Math. Proc. Camb. Phil. Soc. (1984), **95**, 513
Printed in Great Britain

Corrigenda
Volume 92 (1982), 115–119

'Growth conditions on powers of Hermitian elements'

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(*Received* 15 *December* 1983)

The above paper aimed to obtain Banach algebra equivalents of the elegant theorems of Roe [2] and Burkill [1] which gave characterizations of the sine function. Professor John Duncan has kindly pointed out that the paper contains two oversights, with the result that the theorems stated are not the strict Banach algebra equivalents as was claimed. (The assertion on p. 117 that T is Hermitian is false if $\mu > 0$, and at the bottom of p. 117 it follows from Theorem 2 that $F_0(t) = c_1 e^{it} + c_2 e^{it}$ so that $a^2 - e = 0$.) Whilst Theorems 1–3 are correct as stated, the conclusions of Theorems 1 and 3 must be strengthened to provide equivalent versions when $\mu > 0$. In the basic case when $\mu = 0$, the analysis remains as given.

I am most grateful to Professor Duncan for providing detailed comments on these points, and to Professor V. I. Istrăţescu for drawing my attention to [3].

In the restatement of the results below, it is convenient to give two sets of equivalent theorems.

THEOREM 1′. *Let* $(A, \|\ \|)$ *be a complex unital Banach algebra with unit* e, *and let* $a \in A$ *be an invertible Hermitian element such that*

$$\|a^n\| \leqslant M(|n|+1)^\lambda \tag{C 1}$$

for $n \in \mathbb{Z}$, *where* M *and* λ *are non-negative numbers. Then*

$$a^2 = e.$$

THEOREM 2′. (Roe) *Suppose that the complex-valued functions* $F_n (n \in \mathbb{Z})$ *on* $(-\infty, \infty)$ *are such that* $F_n' = F_{n+1}$ *for all* n (*dashes denoting differentiation*). *Suppose that there exist non-negative numbers* M *and* λ *such that*

$$|F_n(t)| \leqslant M(|n|+1)^\lambda \tag{C 2}$$

for all n *and* t. *Then*

$$F_0(t) = c_1 e^{it} + c_2 e^{-it},$$

where c_1 *and* c_2 *are complex numbers.*

THEOREM 3′. *Let* $(X, \|\ \|)$ *be a complex Banach space and let* T *be an invertible Hermitian operator on* X. *Suppose* $x \in X$ *is such that*

$$\|T^n x\| \leqslant M(|n|+1)^\lambda \tag{C 3}$$

for $n \in \mathbb{Z}$, *where* M *and* λ *are non-negative numbers. Then*

$$(T^2 - I)x = 0.$$

THEOREM 4′. *Theorems* 1′, 2′ *and* 3′ *remain true if we merely require conditions* C 1, *C 2 and C 3 respectively to hold for $n = r_m (m \in \mathbb{Z})$, where $\{r_m\}$ is a two way infinite sequence with $r_m \to \infty$ as $m \to \infty$ and $r_m \to -\infty$ as $m \to -\infty$.*

In order to state the second set of theorems, we need to use quasi-Hermitian elements.

An element $a \in A$ is said to be *quasi-Hermitian of order* μ if

$$\|\exp(ita)\| \leqslant M(1+|t|)^{\mu}, \tag{1}$$

for real t, for some constant M. Similarly, a Banach space operator T is *quasi-Hermitian of order* μ if

$$\|\exp(itT)\| \leqslant M(1+|t|)^{\mu}.$$

THEOREM 1″. *If, in Theorem 1′, a is an invertible quasi-Hermitian element of order $\mu \geqslant 0$, then*

$$(a^2-e)^{k+1} = 0 \quad \textit{where} \quad k = \min(\lfloor\lambda\rfloor, \lfloor\mu\rfloor).$$

THEOREM 2″. (Burkill). *If, in Theorem 2′, we have*

$$|F_n(t)| \leqslant M(|n|+1)^{\lambda}(|t|+1)^{\mu},$$

where $\mu > 0$, then

$$F_0(t) = p_1(t)\,e^{it} + p_2(t)\,e^{-it},$$

where p_1 and p_2 are complex polynomials of degrees at most $\min(\lfloor\lambda\rfloor, \lfloor\mu\rfloor)$.

THEOREM 3″. *If, in Theorem 3′, T is an invertible quasi-Hermitian operator of order $\mu \geqslant 0$, then*

$$(T^2-I)^{k+1}x = 0 \quad \textit{where} \quad k = \min(\lfloor\lambda\rfloor, \lfloor\mu\rfloor).$$

For the sake of completeness, we state the following Theorem of Istrăţescu [3] which is of a similar form.

THEOREM 1‴ (Istrătescu). *If, in Theorem 1′, a is an element with real spectrum, then*

$$(a^2-e)^{k+1} = 0 \quad \textit{where} \quad k = \lfloor\lambda\rfloor.$$

The proofs of these theorems and the equivalences follow the lines of those presented before, with the following amendments.

1. To prove Theorem 1′, which is Theorem 1 with the conclusion strengthened to $a^2 = e$, we proceed as before to estimate the resolvent $R(z) = (ze-a)^{-1}$ near $z = \pm 1$. If $w = z-1$ or $w = z+1$ we have

$$\|R(z)\| \leqslant c|w|^{-1} \quad (|\mathrm{Re}\, w| \leqslant |\mathrm{Im}\, w|), \tag{2}$$

$$\|R(z)\| \leqslant c|w|^{-\lambda-1} \quad (|\mathrm{Re}\, w| \geqslant |\mathrm{Im}\, w|), \tag{3}$$

near $w = 0$, where c is independent of z. Thus $R(z)$ has a terminating Laurent expansion at 1 and -1. In particular, the growth of $R(z)$ near these poles is determined by its growth in any segment. Thus (2) holds for any z near $w = 0$. It follows as before, but using this stronger estimate, that $a^2 = e$.

2. Theorem 3′ follows from Theorem 1′ exactly as before but again the conclusion has been strengthened.

3. Theorem 1′ is equivalent to Roe's theorem, Theorem 2′, with proofs as before. In this case, we need only consider differential equations of the form $F'' + F = 0$.

4. Theorem 4′ applies to Theorems 1′, 2′ and 3′, using the Kolmogorov lemma quoted before.

For the second set of theorems we study properties of quasi-Hermitian elements.

5. To prove Theorem 1″ we proceed as for Theorem 1 modified by §1 above. We require estimates for the resolvent $R(z) = (ze-a)^{-1}$ near $z = \pm 1$ when a is quasi-Hermitian. The growth conditions on a ensure that (3) holds. To get an estimate for $R(z)$ off the real axis, we use the Laplace formula

$$(ze-ia)^{-1} = \int_0^\infty \exp(-zt)\exp(ita)\,dt \quad (\operatorname{Re} z > 0). \tag{4}$$

Writing $z = ix-y$, this gives

$$\begin{aligned} \|(x+iy-a)^{-1}\| &\leqslant \int_0^\infty \exp(yt)\,\|\exp(ita)\|\,dt \\ &\leqslant M\int_0^\infty \exp(yt)\,(1+|t|)^\mu\,dt \\ &\leqslant M\,|y|^{-\mu-1}, \end{aligned} \tag{5}$$

if $y < 0$, using (1). A similar estimate in the upper half plane comes from using $\exp(-ita)$ in (4). Combining (5) with (3) we see that $R(z)$ has a terminating Laurent expansion near $z = \pm 1$, so as in §1

$$\|R(z)\| \leqslant c|w|^{-k-1},$$

where $k = \min(\lfloor\lambda\rfloor, \lfloor\mu\rfloor)$. Theorem 1″ now follows from this estimate in the usual way.

6. As before, we see that Theorem 3″ is the operator version of Theorem 1″.

7. The equivalence of Theorem 1″ and Theorem 2″ may be shown as before. To deduce Theorem 2″ from Theorem 1″ we need to use the fact that the operator $TF = iF'$ is quasi-Hermitian of order μ (rather than Hermitian) on the Banach space

$$\{F\colon \|F\| < \infty\},$$

where

$$\|F\| = \sup_{n,t} |F_n(t)|\,(|n|+1)^{-\lambda}\,(|t|+1)^{-\mu}.$$

To see this, observe that if $\exp(i\alpha t)\,F = G$, then $G(t) = F(t-\alpha)$, so that

$$\begin{aligned} \|\exp(i\alpha T)\,F\| &= \sup_{n,t}(1+|n|)^{-\lambda}(1+|t|)^{-\mu}\,|F_n(t-\alpha)| \\ &= \sup_{n,t}(1+|n|)^{-\lambda}(1+|t+\alpha|)^{-\mu}\,|F_n(t)| \\ &= (1+|\alpha|)^{\mu}\,\|F\|. \end{aligned}$$

8. It is not at present clear whether a version of Kolmogorov's theorem for functions of polynomial growth, and thus for quasi-Hermitian operators, is valid. Thus it is unclear whether a version of Theorem 4 holds for the second trio of theorems.

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