Since $\{n_\alpha(p \circ b_{hl}(s_\alpha\eta))n_\alpha^{-1}\}$ are by definition transition functions for ${}^{s_\alpha}\tau_B$, we thus have an isomorphism $\tau_{s_\alpha(B)} \cong {}^{s_\alpha}\tau_B$. □

We keep the notations of the proof of Lemma 3.2. For each positive root $\beta \in R^+$, we shall denote by $\beta_h : U_h \to \mathbf{C}$ the coordinates of the section of $\pi^* K$ over $\widetilde{C}$ given by the composition $d\beta \circ \iota$ (see Section 1). Our next step consists in finding suitable transition functions b_{ji} for P_B on intersections $U_i \cap U_j$ with $j \in J$. Indeed, we will find suitable maps $\mu_j : U_j \to G$ with $j \in J$ defining the completed section $\widehat{\varphi}_B$. We fix nilpotent generators $\{X_\gamma\}_{\gamma\in R^+}$ in the Lie algebra $\mathbf{b}$ of B with $ad\, t(X_\gamma) = d\gamma(t)X_\gamma,\ \forall t \in \mathbf{t}, \forall\gamma \in R^+$. In general, the completion $\widehat{\varphi}_B : \widetilde{C} \to \pi^* P/B$ of our φ above is locally given by holomorphic maps $f_j : U_j \to G$ with $j \in J$ such that

$$\mathrm{Ad}\, f_j(\eta)^{-1} s_j(\pi(\eta)) = \iota_j(\eta) + \sum_{\gamma\in R^+} a_\gamma(\eta) X_\gamma. \tag{3.9}$$

By Remark 1.1, for $j \in J$ the set U_j is a union of open sets $\bigcup_{\beta\in R(j)\cap R^+} U_{j,\beta}$ where $R(j)$ is some W-orbit of roots depending on j and each $U_{j,\beta}$ contains only those ramification points that are zeroes for β_j.

LEMMA 3.3. *There exists a holomorphic map $\mu_j : U_j \to G$ satisfying for each $\beta \in R(j) \cap R^+$ and $\eta \in U_{j,\beta}$*

$$\mathrm{Ad}\, \mu_j(\eta)^{-1} s_j(\pi(\eta)) = \iota_j(\eta) + X_\beta. \tag{3.10}$$

Proof. We construct μ_j separately on each connected component of U_j. By our genericity hypothesis we may assume for every ramification point $p \in U_{j,\beta}$

$$\mathrm{Ad}\, f_j(p)^{-1} s_j(\pi(p)) = \iota_j(p) + X_\beta \tag{3.11}$$

with $\beta_j(p) \equiv d\beta(\iota_j(p)) = 0$.

Let α be the root with minimal height in $R^+ \setminus \{\beta\}$ such that $a_\alpha(\eta)$ in (3.9) is not identically zero. The G-valued map $c_j(\eta) = \exp \frac{a_\alpha(\eta)}{\alpha_j(\eta)} X_\alpha$ is holomorphic on each fixed connected component of $U_{j,\beta}$ and by evaluating $\mathrm{Ad}\, c_j(\eta)$ on the right-hand side of (3.9) we get

$$\mathrm{Ad}\, c_j(\eta)(\iota_j(\eta) + \sum_{\gamma\in R^+} a_\gamma(\eta)X_\gamma) = \iota_j(\eta) + a'_\beta(\eta)X_\beta + \sum_{\substack{\gamma\in R^+\setminus\{\beta\}\\ \gamma>\alpha}} a_\gamma(\eta)X_\gamma.$$

By an induction argument we can then assume

$$\mathrm{Ad}\, f_j(\eta)^{-1} s_j(\pi(\eta)) = \iota_j(\eta) + a_\beta(\eta)X_\beta, \tag{3.12}$$

where $a_\beta(p) = 1$ (since we may multiply f_j by a suitable constant in T). Consider now the map $d_j(\eta) = \exp \frac{a_\beta(\eta)-1}{\beta_j(\eta)} X_\beta$. Since p is a simple zero for β_j, d_j is holomorphic on the connected component of $U_{j,\beta}$ containing p. We have

$$\operatorname{Ad} d_j(\eta)(\iota_j(\eta) + a_\beta(\eta)X_\beta) = \iota_j(\eta) + X_\beta$$

and the claim of our lemma is proved. □

For each $j \in J$, define $u_j : U_j \to B$ by $u_j(\eta) = \exp \frac{X_\beta}{\beta_j(\eta)}$ whenever $\eta \in U_{j,\beta}$. We have

$$\operatorname{Ad} u_j(\eta)^{-1}\iota_j(\eta) = \iota_j(\eta) + X_\beta. \tag{3.13}$$

We may represent the completed section $\widehat{\varphi}_B$ by $\{\mu_h(\eta)B\}$ where the μ_i 's are as in (3.8) for every $i \in I$ and the μ_j 's satisfy (3.10) for every $j \in J$. By substituting (3.8) and (3.10) in (3.7) and replacing $\iota_j(\eta) + X_\beta$ with $\operatorname{Ad} u_j(\eta)^{-1}\iota_j(\eta)$ we obtain transition functions on each nonempty intersection $U_j \cap U_i$

$$b_{ji}(\eta) \equiv \mu_j(\eta)^{-1} g_{ji}(\pi(\eta))\mu_i(\eta) = u_j^{-1}(\eta)t_{ji}(\eta), \tag{3.14}$$

where $t_{ji} : U_i \cap U_j \to T$ is holomorphic (as u_j is holomorphic on $U_i \cap U_j$). Since each element in B can be written uniquely as a product of a unipotent element by an element in T we have $t_{ji} = p \circ b_{ji}$.

We now compare P_B with $P_{s_\alpha(B)}$. By definition we only need to compare them around the ramification points. As set of nilpotent generators in the Lie algebra of $s_\alpha(B)$ we may choose $\{X_\beta\}_{\beta\in R^+\setminus\{\alpha\}} \cup \{\operatorname{Ad} n_\alpha(X_\alpha)\}$. Thus from Lemma 3.3 we may define a section $\widehat{\varphi}_{s_\alpha(B)} : \widetilde{C} \to \pi^*P/s_\alpha(B)$ completing φ by

$$\widehat{\varphi}_{s_\alpha(B)}(\eta) = \mu_j(\eta)s_\alpha(B) \quad \text{for } \eta \in U_j \setminus U_{j,\alpha},$$

$$\widehat{\varphi}_{s_\alpha(B)}(\eta) = \mu_j(s_\alpha\eta)n_\alpha^{-1}s_\alpha(B) \quad \text{for } \eta \in U_{j,\alpha},$$

where the G-valued maps μ_j satisfy (3.10). From this we see that $P_{s_\alpha(B)}$ and P_B are isomorphic on $\widetilde{C} \setminus \mathcal{D}_\alpha$ and that on all intersection sets $U_{j,\alpha} \cap U_i$ with $j \in J$ we have transition functions for $P_{s_\alpha(B)}$ of the form

$$b'_{ji}(\eta) = n_\alpha\mu_j(s_\alpha\eta)^{-1}\mu_j(\eta)b_{ji}(\eta). \tag{3.15}$$

If we apply Lemma 3.3 to the set $s_\alpha(R^+)$ of positive roots corresponding to $s_\alpha(B)$ we obtain on $U_{j,\alpha} \cap U_i$ a factorization $b'_{ji}(\eta) = {u'_j}^{-1}(\eta)t'_{ji}(\eta)$ with $u'_j(\eta) = \exp \frac{\operatorname{Ad} n_\alpha(X_\alpha)}{-\alpha_j(\eta)} = n_\alpha {u_j}^{-1}(\eta)n_\alpha^{-1}$ and $t'_{ji}(\eta) = p' \circ b'_{ji}(\eta)$ (compare with (3.14)). Let us denote by I the identity element in G. From (3.15) and Lemma 3.2 a meromorphic section of ${}^{s_\alpha}\tau_B - \tau_B$ is given by a 0-cochain $\{t_h\}_{h\in H} \in \mathcal{C}^0(\mathcal{U}, T)$ where

$$t_h(\eta) = I \quad \text{whenever } h \in I \quad \text{or } h \in J \quad \text{and} \quad \eta \notin U_{j,\alpha}, \tag{3.16}$$

$$t_j(\eta) = n_\alpha u_j(\eta)^{-1}\mu_j(s_\alpha\eta)^{-1}\mu_j(\eta)u_j(\eta)^{-1} \quad \forall \eta \in U_{j,\alpha}, \quad j \in J. \tag{3.17}$$

By (3.10) on each $U_{j,\alpha}$ the map $h_j(\eta) = \mu_j(s_\alpha\eta)^{-1}\mu_j(\eta)$ satisfies

$$\mathrm{Ad}\, h_j(\eta)(\iota_j(\eta) + X_\alpha) = \iota_j(s_\alpha\eta) + X_\alpha = \mathrm{Ad}\, n_\alpha(\iota_j(\eta)) + X_\alpha. \tag{3.18}$$

Choose $X_{-\alpha} \in \mathbf{g}$ so that $X_\alpha, X_{-\alpha}, h_\alpha := [X_\alpha, X_{-\alpha}] \in \mathbf{t}$ generate a Lie subalgebra $\mathbf{h}_\alpha \subset \mathbf{g}$ with $\mathbf{h}_\alpha \cong sl(2)$ and $d\alpha(h_\alpha) = 2$. Define

$$F_j(\eta) = \exp(\alpha_j(\eta)X_{-\alpha}) \quad \forall \eta \in U_{j,\alpha}.$$

Since $F_j(\eta)$ satisfies $\mathrm{Ad}\, F_j(\eta)(\iota_j(\eta) + X_\alpha) = \mathrm{Ad}\, n_\alpha(\iota_j(\eta)) + X_\alpha$, by (3.18) we have on $U_{j,\alpha}$

$$\mu_j(s_\alpha\eta)^{-1}\mu_j(\eta) = F_j(\eta) \cdot L_j(\eta), \tag{3.19}$$

where for each $\eta \in U_{j,\alpha}$, $L_j(\eta) \in B$ lies in the centralizer of $\iota_j(\eta) + X_\alpha \in \mathbf{b}$. Note that for q any ramification point in $U_{j,\alpha}$ we have by definition

$$L_j(q) = \mathrm{I}. \tag{3.20}$$

In particular the map L_j is holomorphic. Since when $\eta \in U_{j,\alpha}$ is not a ramification point $\iota_j(\eta) + X_\alpha$ is regular semisimple and by (3.13) one has $c\mathbf{g}(\iota_j(\eta) + X_\alpha) = \mathrm{Ad}\, u_j(\eta)^{-1}\mathbf{t}$, the holomorphic T-valued map $l_j(\eta) = p \circ L_j(\eta)$ has the form

$$l_j(\eta) = u_j(\eta)L_j(\eta)u_j(\eta)^{-1}. \tag{3.21}$$

Relation (3.17) becomes

$$t_j(\eta) = z_j(\eta) \cdot l_j(\eta), \tag{3.22}$$

where the map $z_j(\eta) \equiv n_\alpha u_j(\eta)^{-1}F_j(\eta)u_j(\eta)^{-1}$ has values in T and is holomorphic everywhere in $U_{j,\alpha}$ but on the ramification points. The connected subgroup $H_\alpha \subset G$ generated by $\exp(X_\alpha), \exp(X_{-\alpha}), \exp(h_\alpha)$ is isomorphic to a copy of $\mathrm{Sl}(2)$ or $\mathrm{PGl}(2)$ in G and one can compute $z_j(\eta)$ directly in terms of two by two matrices. In the $\mathrm{Sl}(2)$ case, denoting by ϱ the isomorphism: $H_\alpha \to \mathrm{Sl}(2)$, one has for some $c \in \mathbf{C}^*$

$$\begin{aligned}\varrho(z_j(\eta)) &\\ &= \mp \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -c/\alpha_j(\eta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_j(\eta)/c & 1 \end{pmatrix} \begin{pmatrix} 1 & -c/\alpha_j(\eta) \\ 0 & 1 \end{pmatrix} \\ &= \pm\, \mathrm{diag}(c^{-1}\alpha_j(\eta), c\alpha_j(\eta)^{-1}),\end{aligned} \tag{3.23}$$

where $\alpha_j(\eta)$ are the coordinates of the section $d\alpha \circ \iota$, according to our previous notations. As for $H_\alpha \overset{\varrho}{\cong} \mathrm{PGl}(2)$ one gets

$$\varrho(z_j(\eta)) = \overline{\mathrm{diag}(c^{-1}\alpha_j(\eta), c\alpha_j(\eta)^{-1})}, \tag{3.24}$$

where the bar indicates the image under the factor map: $\mathrm{Gl}(2) \to \mathrm{PGl}(2)$. Let now $T_\alpha \subset T$ be the identity component of the subgroup $\mathrm{Ker}(\alpha) = \{t \in T \mid \alpha(t) = 1\}$. The centralizer Z_α in G of T_α is a reductive group of semisimple rank 1 having Lie algebra $\mathbf{z} = \mathbf{t} \oplus \mathbf{C}X_\alpha \oplus \mathbf{C}X_{-\alpha}$ and it is known that such a group is a product $T' \times H$, T' being a torus and H being a copy of $\mathrm{Sl}(2)$, $\mathrm{PGl}(2)$ or $\mathrm{Gl}(2)$. The case $H = \mathrm{Sl}(2)$ is characterized by the group of characters $X(T)$ being an orthogonal direct sum $\mathbf{Z}\chi_1 \oplus X'$, with $\chi_1 = \sqrt{\alpha}$. If we compose any $\lambda \in X'$ with the 0-chain $\{t_h\}_{h \in H}$ defined by (3.16) and (3.17) we obtain a nowhere vanishing holomorphic section of the line bundle $({}^{s_\alpha}\tau_B - \tau_B) \times_\lambda \mathbf{C}$. If instead we compose χ_1 to $\{t_h\}_{h\in H}$, by (3.22) and (3.23) we get a holomorphic section of $({}^{s_\alpha}\tau_B - \tau_B) \times_{\chi_1} \mathbf{C}$ having simple zeroes exactly on the locus $\mathcal{D}_\alpha$. Thus relation (3.6) is satisfied (see Remark 1.1).

The case $H = \mathrm{PGl}(2)$ is characterized by $X(T)$ being an orthogonal direct sum $\mathbf{Z}\alpha \oplus X'$. For $\lambda \in X'$, we get the same result as for the $\mathrm{Sl}(2)$ case. For $\lambda = \alpha$ we find instead a holomorphic section of $({}^{s_\alpha}\tau_B - \tau_B) \times_\lambda \mathbf{C}$ having zeroes of multiplicity two on $\mathcal{D}_\alpha$. This proves (3.6).

In case $H = \mathrm{Gl}(2)$, we have an orthogonal direct sum $X(T) = X' \oplus \mathbf{Z}\chi_1 \oplus \mathbf{Z}\chi_2$ with $\alpha = \chi_1 \cdot \chi_2^{-1}$. Composing $\lambda \in X'$ gives us again ${}^{s_\alpha}\tau_B \times_\lambda \mathbf{C} \cong \tau_B \times_\lambda \mathbf{C}$ as in the previous cases. If we compose χ_1 we obtain a holomorphic section of $({}^{s_\alpha}\tau_B - \tau_B) \times_{\chi_1} \mathbf{C}$ having simple zeroes exactly on $\mathcal{D}_\alpha$. If we compose χ_2 we obtain a meromorphic section of $({}^{s_\alpha}\tau_B - \tau_B) \times_{\chi_2} \mathbf{C}$ having simple poles exactly on $\mathcal{D}_\alpha$. Thus relation (3.6) holds also in this case and Theorem 3.1 is proved. □

We thus have a map

$$\begin{aligned} \mathcal{T}\colon \mathcal{H}^{-1}(\phi) &\to \widehat{\mathcal{P}} \equiv [\mathrm{Pic}(\widetilde{C}) \otimes X(T)^*]^W, \\ (P, s) &\longmapsto \tau(P, s) + K_\rho. \end{aligned}$$

Note that from (3.5) and Lemma 3.2 $\mathcal{T}$ does not depend on the choice of the Borel subgroup $B \supset T$ (or of the subset of positive roots in R).

DEFINITION 3.4. Let $\mathcal{H}^{-1}(\phi)_c$ be some connected component of $\mathcal{H}^{-1}(\phi)$. For a fixed point $(P', s') \in \mathcal{H}^{-1}(\phi)_c$ we define $\mathcal{F}\colon \mathcal{H}^{-1}(\phi)_c \to \mathcal{P}_0$ by

$$\mathcal{F}(P, s) = \mathcal{T}(P, s) - \mathcal{T}(P', s') \equiv \tau(P, s) - \tau(P', s').$$

Such definition does not depend on our previous choice of the theta characteristic $\frac{1}{2}K$. We now want to study the fibers of $\mathcal{T}$. First we make the following

Remark 3.1. For $i \in I$, the maps $\mu_i(\eta)$ in (3.8) are defined up to multiplication to the right by some holomorphic map $m_i : U_i \to T$. As for $j \in J$, any other holomorphic map $\mu'_j(\eta)$ satisfying (3.10) has the form $\mu'_j(\eta) = \mu_j(\eta) M_j(\eta)$ where, for every $\alpha \in R(j) \cap R^+$, $M_j : U_{j,\alpha} \to B$ is holomorphic and such that $M_j(\eta) \in c_G(\iota_j(\eta) + X_\alpha)$. If we replace μ_j and μ_i with the new maps $\mu'_j(\eta)$ and $\mu'_i(\eta) = \mu_i(\eta) m_i(\eta)$, we obtain from (P, s) and B an equivalent cocycle $\{m_h^{-1} t_{hi} m_i\}$

representing τ_B. Since, for every $j \in J$ and $q \in U_j \cap \mathcal{D}_\alpha$, $\iota_j(q) + X_\alpha \in \mathbf{b}$ is regular, we have $c_G(\iota_j(q) + X_\alpha) = T_\alpha \mathcal{U}_\alpha$, where T_α is the identity component of $\mathrm{Ker}(\alpha\colon T \to \mathbf{C}^*)$ and $\mathcal{U}_\alpha$ is the unipotent 1-dimensional subgroup corresponding to the root α. Hence the T-valued map $m_j(\eta) := p \circ M_j(\eta) \equiv u_j(\eta) M_j(\eta) u_j(\eta)^{-1}$ satisfies for every $\alpha \in R(j) \cap R^+$

$$\alpha(m_j(q)) = 1 \quad \forall q \in U_j \cap \mathcal{D}_\alpha. \tag{3.25}$$

LEMMA 3.5. *Let $(P, s), (Q, v)$ be pairs in $\mathcal{H}^{-1}(\phi)$ such that $\tau(P, s)$ and $\tau(Q, v)$ are isomorphic. Let $\{t_{hl}\}$ and $\{\widetilde{t}_{hl}\}$ with $h, l \in H$ be cocycles representing $\tau(P, s)$ and $\tau(Q, v)$ respectively and suppose*

$$\widetilde{t}_{hl} = m_h^{-1} t_{hl} m_l, \tag{3.26}$$

where the maps $m_h : U_h \to T$ are holomorphic and satisfy condition (3.25) *for every $j \in J$ and $\alpha \in R(j) \cap R^+$. Then Q is isomorphic to P and $v = s$.*

Proof. For what concerns P and the construction of $\tau(P, s)$ we keep the notations used in the proof of Theorem 3.1. In particular we still consider a Cěch covering $\mathcal{U} = \{U_h\}_{h\in H}$ of $\widetilde{C}$ consisting of W-invariant open sets as it was first defined in the proof of Lemma 3.2. For each nonempty intersection $U_h \cap U_l$ we have transition functions for the B-reduction Q_B of $\pi^* Q$ having the form:

$$\widetilde{b}_{ji}(\eta) = \widetilde{\mu}_j(\eta)^{-1} \widetilde{g}_{ji}(\pi(\eta)) \widetilde{\mu}_i(\eta) = u_j(\eta)^{-1} \widetilde{t}_{ji}(\eta) \quad \forall j \in J, i \in I, \tag{3.27}$$

$$\widetilde{b}_{hi}(\eta) = \widetilde{\mu}_h(\eta)^{-1} \widetilde{g}_{hi}(\pi(\eta)) \widetilde{\mu}_i(\eta) = \widetilde{t}_{hi}(\eta) \quad \forall i, h \in I, \tag{3.28}$$

where $\{\widetilde{g}_{hl}\}_{h,l\in H}$ are transition functions for the G-bundle Q and $\widetilde{\mu}_i$, $\widetilde{\mu}_j$ are defined analogously as μ_i and μ_j in (3.14). For $j \in J$, define $M_j : U_j \to B$ by

$$M_j := u_j^{-1} m_j u_j \quad \text{(see Remark 3.1).} \tag{3.29}$$

The hypothesis of the lemma provide that M_j is holomorphic on $U_{j,\alpha}$ for each $\alpha \in R(j) \cap R^+$ and we have $M_j(\eta) \in c_G(\iota_j(\eta) + X_\alpha)$ $\forall \eta \in U_{j,\alpha}$ by definition of u_j. Define the holomorphic maps

$$\Gamma_i = \mu_i m_i \widetilde{\mu}_i^{-1} \quad \forall i \in I \quad \text{and}$$
$$\Gamma_j = \mu_j M_j \widetilde{\mu}_j^{-1} \quad \forall j \in J.$$

From (3.27), (3.14) and (3.26) we obtain the equivalence condition between cocycles on $\widetilde{C}$:

$$\widetilde{g}_{hl}(\pi(\eta)) = \Gamma_h(\eta)^{-1} g_{hl}(\pi(\eta)) \Gamma_l(\eta) \quad \forall \eta \in U_h \cap U_l \quad \forall h, l \in H.$$

The claim of the lemma is then proved provided we show that the maps Γ_l are invariant with respect to the action of W on the sheets of $\widetilde{C}$. In fact if we indicate by $\{v_h\}_{h\in H}$ the coordinates of v so that $v_h = \mathrm{Ad}\, \widetilde{g}_{hl} \cdot k_{hl} v_l$, by our definition of the maps $\widetilde{\mu}_l$, $\widetilde{\mu}_h$ we have:

$$\mathrm{Ad}\, \Gamma_l v_l = s_l \quad \forall l \in H.$$

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Since W is generated by the simple reflections, it suffices to show $\Gamma_l(s_\alpha\eta) = \Gamma_l(\eta)$ for every simple reflection s_α. From (3.3) we have for each $i \in I$

$$\mu_i(s_\alpha\eta)^{-1}\mu_i(\eta) = n_\alpha l_i(\eta) \tag{3.30}$$

for suitable holomorphic maps $l_i\colon U_i \to T$. By evaluating the transition functions $t_{hi} = \mu_h^{-1}g_{hi}\mu_i$ with $h, i \in I$ on $s_\alpha\eta$ and replacing $\mu_i(s_\alpha\eta)$ with $\mu_i(\eta)l_i(\eta)^{-1}n_\alpha^{-1}$ and $\mu_h(s_\alpha\eta)$ with $\mu_h(\eta)l_h(\eta)^{-1}n_\alpha^{-1}$ we obtain

$$t_{hi}(s_\alpha\eta) = n_\alpha l_h(\eta)t_{hi}(\eta)l_i(\eta)^{-1}n_\alpha^{-1}. \tag{3.31}$$

Analogously, if we define $\widetilde{l}_i\colon U_i \to T$ by

$$\widetilde{\mu}_i(s_\alpha\eta)^{-1}\widetilde{\mu}_i(\eta) = n_\alpha\widetilde{l}_i(\eta), \tag{3.32}$$

we have

$$\widetilde{t}_{hi}(s_\alpha\eta) = n_\alpha\widetilde{l}_h(\eta)\widetilde{t}_{hi}(\eta)\widetilde{l}_i(\eta)^{-1}n_\alpha^{-1}. \tag{3.33}$$

By replacing $\widetilde{t}_{hi}$ with $m_h^{-1}t_{hi}m_i$ in both sides of (3.33) and substituting (3.31) in the left-hand side, we obtain an equality both sides of which contain only factors with values in T. We cancel $t_{hi}(\eta)$ and obtain

$$\begin{aligned} &m_h(\eta)\cdot n_\alpha^{-1}m_h(s_\alpha\eta)^{-1}n_\alpha\cdot\widetilde{l}_h(\eta)^{-1}\cdot l_h(\eta)\\ &\quad= m_i(\eta)\cdot n_\alpha^{-1}m_i(s_\alpha\eta)^{-1}n_\alpha\cdot\widetilde{l}_i(\eta)^{-1}\cdot l_i(\eta)\end{aligned}$$

for every $\eta \in U_h \cap U_i$, $i, h \in I$. We can repeat the same calculation on intersection sets $U_i \cap U_j$ with $j \in J$ and $i \in I$. What we need is the analog for $j \in J$ of the relations (3.30) and (3.32). On each open set $U_{j,\alpha}$ the map $\mu_j(\eta)$ is related with $\mu_j(s_\alpha\eta)$ via the identity (3.19). If for each $\beta \in R^+ \setminus \{\alpha\}$ we define $n_{\alpha\beta} \in N(T)$ to be the representative of s_α satisfying $\operatorname{Ad} n_{\alpha,\beta}(X_\beta) = X_{s_\alpha(\beta)}$, by construction of the maps μ_j in Lemma (3.3) we have for $\eta \in U_{j,\beta}$

$$\mu_j(s_\alpha\eta)^{-1}\mu_j(\eta) = n_{\alpha,\beta}L_j(\eta), \tag{3.34}$$

where $L_j(\eta)$ is a suitable element in the centralizer of $\iota_j(\eta) + X_\beta$. We analogously define $\widetilde{L}_j\colon U_j \to B\ \forall j \in J$ by

$$\widetilde{\mu}_j(s_\alpha\eta)^{-1}\widetilde{\mu}_j(\eta) = F_j(\eta)\widetilde{L}_j(\eta) \quad \text{for } \eta \in U_{j,\alpha}, \tag{3.35}$$

$$\widetilde{\mu}_j(s_\alpha\eta)^{-1}\widetilde{\mu}_j(\eta) = n_{\alpha,\beta}\widetilde{L}_j(\eta) \quad \text{for } \eta \in U_{j,\beta} \quad \text{with } \beta \neq \alpha \tag{3.36}$$

and set for each $\eta \in U_j$

$$l_j(\eta) := p \circ L_j(\eta) = u_j(\eta)L_j(\eta)u_j(\eta)^{-1}, \tag{3.37}$$

$$\widetilde{l}_j(\eta) := p \circ \widetilde{L}_j(\eta) = u_j(\eta)\widetilde{L}_j(\eta)u_j(\eta)^{-1}. \tag{3.38}$$

One uses (3.19), (3.35) and the fact that the map $z_j(\eta) = n_\alpha u_j^{-1}(\eta) F_j(\eta) u_j^{-1}(\eta)$ (see (3.22)) is holomorphic T-valued outside the ramification points (hence it commutes with any other map with values in T), to obtain by the same procedure described above for all pairs of indices $h, i \in I$

$$\begin{aligned} &m_j(\eta) \cdot n_\alpha^{-1} m_j(s_\alpha \eta)^{-1} n_\alpha \cdot \widetilde{l}_j(\eta)^{-1} \cdot l_j(\eta) \\ &\quad = m_i(\eta) \cdot n_\alpha^{-1} m_i(s_\alpha \eta)^{-1} n_\alpha \cdot \widetilde{l}_i(\eta)^{-1} \cdot l_i(\eta) \end{aligned}$$

for each $\eta \in U_{j,\alpha} \cap U_i$. One uses (3.34) and (3.36) to prove the same identity for all $\eta \in U_{j,\beta} \cap U_i$ with $\beta \neq \alpha$. In conclusion, the maps $m_h(\eta) \cdot n_\alpha^{-1} m_h(s_\alpha \eta)^{-1} n_\alpha \cdot \widetilde{l}_h(\eta)^{-1} \cdot l_h(\eta) : U_h \to T$ with $h \in H$ are the restriction to U_h of a global holomorphic map on $\widetilde{C}$, hence are equal to some constant **c**. We compute such map on one ramification point $q \in U_{j,\alpha}$. Since we have $l_j(q) = \widetilde{l}_j(q) =$ I (compare with (3.20)) and $\alpha(m_j(q)) = 1$ by hypothesis, we obtain **c** = I, i.e.

$$m_h(s_\alpha \eta) = n_\alpha m_h(\eta) \cdot l_h(\eta) \cdot \widetilde{l}_h(\eta)^{-1} n_\alpha^{-1} \quad \forall h \in H. \tag{3.39}$$

By use of (3.30), (3.32) and this last identity we find $\Gamma_i(s_\alpha \eta) = \Gamma_i(\eta)$ for each $\eta \in U_i$, $i \in I$. As for $j \in J$, if η is in $U_{j,\alpha}$ we have by (3.19) and (3.35), by the definition of M_j, l_j and $\widetilde{l}_j$ and by (3.39)

$$\begin{aligned} &\Gamma_j(s_\alpha \eta) \\ &\quad = \mu_i(\eta) u_j(\eta)^{-1} l_j(\eta)^{-1} z_j(\eta)^{-1} m_j(\eta) l_j(\eta) \widetilde{l}_j(\eta)^{-1} z_j(\eta) \widetilde{l}_j(\eta) u_j(\eta) \widetilde{\mu}_j(\eta)^{-1} \\ &\quad = \Gamma_j(\eta). \end{aligned}$$

If η is in $U_{j,\beta}$, one proves $\Gamma_j(s_\alpha \eta) = \Gamma_j(\eta)$ by using (3.34), (3.36), (3.39) and the identity (following from the above definition of $n_{\alpha,\beta}$) $n_{\alpha,\beta} u_j(s_\alpha \eta) n_{\alpha,\beta}^{-1} = u_j(\eta)$. □

LEMMA 3.6. *Let $(P, s), (Q, v)$ be pairs in $\mathcal{H}^{-1}(\phi)$ such that $\tau(P, s)$ and $\tau(Q, v)$ are isomorphic. Let $\{t_{hl}\}$ and $\{\widetilde{t}_{hl}\}$ with $h, l \in H$ be cocycles representing $\tau(P, s)$ and $\tau(Q, v)$ respectively and write*

$$\widetilde{t}_{hl} = m_h^{-1} t_{hl} m_l \tag{3.40}$$

for suitable holomorphic maps $m_h : U_h \to T$ with $h \in H$. Up to multiplying each m_h by one and the same suitably chosen element in T, the following holds:

(i) *for each positive root $\alpha \in R^+$ and $q \in U_j \cap \mathcal{D}_\alpha$ we have $\alpha(m_j(q)) = \mp 1$.*
(ii) *if for $\alpha \in R^+$ there exists some character $\lambda \in X(T)$ such that*

$$\langle \lambda, \alpha \rangle = 1, \tag{3.41}$$

we have $\alpha(m_j(q)) = 1 \; \forall q \in U_j \cap \mathcal{D}_\alpha$.

Proof. Choose one ramification point $q_\alpha \in \mathcal{D}_\alpha$ for each $\alpha \in \Delta$, $q_\alpha \in U_{j(\alpha)}$ for suitable $j(\alpha) \in J$. Up to multiplying the maps $\{m_h\}_{h\in H}$ by a suitable element in T we may assume

$$\alpha(m_{j(\alpha)}(q_\alpha)) = 1 \quad \forall \alpha \in \Delta. \tag{3.42}$$

We keep the same notation as before. We consider the maps $\{l_h\}$ and $\{\widetilde{l}_h\}$, $h \in H$ as in (3.30), (3.32), (3.37) and (3.38) and let α be some simple root. From the proof of Lemma (3.5) one has that the maps $m_h(\eta) \cdot n_\alpha^{-1} m_h(s_\alpha \eta)^{-1} n_\alpha \cdot \widetilde{l}_h(\eta)^{-1} \cdot l_h(\eta)\colon U_h \to T$ are the restriction of a global holomorphic map on $\widetilde{C}$. Computing such map on q_α gives us by (3.42) and the fact that we have $l_j(q) = \widetilde{l}_j(q) = \mathrm{I}\ \forall q \in \mathcal{D}_\alpha \cap U_j$

$$m_j(q) \cdot n_\alpha^{-1} m_j(s_\alpha q)^{-1} n_\alpha \cdot \widetilde{l}_j(q)^{-1} \cdot l_j(q) = \mathrm{I} \quad \forall q \in \mathcal{D} \cap U_j,\ j \in J \tag{3.43}$$

and

$$m_j(q) = n_\alpha^{-1} m_j(q) n_\alpha \quad \forall q \in \mathcal{D}_\alpha \cap U_j, \quad j \in J.$$

By evaluating $\alpha\colon T \to \mathbf{C}^*$ on both sides of this last identity we obtain

$$\alpha^2(m_j(q)) = 1.$$

If moreover α satisfies condition (3.41), evaluating λ on both sides of the same identity gives $\lambda(m_j(q)) = \lambda(m_j(q)) \cdot \alpha^{-1}(m_j(q))$, or

$$\alpha(m_j(q)) = 1.$$

The claim of the theorem is thus proved for every simple root. Consider now $q \in \mathcal{D}_\beta$ with $\beta \in R^+ \setminus \Delta$. Note that for $q \in U_j$, from the definition of l_j and $\widetilde{l}_j$ and the fact that $L_j(q)$ and $\widetilde{L}_j(q)$ belong to the centralizer in G of $\iota_j(q) + X_\beta$ we have

$$\beta(l_j(q)) = \beta(\widetilde{l}_j(q)) = 1 \tag{3.44}$$

(compare with (3.25) in Remark 3.1). By evaluating $\beta\colon T \to \mathbf{C}^*$ on both sides of (3.43) as α runs over all simple roots we obtain $\beta(m_j(q)) = \beta(n_\alpha^{-1} m_j(s_\alpha q) n_\alpha)$ $\forall \alpha \in \Delta$, hence

$$\beta(m_j(q)) = \beta(n_w^{-1} m_j(wq) n_w) \quad \forall w \in W.$$

On the other hand, we know that there exist $\alpha \in \Delta$ and $u \in W$ with $u(\alpha) = \beta$. We thus have

$$\beta(m_j(q)) = \beta(n_u m_j(u^{-1}q) n_u^{-1}) = \alpha(m_j(u^{-1}q)) = \mp 1. \qquad \square$$

THEOREM 3.7. *Suppose G has one of the following properties:*

(a) *the commutator group (G, G) is simply connected;*
(b) *the Dynkin diagram of G has no component of type $B_l, l \geqslant 1$.*

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Then the map $\mathcal{T}\colon \mathcal{H}^{-1}(\phi) \to \widehat{\mathcal{P}}$ *is injective.*

Proof. In case (G, G) is simply connected the fundamental weights are elements in $X(T)$; in particular condition (3.41) in Lemma 3.6 is satisfied for every root $\alpha \in R^+$ and our claim follows from Lemma 3.5. As for the case G satisfies condition (b), we see from the Dynkin diagram of all simple groups of type different from B_l, $l \geqslant 1$ and G_2 that for every $\alpha \in R^+$ there exists another root β with $\langle \beta, \alpha \rangle = 1$. On the other hand the type G_2 is simply connected. □

THEOREM 3.8. *Let* $a \geqslant 1$ *be the cardinality of the subset* $A \subset R^+$ *of those roots which do not satisfy condition* (3.41) *in Lemma* 3.6. *If* d *denotes the degree of* $\pi^* K$, *the fibre of* $\mathcal{T}$ *consists of at most* $2^{a(d-1)}$ *points.*

Proof. Let $(P, s) \in \mathcal{H}^{-1}(\phi)$, $\tau(P, s)$ be as in Theorem 3.1 and suppose there exists a pair $(Q, v) \in \mathcal{H}^{-1}(\phi)$ such that $\tau(Q, v) \cong \tau(P, s)$. Let $\{t_{hl}\}_{h,l \in H}$ and $\{\widetilde{t}_{hl}\}_{h,l \in H}$ be cocycles representing $\tau(P, s)$ and $\tau(Q, v)$ respectively and write $\widetilde{t}_{hl} = m_h^{-1} t_{hl} m_l$ for suitable holomorphic maps $m_h : U_h \to T$ with $h \in H$. From the proof of Lemma 3.6 we can assume that for a chosen ramification points $q \in \mathcal{D}_\beta$, one for each $\beta \in A$, and every other ramification point $q \in \mathcal{D}_\beta$ with $\beta \notin A$, condition $\beta(m_j(q)) = 1$ (for suitable $j \in J$) holds. If (Q, v) is distinct from (P, s), by Lemmas 3.5 and 3.6 there exists some $\alpha \in A$ and some $p_\alpha \in U_j \cap \mathcal{D}_\alpha$ (with suitable $j \in J$) such that condition

$$\alpha(m_j(p_\alpha)) = -1 \tag{3.45}$$

is satisfied. Moreover, two pairs for which relation (3.45) holds for exactly the same set of ramification points coincide by Remark 3.1. □

From Theorems 3.7 and 3.8 and from Proposition 2.1 we obtain the following

COROLLARY 3.9. *The image under* $\mathcal{F}$ *of* $\mathcal{H}^{-1}(\phi)_c$ *contains a Zariski open set in* $\mathcal{P}_0$.

3.1. The PGl(2) case

Let $\phi \in H^0(C, K^2)$ be generic. Let P be a PGl(2)-bundle over C and $s \in H^0(C, \operatorname{ad} P \otimes K)$ such that $\mathcal{H}(P, s) = \phi$. We indicate by $pr : \mathrm{Gl}(2) \to \mathrm{PGl}(2) = \mathrm{Gl}(2)/\mathbf{C}^*$ the factor map and as maximal torus $T \subset \mathrm{PGl}(2)$ we choose the one obtained by restricting pr to the maximal torus $\widetilde{T} \subset \mathrm{Gl}(2)$ given by all diagonal matrices. We also set $\mathbf{t} = \operatorname{Lie} T, \widetilde{\mathbf{t}} = \operatorname{Lie} \widetilde{T}$. In this setting, $\widetilde{C} = \phi^*(\mathbf{t} \otimes K)$ is a ramified double covering of C whose ramification divisor $\mathcal{D}$ satisfies by definition $\mathcal{O}(\mathcal{D}) \cong \pi^* K$.

Let $\{V_h\}_{h \in H}$ and $\{U_h\}_{h \in H}$ be open coverings of C and $\widetilde{C}$ defined as before. If $\{g_{hl} : V_h \cap V_l \to \mathrm{PGl}(2)\}_{h,l \in H}$, are transition functions for P, it is known that there exists some rank 2 vector bundle F, hence some principal Gl(2)-bundle $\widetilde{P}$, with transition functions $\widetilde{g}_{hl}$ satisfying

$$pr \circ \widetilde{g}_{hl} = g_{hl} \quad \forall h, l \in H. \tag{3.46}$$

Moreover, any other rank 2 vector bundle F' has the same property if and only if $F' \cong F \otimes L$ for some line bundle $L \in \mathrm{Pic}(C)$. Note also that this implies $\deg F \equiv \deg F' (\mathrm{mod}\, 2)$ (since $deg(F \otimes L) = \deg F \cdot \deg L^2$). For the sake of simplicity for any F satisfying relation (3.46) we write $P = pr(F)$. For $\widetilde{P}$ as above, we clearly have an isomorphism ad $\widetilde{P} \otimes K \cong (\mathrm{ad}\, P \otimes K) \oplus K$ and given some fixed generic section $x : C \to K$ we may define $\widetilde{s} \in H^0(ad\widetilde{P} \otimes K)$ by $\widetilde{s} = s \oplus x$. We set $\widetilde{\phi} = \mathcal{H}_{\mathrm{Gl}(2)}(\widetilde{P}, \widetilde{s}) \in H^0(C, K \oplus K^2)$ (the subscript indicating that we are in the Gl(2) setting) and observe that the covering $\widetilde{\phi}^*(\widetilde{\mathbf{t}} \otimes K)$ of C coincides with $\widetilde{C}$. Then it is clear from the argument above that we have a surjective map

$$\text{‘}pr\text{’}\colon \mathcal{H}^{-1}_{\mathrm{Gl}(2)}(\widetilde{\phi}) \to \mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi).$$

This also shows that $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)$ has two components $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_0$, $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_1$: namely $(Q, v) \in \mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)$ is contained in $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_0$ or $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_1$ depending on the parity of the degree of those F which satisfy $pr(F) = Q$.

We now look at our construction in the Gl(2) case. If we indicate by χ_1 and χ_2 the coordinate functions on $\widetilde{T}$ and set $\widetilde{\alpha} = \chi_1 \cdot \chi_2^{-1}$, $\sigma = s_{\widetilde{\alpha}}$, we have by definition

$$\mathcal{P}_{\mathrm{Gl}(2)} = \{Q \otimes \chi_1' \oplus \sigma^* Q \otimes \chi_2' \mid Q \in J(\widetilde{C})\} \equiv J(\widetilde{C})$$

(the one parameter subgroups χ_i' being defined by $\chi_i(\chi_j') = (\chi_i, \chi_j), j = 1, 2$) and

$$\widehat{\mathcal{P}}_{\mathrm{Gl}(2)} = \mathrm{Pic}(\widetilde{C}).$$

The map $\mathcal{T}\colon \mathcal{H}^{-1}_{\mathrm{Gl}(2)}(\widetilde{\phi}) \to \mathrm{Pic}(\widetilde{C})$ is injective (see Theorem 3.7), dominant and by Hitchin's theory (see [Hi]) it preserves the parity of the degrees. By the argument above the generic fibre of the map ‘pr’ is a principal homogeneous space with respect to $\Lambda = \{M \in \mathrm{Pic}(\widetilde{C}) \mid M = \pi^* L, L \in \mathrm{Pic}(C)\}$. In this setting the map $\pi^*\colon \mathrm{Pic}(C) \to \mathrm{Pic}(\widetilde{C})$ is injective (since $\widetilde{C} \to C$ is a ramified covering: see e.g [M]), hence Λ coincides with $\mathrm{Pic}(C)$. Since $\mathrm{Pic}(\widetilde{C})^{\mathrm{even}}/\mathrm{Pic}(C)$ and $\mathrm{Pic}(\widetilde{C})^{\mathrm{odd}}/\mathrm{Pic}(C)$ are both principal homogeneous spaces with respect to the connected group $J(\widetilde{C})/J(C)$, it follows that the components $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_0$, $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_1$ are connected. Now, let χ' be the one parameter subgroup in $T \subset \mathrm{PGl}(2)$ given by composing pr with χ_1' (we have $X(T)^* = \mathbf{Z}\chi'$). By definition, we have $\widehat{\mathcal{P}}_{\mathrm{PGl}(2)} = \mathcal{P}_{\mathrm{PGl}(2)} = \{Q \otimes \chi' \mid Q \in J(\widetilde{C}), \sigma^* Q \cong Q^{-1}\}$ and, since $\pi^*\colon J(C) \to J(\widetilde{C})$ is injective, this is just the Prym variety $P(\widetilde{C}, \sigma) \subset J(\widetilde{C})$. From Theorem 3.1 the $\widetilde{T}$-bundle $\widetilde{\tau} = \tau(\widetilde{P}, \widetilde{s})$ has transition functions $t_{hl} : U_h \cap U_l \to \widetilde{T}$ of the form

$$t_{hl}(\eta) = \mathrm{diag}(q_{hl}(\eta), \sigma^* q_{hl}(\eta) \cdot k_{hl}(\pi(\eta))).$$

One can easily check that the maps

$$pr \circ t_{hl}(\eta) = q_{hl}(\eta) \cdot \sigma^* q_{hl}(\eta)^{-1} \cdot k_{hl}(\pi(\eta))^{-1} \colon U_h \cap U_l \to \mathbf{C}^*$$

are transition functions for $\tau = \tau(P, s)$. In other words, if we use the additive notation, we have $\mathcal{T}_{\mathrm{PGl}(2)}(P, s) = (1 - \sigma^*) \circ \mathcal{T}_{\mathrm{Gl}(2)}(\widetilde{P}, \widetilde{s})$. Moreover, if $\widetilde{P}'$ is another Gl(2)-bundle inducing via the factor map pr the same PGl(2)-bundle P, we have that $\tau(\widetilde{P}', \widetilde{s})$ has transition functions $t_{hr}(\eta) \cdot l_{hr}(\pi(\eta))$, where $\{l_{hr} : V_h \cap V_r \to \mathbf{C}^*\}_{h,r \in H}$ define some line bundle L over C. We thus have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Pic}(\widetilde{C}) & \xrightarrow{(1-\sigma^*)} & P(\widetilde{C}, \sigma) \\
\uparrow \mathcal{T}_{\mathrm{Gl}(2)} & & \uparrow \mathcal{T}_{\mathrm{PGL}(2)} \\
\mathcal{H}^{-1}_{\mathrm{Gl}(2)}(\widetilde{\phi}) & \xrightarrow{\text{`}pr\text{'}} & \mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_0 \coprod \mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_1
\end{array}$$

If we set $\Lambda' = \{N \in \mathrm{Pic}(\widetilde{C}) \mid N = \sigma^* N\}$, we see that all sufficiently general fibres of the dominant map $\mathcal{T}_{\mathrm{PGl}(2)}$ are principal homogeneous spaces with respect to Λ'/Λ. It is known (see [M]) that Λ'/Λ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{(d-1)}$, d being the number of ramification points of $\widetilde{C}$ or, in this setting, the degree of $\pi^* K$. Note here that the number of $\mathbf{Z}/2\mathbf{Z}$ factors reaches its maximum with respect to the estimate given in Theorem 3.8. Since each component $\mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_c$, $c = 0, 1$, is connected, we have that the generic fibre of $\mathcal{F}\colon \mathcal{H}^{-1}_{\mathrm{PGl}(2)}(\phi)_c \to P(\widetilde{C}, \sigma)$ consists of $2^{(d-2)}$ points.

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