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# THE GROUP OF UNITS IN $K$-THEORY MODULO AN ODD PRIME

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**1. Introduction.** There are several multiplicative cohomology theories for which the group of units in the zeroth term is the zeroth term of another cohomology theory. Examples, due to Segal, May and others, are given by ordinary cohomology with rather general graded coefficients, real and complex $K$-theory with integral coefficients, and various bordism theories, also with integral coefficients [**8**, **7**, 2, **5**, IV]. The object of this paper is to show that complex $K$-theory modulo an odd prime $p$ provides a counter-example.

To state the theorem precisely we recall the result of Araki and Toda that there is a unique anticommutative associative admissible multiplication in $K^*(\quad;\mathbf{Z}/p)$ for $p$ an odd prime [**3**, 3, 7, 10]; *admissible* is defined in [**3**] and means essentially that the reduction homomorphism $K^*(\quad) \to K^*(\quad;\mathbf{Z}/p)$ preserves products. Now $K^0$ (point; $\mathbf{Z}/p$) is the ring $\mathbf{Z}/p$, so $K^0(\quad;\mathbf{Z}/p)$ is represented by a space $\mathbf{Z}/p \times BU_p$ with $BU_p$ connected and the group of units is represented by $(\mathbf{Z}/p)^* \times BU_p$, where $(\mathbf{Z}/p)^*$ is the group of units in $\mathbf{Z}/p$. As an $H$-space, $(\mathbf{Z}/p)^* \times BU_p$ is the product of $(\mathbf{Z}/p)^*$ with the $H$-space $\{1\} \times BU_p = BU_p^{\otimes}$, say. To prove our theorem, it suffices to show that $BU_p^{\otimes}$ is not an infinite loop space. In fact we shall prove the following theorem.

THEOREM. *The $H$-space $BU_p^{\otimes}$ is not a fourth loop space.*

The method of proof is to compute the $\mathbf{Z}/p$-homology of $BU_p^{\otimes}$ and its loop space $U_p$ (which represents $K^{-1}(\quad;\mathbf{Z}/p)$) and to show that they do not admit Dyer-Lashof operations satisfying all the formulae that they should; the formulae are given by Cohen in [**4**, III, 1].

The result is perhaps not very surprising, as the construction of the multiplication in $K^*(\quad;\mathbf{Z}/p)$ is rather artificial. What is perhaps surprising is the difficulty of the proof, at least by the method used here. For one thing the Hopf algebra $H_*(BU_p^{\otimes};\mathbf{Z}/p)$ is isomorphic to the Hopf algebra $H_*(BU_p^{\oplus};\mathbf{Z}/p)$ (see 4.1 below), where $BU_p^{\oplus}$ is the $H$-space $\{0\} \times BU_p \subset \mathbf{Z}/p \times BU_p$ with product representing addition, and $BU_p^{\oplus}$ of course is an infinite loop space. We need the duals of Steenrod operations to distinguish these homologies. For another thing (3.3, 4.1) it turns out that $p$th powers of positive degree elements in $H_*(BU_p^{\otimes};\mathbf{Z}/p)$ all vanish, which makes it appear possible for $H_*(BU_p^{\otimes};\mathbf{Z}/p)$ to admit trivial Dyer-Lashof operations. It is to rule out this possibility that we compute $H_*(U_p;\mathbf{Z}/p)$ as well.

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It seems difficult to adapt the proof to other circumstances. I have tried and failed to apply it to complex bordism and to complex $K$-theory modulo a composite number (by [**3**], $K^*( \quad ; \mathbf{Z}/q)$ has an anticommutative associative admissible multiplication if and only if $q \not\equiv 2$ modulo 4). Also $K$-theory localized at the odd prime $p$ has as a factor the multiplicative cohomology theory given by real $K$-theory localized at $p$ and a smaller multiplicative factor given by Adams [**1**, 4]. These have their coefficient groups in degrees which are multiples of 4 and $2(p - 1)$ respectively, and yield $H$-space factors of $BU_p^{\otimes}$ which are at least 3-connected. The proof that $BU_p^{\otimes}$ is not an infinite loop space uses homology classes of degree 2, so does not apply to these factors. A more efficient method might also determine whether $BU_p^{\otimes}$ is a loop space at all (there seems no good reason why it should be).

We make the following conventions for the whole paper. All homology and cohomology groups have coefficients $\mathbf{Z}/p$, $p$ being the odd prime of the theorem. When a space $X$ has two products, one representing addition and one representing multiplication in $K$-theory, then we shall use $X^{\oplus}$ and $X^{\otimes}$ for the two $H$-spaces and $X$ for the underlying space. Beside $BU_p$, for which this notation was used above, this will be used for $BU$ and $BSU$, the classifying spaces of the infinite unitary and special unitary groups $U$ and $SU$.

The successive sections of the paper compute $H_*(U_p)$, $H_*(BU_p^{\oplus})$, $H_*(BU_p^{\otimes})$, and give the proof of the theorem.

The material is taken from my doctoral thesis at the University of Cambridge. I am grateful to my supervisors V. P. Snaith and J. F. Adams; to Professor Snaith for posing the problem and encouraging me to work on it, and to Professor Adams for help with technical details.

**2. The homology of** $U_p$. From the Bockstein sequence

$$K^{-1}( \quad ) \longrightarrow K^{-1}( \quad ; \mathbf{Z}/p) \longrightarrow K^0( \quad ) \xrightarrow{\times p} K^0( \quad )$$

we obtain a fibration sequence of representing spaces

$$U \to U_p \to \mathbf{Z} \times BU \to \mathbf{Z} \times BU$$

and by taking connected components a homotopy-commutative diagram of $H$-spaces

$$\begin{array}{ccc} U & \xrightarrow{=} & U \\ {\scriptstyle s}\downarrow & & \downarrow \\ U_p & \longrightarrow & * \\ {\scriptstyle r}\downarrow & & \downarrow \\ BU^{\oplus} & \xrightarrow{p} & BU^{\oplus}, \end{array}$$

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whose columns are fibrations. We shall compute $H^*(U_p)$ from the cohomology Serre spectral sequences of these fibrations, and then obtain $H_*(U_p)$ by dualization.

Thus we have a morphism $E_r \to \bar{E}_r$ of spectral sequences of algebras. They begin with

$$E_2^{i,j} = H^i(BU) \otimes H^j(U) \xrightarrow{p^* \otimes 1} H^i(BU) \otimes H^j(U) = \bar{E}_2^{i,j}$$

and $E_r \Rightarrow \mathbf{Z}/p$, $\bar{E}_r \Rightarrow H^*(U_p)$. The behaviour of $E_r$ is well-known. We have

(2.1) $\quad H^*(BU) = \mathbf{Z}/p[c_1, c_2, \ldots]$ with $\deg(c_k) = 2k$,

the polynomial algebra on the modulo $p$ reductions $c_k$ of the universal Chern classes, and

(2.2) $\quad H^*(U) = \Lambda[u_1, u_2, \ldots]$ with $\deg(u_k) = 2k - 1$,

the exterior algebra on classes $u_k$ defined by

(2.3) $\quad \sigma^* c_k = u_k$,

where $\sigma^*: \tilde{H}^*(BU) \to H^*(U) = H^*(\Omega BU)$ is the cohomology suspension.

From (2.1) – (2.3) we obtain a description of $E_r$: we find that

$$E_{2r-1} = E_{2r} = \mathbf{Z}/p[c_k: k \geqq r] \otimes \Lambda[u_k: k \geqq r]$$

with $\operatorname{bideg}(c_k \otimes 1) = (2k, 0)$ and $\operatorname{bideg}(1 \otimes u_k) = (0, 2k - 1)$; the differentials are given by

$$d_r(c_k \otimes 1) = 0 \quad \text{for all } r \text{ and } k,$$
$$d_{2k}(1 \otimes u_k) = c_k \otimes 1,$$
$$d_r(1 \otimes u_k) = 0 \text{ for all other } r \text{ and } k.$$

Now the morphism $E_2 \to \bar{E}_2$ sends $c_k \otimes 1$ to $p^* c_k \otimes 1$ and $1 \otimes u_k$ to $1 \otimes u_k$. To compute $p^*$ we recall that the coproduct in $H^*(BU^{\oplus})$ is given by

(2.4) $\quad \phi^*(c_k) = \sum_{i+j=k} c_i \otimes c_j \quad (c_0 = 1),$

where $\phi$ is the product in $BU^{\oplus}$. Therefore

$$\phi^*(1 + c_1 + c_2 + \ldots) = (1 + c_1 + c_2 + \ldots) \otimes (1 + c_1 + c_2 + \ldots),$$
$$p^*(1 + c_1 + c_2 + \ldots) = (1 + c_1 + c_2 + \ldots)^p,$$
$$p^* c_k = 0 \text{ if } p \nmid k,$$
$$p^* c_{pj} = c_j^p.$$

We now obtain $\bar{E}_r$ by induction on $r$:

$$\bar{E}_{2p(j-1)+1} = \bar{E}_{2p(j-1)+2} = \ldots = \bar{E}_{2pj} = \mathbf{Z}/p[c_k: k \geqq 1]/ (c_1^p, \ldots, c_{j-1}^p) \otimes \Lambda[u_k: k \neq p, 2p, \ldots, (j-1)p]$$

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and

$$\bar{d}_r(c_k \otimes 1) = 0 \quad \text{for all } r \text{ and } k,$$
$$\bar{d}_{2pj}(1 \otimes u_{pj}) = c_j^p \otimes 1,$$
$$\bar{d}_r(1 \otimes u_k) = 0 \quad \text{for all other } r \text{ and } k.$$

For the only differentials among these which can be non-zero are those whose images lie in the first quadrant, and they are determined by the differentials in $E_r$.

Consequently,

$$\begin{aligned}\bar{E}_\infty &= \mathbf{Z}/p[c_k\colon k \geqq 1]/(c_k^p\colon k \geqq 1) \otimes \Lambda[u_k\colon p \nmid k] \\ &= \mathbf{Z}/p[c_k\colon k \geqq 1]//\xi \otimes \Lambda[u_k\colon p \nmid k],\end{aligned}$$

where $\xi$ denotes the Frobenius homomorphism $x \mapsto x^p$ in a $\mathbf{Z}/p$-algebra. Therefore $H^*(U_p)$ is obtained from

$$r^*H^*(BU) = \mathbf{Z}/p[c_k'\colon k \geqq 1]//\xi \quad (c_k' = r^*c_k)$$

by adjoining for each $k$ with $p \nmid k$ an indecomposable $u_k'$ of degree $2k - 1$ with $s^*u_k' = u_k$. By anticommutativity, $u_k'^2 = 0$. Now in a bicommutative biassociative connected Hopf algebra $A$ of finite type over $\mathbf{Z}/p$ we have a Milnor-Moore exact sequence [**6**, 4.23]

$$(2.5) \qquad 0 \to P\xi A \to PA \to QA \to [P\xi(A^*)]^* \to 0, \text{ where } PA = [Q(A^*)]^* \text{ and } QA = [P(A^*)]^*;$$

here $P$ denotes primitive submodule, $Q$ denotes indecomposable quotient, asterisks denote vector space duals, $P\xi A \to PA$ is the inclusion, $QA \to [P\xi(A^*)]^*$ is the dual of the inclusion $P\xi(A^*) \to P(A^*)$, and $PA \to QA$ is the obvious homomorphism, dual to the obvious homomorphism $P(A^*) \to Q(A^*)$. In particular we see that $PA_k \cong QA_k$ if $k$ is not a multiple of $2p$. We may therefore specify the indecomposable $u_k'$ in $H^{2k-1}(U_p)$ with $s^*u_k' = u_k$ uniquely by requiring it to be primitive, since $u_k \in H^{2k-1}(U_p)$ is primitive (from (2.3), as the image of the cohomology suspension is contained in the primitives).

We deduce the following description of $H^*(U_p)$.

PROPOSITION 2.6.

$$H^*(U_p) = \mathbf{Z}/p[c_k'\colon k \geqq 1]//\xi \otimes \Lambda[u_k'\colon p \nmid k] \quad \textit{with} \quad \deg(c_k') = 2k, \quad \deg(u_k') = 2k - 1.$$

*Set* $c_0' = 1$. *Under* $r\colon U_p \to BU$ *and* $s\colon U \to U_p$ *we have*

$$r^*c_k = c_k' \textit{ for } k \geqq 0, \quad s^*c_k' = 0 \textit{ for } k \geqq 1, \quad s^*u_k' = u_k \textit{ for } p \nmid k.$$

*Let* $\phi$ *be the product in* $U_p$; *then*

$$\phi^*c_k' = \sum\nolimits_{i+j=k} c_i' \otimes c_j', \quad \phi^*u_k' = u_k' \otimes 1 + 1 \otimes u_k'.$$

Indeed the formula for $\phi^*c_k'$ follows by naturality from (2.4) as $r\colon U_p \to BU^{\oplus}$

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is an $H$-map, $s^*c_k' = s^*r^*c_k = 0$ for $k \geqq 1$ as $rs$ is null-homotopic, and the rest of 2.6 has already been proved.

Next we compute $H_*(U_p)$ by dualization. First we recall the homology of $BU^\oplus$ and $U$.

(2.7) $H_*(BU^\oplus) = \mathbf{Z}/p[b_1, b_2, \ldots]$ with $\deg(b_k) = 2k$,

where

$$\langle c_1^k, b_k\rangle = 1, \quad \langle m, b_k\rangle = 0 \text{ for other monomials } m \text{ in the } c_k,$$
$$\langle c_k, b_1^k\rangle = 1, \quad \langle c_k, m\rangle = 0 \text{ for other monomials } m \text{ in the } b_k.$$

The diagonal $\Delta_*$ is given by

$$\Delta_*(b_k) = \sum_{i+j=k} b_i \otimes b_j \quad (b_0 = 1).$$

(2.8) $H^*(U) = \Lambda[v_1, v_2, \ldots]$ with $\deg(v_k) = 2k - 1$, $\langle u_k, v_k\rangle = 1$, $v_k$ primitive.

It follows from 2.6 that $H_*(U_p)$ is a tensor product of Hopf algebras $A' \otimes s_*H_*(U)$ with $r_*$ mapping $A'$ isomorphically onto $A \subset H_*(BU^\oplus)$, where $A$ is the annihilator of the ideal $(\xi\tilde{H}^*(BU)) = (c_1^p, c_2^p, \ldots)$. It is clear that the kernel of $s_*$ is the ideal $(v_p, v_{2p}, \ldots)$, and it remains to compute $A$. We proceed as follows. For $k = 1, 2, \ldots$ let $a_k \in H_{2k}(BU^\oplus)$ be the $k$th Newton polynomial in the $b_k$. Since we are working modulo $p$ we find that

(2.9) $a_{kp} = a_k^p$;

on the other hand

(2.10) $a_k \equiv (-1)^{k-1}kb_k$ modulo decomposables,

so the $a_k$ with $p \nmid k$ generate a polynomial algebra

$$P = \mathbf{Z}/p[a_k\colon p \nmid k] \subset H_*(BU^\oplus).$$

We claim that $A = P$. Indeed each $a_k$ is in $A$ because it is primitive, so annihilates decomposables; thus $P \subset A$. To establish equality, we show that the Euler-Poincaré polynomials

$$f(P) = \sum_{k=0} (\dim P_k)t^k$$

and

$$f(A) = \sum_{k=0} (\dim A_k)t^k$$

are equal. Indeed

$$f(P) = \prod_{p \nmid k} (1 + t^{2k} + t^{4k} + \ldots) = \prod_{p \nmid k} 1/(1 - t^{2k}),$$

while

$$f(A) = \prod_{k\geqq 1}(1 + t^{2k} + t^{4k} + \ldots + t^{2(p-1)k}) = \prod_{k\geqq 1} (1 - t^{2pk})/(1 - t^{2k}),$$

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since $A$ is dual to $\mathbf{Z}/p[c_k: k \geqq 1]/(c_k^p: k \geqq 1)$. Plainly $f(P) = f(A)$, so $P = A$ as claimed.

We obtain the following description of $H_*(U_p)$.

PROPOSITION 2.11.

$$H_*(U_p) = \mathbf{Z}/p[a_k': p \nmid k] \otimes \Lambda[v_k': p \nmid k] \text{ with } \deg(a_k') = 2k, \quad \deg(v_k') = 2k - 1,$$

*$a_k'$ and $v_k'$ primitive. Under the H-maps $r: U_p \to BU^\oplus$ and $s: U \to U_p$ we have:*

*for $p \nmid k$, $r_* a_k' = a_k$, $r_* v_k' = 0$, $s_* v_k = v_k'$;*

*for $p \mid k$, $s_* v_k = 0$.*

**3. The homology of** $BU_p^\oplus$. In this section we compute $H_*(BU_p^\oplus)$ as a preliminary to computing $H_*(BU_p^\otimes)$. We also compute the Bockstein and the duals of the Steenrod operations in $PH_*(BU_p)$ (note that $PH_*(BU_p)$ does not depend on the product in $BU_p$ that we use).

The computation of $H_*(BU_p^\oplus)$ is similar to the computation of $H^*(U_p)$ in the last section. The Bockstein sequence in $K$-theory yields a fibration sequence

$$\mathbf{Z} \times BU^\oplus \xrightarrow{\times p} \mathbf{Z} \times BU^\oplus \longrightarrow \mathbf{Z}/p \times BU_p^\oplus \longrightarrow U$$

of $H$-spaces, hence, by killing low-degree homotopy groups, a homotopy-commutative diagram of $H$-spaces

$$\begin{tikzcd} BU^\oplus \arrow[r, "p"] \arrow[d] & BU^\oplus \arrow[d, "\bar{s}"] \\ * \arrow[r] \arrow[d] & BU_p^\oplus \arrow[d, "\bar{r}"] \\ SU \arrow[r, "="] & SU, \end{tikzcd}$$

whose columns are fibrations (note that as an $H$-space $U$ is the product of $SU$ with the circle $S^1$). Here $\Omega\bar{r} \simeq r$, $\Omega\bar{s} \simeq s$, where $r$ and $s$ are as in the last section.

The homology Serre spectral sequences of these fibrations behave in the same way as the cohomology spectral sequences considered in the last section. To see this we observe from (2.7) that $H_*(BU^\oplus) = \mathbf{Z}/p[b_1, b_2, \ldots]$ with $\deg(b_k) = 2k$ and $\Delta_* b_k = \sum_{i+j=k} b_i \otimes b_j$, that we may make an identification

(3.1) $\quad H_*(SU) = \Lambda[v_2, v_3, \ldots] \subset H_*(U)$ with $\deg(v_k) = 2k - 1$, $v_k$ primitive,

from (2.8), and that the homology suspension $\sigma_*: \tilde{H}_*(BU^\oplus) \to H_*(SU)$ is given by

(3.2) $\sigma_*(b_k) = (-1)^k v_{k+1}$

(the sign, which is not very important, is obtained by looking at the effect of the Bott periodicity map $\Sigma^2 BU \to BU$ on the Chern character). Arguments like those in the last section now show that $H_*(BU_p^{\oplus})$ has the following description.

PROPOSITION 3.3.

$$H_*(BU_p^{\oplus}) = \mathbf{Z}/p[b_k'': k \geqq 1]//\xi \otimes \Lambda[x_k'': p \nmid k],$$

*a tensor product of Hopf algebras, with* $\deg(b_k'') = 2k$, $\deg(x_k'') = 2k + 1$. *Under the H-maps* $\bar{r}: BU_p^{\oplus} \to SU$ *and* $\bar{s}: BU^{\oplus} \to BU_p^{\oplus}$ *we have*

$$\bar{r}_* b_k'' = 0 \text{ for } k \geqq 1, \quad \bar{r}_* x_k'' = v_{k+1}, \quad \bar{s}_* b_k = b_k''.$$

*The* $x_k''$ *are primitive. The primitive submodule of* $\mathbf{Z}/p[b_k'': k \geqq 1]$ *has a base consisting of one element* $a_k'' = \bar{s}_* a_k$ *in each degree* $2k$ *with* $p \nmid k$.

To justify the last sentence, we note that the $a_k''$ are primitive by naturality, that $a_k'' \neq 0$ for $p \nmid k$ as $a_k \equiv (-1)^{k-1} k b_k$ modulo decomposables in $H_*(BU^{\oplus})$ by (2.10), and that the $a_k''$ and $x_k''$ together span $PH_*(BU_p)$ because the dual space $QH^*(BU_p)$ has, by analogy with 2.11, dimension 1 in degrees $2k$ and $2k + 1$ with $p \nmid k$ and dimension 0 in other degrees. (Note that $\bar{s}_* a_k = 0$ if $k = pj$ is a multiple of $p$, for (2.9) then gives $a_k = \xi(a_j)$.)

The suspension $\sigma_*: QH_*(U_p) \to PH_*(BU_p)$ is given by

(3.4) $\sigma_* v_k' = a_k''$ and $\sigma_* a_k' = -k x_k''$ for $p \nmid k$.

For dualizing (2.3) and using (2.7) and (2.10) shows that

(3.5) $\sigma_* v_k = a_k$

under $\sigma_*: QH_*(U) \to PH_*(BU)$, whence $\sigma_* v_k' = a_k''$ by naturality. As for $\sigma_* a_k' = -k x_k''$, (2.7) and (3.2) give $\sigma_* a_k = -k v_{k+1}$, so $\bar{r}_* \sigma_* a_k' = \sigma_* r_* a_k' = \sigma_* a_k = -k v_{k+1} = \bar{r}_*(-k x_k'')$. But $\bar{r}_*: PH_{2k+1}(U_p) \to PH_{2k+1}(SU)$ is a monomorphism, so $\sigma_* a_k' = -k x_k''$ as claimed.

Next we give the Bockstein $\beta$ on primitive elements of $H_*(BU_p)$. Clearly $\beta a_k'' = 0$ by naturality, as $\beta$ vanishes in $H_*(BU)$. We also have

(3.6) $\beta x_k'' = \epsilon_p k^{-1} a_k''$ for $p \nmid k$,

where $\epsilon_p = \pm 1$ and depends only on $p$, not on $k$. Essentially I owe this result to Professor Adams.

The proof of (3.6) is as follows. The modulo $p$ Bockstein sequence of a cohomology theory represented by a spectrum $E$ may be obtained by smashing $E$ in the stable category with the cofibration sequence

$$\ldots \longrightarrow S^0 \xrightarrow{\times p} S^0 \longrightarrow Y \longrightarrow S^1 \xrightarrow{\times p} S^1 \longrightarrow \ldots$$

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and the portion of this sequence displayed is the first desuspension of what is obtained from the cofibration sequence of *spaces*

$$S^1 \xrightarrow{\times p} S^1 \xrightarrow{\iota} M \xrightarrow{\pi} S^2 \xrightarrow{\times p} S^2$$

by applying the suspension spectrum functor. Connective $K$-theory is represented by the $\Omega$-spectrum $(\ldots, U, BU, SU, \ldots)$, so we have a homotopy-commutative diagram

$$\begin{tikzcd} U \wedge S^1 \arrow[r, "1 \wedge \iota"] \arrow[d, "e_1"'] & U \wedge M \arrow[r, "1 \wedge \pi"] \arrow[d, "e"'] & U \wedge S^2 \arrow[d, "e_2"'] \\ BU \arrow[r, "\bar{s}"] & BU_p \arrow[r, "\bar{r}"] & SU, \end{tikzcd}$$

with $e_1$ and $e_2$ the obvious evaluation maps arising from $U \simeq \Omega BU$, $U \simeq \Omega^2 SU$.

Let $g_1 \in H_1(S^1)$ and $h_2 \in H_2(S^2)$ be the standard generators. By the definition of the homology suspension we have

$$e_{1*}(z \wedge g_1) = \sigma_* z, \qquad e_{2*}(z \wedge h_2) = \sigma_* \sigma_* z$$

for $z \in \tilde{H}_*(U)$. Also $H_*(M)$ has a base consisting of $g = \iota_* g_1$ of degree 1 and $h$ of degree 2 with $\pi_* h = h_2$ and $\beta g = \epsilon_p h$, the sign $\epsilon_p$ depending on sign-conventions. For $p \nmid k$, let $z = -k^{-1} v_k \in H_{2k-1}(U)$. By (3.5), (3.2) and (2.10),

$$e_{1*}(z \wedge g_1) = \sigma_* z = -k^{-1} a_k, \quad e_{2*}(z \wedge h_2) = \sigma_* \sigma_* z = v_{k+1}.$$

Therefore

$$\bar{r}_* x_k'' = v_{k+1} = e_{2*}(z \wedge h_2) = e_{2*}(1 \wedge \pi)_*(z \wedge h) = \bar{r}_* e_*(z \wedge h).$$

Since $\bar{r}_*$ induces an isomorphism from $QH_{2k+1}(BU_p^{\oplus})$ to $QH_{2k+1}(SU)$,

$$x_k'' \equiv e_*(z \wedge h) \text{ modulo decomposables in } H_*(BU_p^{\oplus}).$$

Since $\beta$ vanishes in $H_*(U)$, we have

$$\begin{aligned} \beta x_k'' &\equiv \beta e_*(z \wedge h) \equiv -e_*(z \wedge \beta h) \equiv -\epsilon_p e_*(z \wedge g) \\ &\equiv -\epsilon_p e_*(1 \wedge \iota)_*(z \wedge g_1) \equiv -\epsilon_p \bar{s}_* e_{1*}(z \wedge g_1) \\ &\equiv \epsilon_p \bar{s}_* k^{-1} a_k \equiv \epsilon_p k^{-1} a_k'' \quad \text{modulo decomposables in } H_*(BU_p^{\oplus}). \end{aligned}$$

Since $\beta x_k''$ is primitive, we must have $\beta x_k'' = \epsilon_p k^{-1} a_k''$. This completes the proof of (3.6).

From (3.6) we obtain a formula concerning the duals of the Steenrod operations in $H_*(BU_p)$:

(3.7) $\quad P^i{}_* \beta x_k'' = \beta P^i{}_* x_k''$ for $p \nmid k$ and $p \nmid k - (p-1)i$.

Conceptually, (3.6) relates these Bocksteins to suspensions; Steenrod opera-

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tions commute with suspension, so they should here commute with the Bockstein.

To prove (3.7), let $l = k - (p-1)i$. Since $P^i{}_*a_k$ is primitive in $H_*(BU)$ we must have $P^i{}_*a_k = \lambda a_l$ for some $\lambda \in \mathbf{Z}/p$. By (2.10) and (3.2), $\sigma_*a_k = -kv_{k+1}$, $\sigma_*a_l = -lv_{l+1}$. As $P^i{}_*$ commutes with suspension, $P^i{}_*v_{k+1} = \lambda lk^{-1}v_{l+1}$; that is,

$$\bar{r}_*P^i{}_*x_k'' = \bar{r}_*\lambda lk^{-1}x_l''.$$

Since $P^i{}_*x_k''$ and $x_l''$ are primitive and $\bar{r}_*: PH_{2l+1}(BU_p) \to PH_{2l+1}(SU)$ is an isomorphism, $P^i{}_*x_k'' = \lambda lk^{-1}x''$; by (3.6),

$$\beta P^i{}_*x_k'' = \lambda\epsilon_p k^{-1}a_l''.$$

Using (3.6) again then shows that

$$P^i{}_*\beta x_k'' = \epsilon_p k^{-1}P^i{}_*a_k'' = \epsilon_p k^{-1}P^i{}_*\bar{s}_*a_k = \lambda\epsilon_p k^{-1}\bar{s}_*a_l$$
$$= \lambda\epsilon_p k^{-1}a_l'' = \beta P^i{}_*x_k'',$$

as required.

**4. The homology of** $BU_p^{\otimes}$. As announced in the introduction, we have the following result.

PROPOSITION 4.1. *There is an isomorphism* $\theta: H_*(BU_p^{\oplus}) \to H_*(BU_p^{\otimes})$ *of Hopf algebras restricting to the identity on* $PH_*(BU_p)$.

This of course does not imply that $BU_p^{\oplus}$ and $BU_p^{\otimes}$ are equivalent $H$-spaces, for we do not say that $\theta$ is induced by a map from $BU_p^{\oplus}$ to $BU_p^{\otimes}$. On the contrary, $BU_p^{\oplus}$ is an infinite loop space and $BU_p^{\otimes}$, as we shall show eventually, is not. The arguments of the next section show in a roundabout way that $\theta$ does not commute with Steenrod operations.

*Proof.* It suffices to show that there is an isomorphism $\theta^*: H^*(BU_p^{\otimes}) \to H^*(BU_p^{\oplus})$ of Hopf algebras inducing the identity on $QH^*(BU_p)$. The algebra structure of $H^*(BU_p)$ (which does not depend on any product in $BU_p$) may be obtained by comparing 2.6, 2.11 and 3.3: like $H_*(U_p)$ it is a polynomial algebra on generators of degrees $2k$ with $p \nmid k$ tensored with an exterior algebra on generators of odd degrees. By the Milnor-Moore exact sequence (2.5) the canonical maps $PH^*(BU_p^{\otimes}) \to QH^*(BU_p)$ and $PH^*(BU_p^{\oplus}) \to QH^*(BU_p)$ are both surjective. So we can define an algebra isomorphism $\theta^*: H^*(BU_p^{\otimes}) \to H^*(BU_p^{\oplus})$ inducing the identity of $QH^*(BU_p)$ and sending primitive generators to primitive generators. The last point makes $\theta^*$ a morphism of Hopf algebras. This completes the proof.

Now consider the map $\mathbf{Z} \times BU \to \mathbf{Z}/p \times BU_p$ representing the reduction homomorphism $K^0(\quad) \to K^0(\quad; \mathbf{Z}/p)$. Because reduction preserves addition, the map must be homotopic to

$$\rho \times \bar{s}: \mathbf{Z} \times BU \to \mathbf{Z}/p \times BU_p$$

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with $\rho:\mathbf{Z} \to \mathbf{Z}/p$ the reduction homomorphism and $\bar{s}:BU \to BU_p$ the map of the last section. Because reduction preserves multiplication, $\bar{s}:BU^{\otimes} \to BU_p{}^{\otimes}$ (the restriction of $\rho \times \bar{s}$ to the 1-components) must be an $H$-map. Therefore $H_*(BU_p{}^{\otimes})$ contains a Hopf subalgebra $\bar{s}_*H_*(BU^{\otimes})$.

Consider also the odd degree primitive elements $x_k''$ for $p \nmid k$ in $H_*(BU_p)$ given in 3.3. Recall that $\deg(x_k'') = 2k + 1$. Combined with the last paragraph they yield a Hopf algebra homomorphism

$$\alpha:\bar{s}H_*(BU^{\otimes}) \otimes \Lambda[x_k'':p \nmid k] \to H_*(BU_p{}^{\otimes}).$$

We claim that $\alpha$ is an isomorphism. Indeed $\alpha$ clearly restricts to a monomorphism on primitive elements, so its dual $\alpha^*$ induces an epimorphism on indecomposables, $\alpha^*$ is itself an epimorphism, and $\alpha$ is a monomorphism. A dimension count using 3.3 shows that $\alpha$ is an isomorphism, as required.

So the structure of $H_*(BU_p{}^{\otimes})$ may be described as follows.

(4.2) $$H_*(BU_p{}^{\otimes}) = \bar{s}_*H_*(BU^{\otimes}) \otimes \Lambda[x_k'': p \nmid k] \quad \text{with } \deg(x_k'') = 2k+1,\ x_k'' \text{ primitive.}$$

We next compute Steenrod operations in $Q\bar{s}_*H_*(BU^{\otimes}) \subset QH_*(BU_p{}^{\otimes})$. We claim that $Q\bar{s}_*H_*(BU^{\otimes})$ has a base

(4.3) $$\{f_k: k \text{ not a power of } p\} \cup \{g_1, g_p, g_{p^2}, \ldots\}$$

with $\deg(f_k) = 2k$, $\deg(g_k) = 2k$, such that in $QH_*(BU_p{}^{\otimes})$ (that is, modulo decomposables)

(4.4) for $k$ not a power of $p$

$$P^i{}_*f_k = \begin{cases} (i, k - pi)f_{k-(p-1)i} & \text{if } k-(p-1)i \text{ is not a power of } p, \\ 0 & \text{if } k-(p-1)i \text{ is a power of } p; \end{cases}$$

$$P^i{}_*g_{p^m} = \begin{cases} g_{p^m} & \text{if } i = 0,\quad m \geqq 0, \\ g_{p^{m-1}} & \text{if } i = p^{m-1},\quad m \geqq 1, \\ 0 & \text{otherwise} \end{cases}$$

(the notation $(i, k - pi)$ means a binomial coefficient).

To see that (4.3) and (4.4) are true we observe from 3.3, 4.1 and (4.2) that $Q\bar{s}_*H_*(BU^{\otimes})$ has the same dimensions as $Q\bar{s}_*H_*(BU^{\oplus}) = H_*(BU^{\oplus})//\xi$; that is, 1 in degrees 2, 4, 6, . . . and 0 in other degrees. So the base proposed in (4.3) is at any rate the right size. Let us write $A$ for the Hopf algebra $\bar{s}_*H_*(BU^{\otimes})$; then $\bar{s}^*$ maps $A^*$ monomorphically into $H^*(BU^{\otimes})$. It is well known that $BU^{\otimes}$ is as an $H$-space the product of $BSU^{\otimes}$ with infinite complex projective space $\boldsymbol{C}P^{\infty}$; the inclusion $i: BSU \to BU$ also gives an $H$-map from $BSU^{\oplus}$ to $BU^{\oplus}$; and $BSU^{\otimes}$ and $BSU^{\oplus}$ are equivalent $H$-spaces after localization or completion at $p$ by the theorem of Adams and Priddy [**2**]. Putting all this together we see that there is a monomorphism

$$\gamma: A^* \to H^*(BSU^{\oplus}) \otimes H^*(\boldsymbol{C}P^{\infty})$$

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and an epimorphism

$$i^*: H^*(BU^{\oplus}) \to H^*(BSU^{\oplus});$$

these are both morphisms of Hopf algebras and commute with the Steenrod operations. To compute the dual Steenrod operations in $QA$ it suffices to compute the Steenrod operations in $PA^*$. We therefore consider $PH^*(BSU^{\oplus})$, $PH^*(\boldsymbol{C}P^{\infty})$, and the monomorphism

$$\gamma: PA^* \to PH^*(BSU^{\oplus}) \otimes PH^*(\boldsymbol{C}P^{\infty}).$$

We know that $i^*$ identifies $H^*(BSU^{\oplus})$ with

$$H^*(BU^{\oplus})/(c_1) \cong \mathbf{Z}/p[c_2, c_3, \ldots].$$

It follows that the Frobenius homomorphism $\xi$ acts monomorphically on $H^*(BSU)$; it also acts monomorphically on $H_*(BSU^{\oplus})$ as this is contained in the polynomial algebra $H_*(BU^{\oplus})$. Using the Milnor-Moore exact sequence (2.5) and induction on degree we find that $PH^*(BSU^{\oplus})$ has dimension 1 in degrees 4, 6, 8, . . . and dimension 0 in other degrees. Also if $k$ is not a power of $p$ then $PH^{2k}(BSU^{\oplus})$ is generated by $i^*d_k$ where $d_k \in H^{2k}(BU)$ is the $k$th Newton polynomial in the $c_k$. For $d_k$ is known to be primitive in $H^*(BU^{\oplus})$ and $i^*d_k \neq 0$ for $k$ not a power of $p$ since $d_k \equiv (-1)^{k-1}kc_k$ modulo decomposables and $d_{pj} = d_j^p$, analogous to (2.9) and (2.10). As for $H^*(\boldsymbol{C}P^{\infty})$, we have $H^*(\boldsymbol{C}P^{\infty})$ identified with $\mathbf{Z}/p[c_1] \subset H^*(BU)$, so, again using (2.5), $PH^*(\boldsymbol{C}P^{\infty})$ has a base $\{c_1, c_1^p, c_1^{p^2}, \ldots\}$.

It follows that $PA^*$ has a base consisting of elements $f_k^*$ with $k$ not a power of $p$ and $g_k^*$ with $k$ a power of $p$ such that $\gamma f_k^* = i^*d_k$, $\gamma g_1^* = c_1$, and $g_{p^m}^* = g_1^{*p^m}$, whence $\gamma g_{p^m} = c_1^{p^m}$. We shall let $\{f_k, g_k\}$ be the dual base for $QA$. The verification of (4.4) now amounts to computing the Steenrod operations on the $d_k$ and on the powers of $c_1$. On the powers of $c_1$ the computation is elementary; to compute the operations on the $d_k$ we identify the $c_k$ with the elementary symmetric functions on indeterminates $t_1, t_2, \ldots$ of degree 2. The Newton polynomial $d_k$ is thereby identified with the sum of the $k$th powers of the $t_n$, whence $P^id_k = (i, k-i)d_{k+(p-1)i}$. From these computations follows (4.4).

Finally in this section we compute the Bockstein in $QH_*(BU^{\otimes})$:

(4.5) $\beta x_1''$ is a non-zero multiple of $g_1$,

$\beta x_k''$ is a non-zero multiple of $f_k$ for $k \geqq 2$ and $p \nmid k$,

all the $\beta f_k$ and $\beta g_k$ vanish.

For $\beta x_k'' \neq 0$ for $p \nmid k$ by (3.6) and is primitive, so indecomposable by the Milnor-Moore exact sequence (2.5), while the $\beta f_k$ and $\beta g_k$ vanish since $f_k$ and $g_k$ lie in the image of $H_*(BU)$ under $\bar{s}_*$.

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**5. Proof of the theorem.** In this section we shall suppose that $BU_p^{\otimes}$ is a fourth loop space and obtain a contradiction, thereby proving the theorem. We first recall the structure on the homology of an $(n + 1)$*th* loop space $X$ as given by Cohen [**4**, III, 1].

For $s \geqq 0$ and $2s - q < n$ there is a homomorphism $Q^s: H_q(X) \to H_{q+2(p-1)}(X)$ called a Dyer-Lashof operation. If $2s = q$, then $Q^s$ is the $p$th power; if $2s < q$ then $Q^s$ vanishes. The $Q^s$ are stable; that is, they commute with the suspension $\sigma_*: \tilde{H}_*(\Omega X) \to H_*(X)$. They satisfy Cartan formulae, which suffice to show that they map primitives to primitives and decomposables to decomposables. They are related to the Steenrod operations by the Nishida relations:

$$
\begin{aligned}
(5.1)\qquad P^r{}_*Q^s &= \textstyle\sum_i(-1)^{r+i}(r - pi,\ (p-1)s - pr + pi)Q^{s-r+i}P^i{}_*,\\
P^r{}_*\beta Q^s &= \textstyle\sum_i(-1)^{r+i}(r - pi,\ (p-1)s - pr + pi - 1)\beta Q^{s-r+i}P^i{}_*\\
&\quad + \textstyle\sum_i(-1)^{r+i}(r - pi - 1,\ (p-1)s - pr + pi)Q^{s-r+i}P^i{}_*\beta.
\end{aligned}
$$

There is also a "top operation" $\xi_n$, not a homomorphism, which maps $H_q(X)$ to $H_{pq+n(p-1)}(X)$ for $n + q$ even. It may be regarded as a substitute for $Q^s$ with $2s - q = n$. In particular there is a Cartan formula showing that $\xi_n$ maps primitives to primitives. The analogues of the Nishida relations (5.1) are complicated, but fortunately we shall use only the simple special cases given in the following lemma.

LEMMA 5.2. *If $X$ is a fourth loop space and $x \in H_3(X)$, then the formulae for $P^1{}_*\xi_3x$ and $P^1{}_*\beta\xi_3x$ are those given by* (5.1) *for $P^1{}_*Q^3x$ and $P^1{}_*\beta Q^3x$ respectively.*

*Proof.* First consider $P^1{}_*\xi_3x$. By [**4**, III, 1.3(3)] the formulae for $P^1{}_*\xi_3x$ and $P^1{}_*Q^3x$ differ by an error term of the form

$$L(P^1{}_*x, x, \dots, x) \qquad (p - 1 \text{ components } x)$$

where $L: H_*(X)^p \to H_*(X)$ is a multilinear function of degree $3(p - 1)$ made out of Browder operations. Since $P^1{}_*x$ has negative degree, the error term vanishes.

As for $P^1{}_*\beta\xi_3x$, in the notation of [**4**, III, 1] we have

$$P^1{}_*\beta\xi_3x = P^1{}_*\zeta_3x + P^1{}_*\,\mathrm{ad}_3{}^{p-1}(x)(\beta x)$$

by the definition of $\zeta_3$ [**4**, III, 1.3]. By [**4**, III, 1.3(3)], $P^1{}_*\zeta_3x$ is $P^1{}_*\beta Q^3x$ as given by (5.1), so we need to show that $P^1{}_*\mathrm{ad}_3{}^{p-1}(x)(\beta x)$ vanishes. By definition [**4**, III, 1.3].

$$\mathrm{ad}_3{}^{p-1}(x)(\beta x) = L'(x, \dots, x, \beta x) \qquad (p - 1 \text{ components } x)$$

for $L': H_*(X)^p \to H_*(X)$ a multilinear function of degree $3(p - 1)$ made out of Browder operations. Using the precise definition of $L'$ and [**4**, III, 1.2(7)] we see that

$$P^1{}_*L' = L'(P^1{}_* \times 1 \times \dots \times 1) + \dots + L'(1 \times \dots \times 1 \times \mathrm{P}^1{}_*),$$

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just as if $L'$ were an iterated cup-product. But $P^1{}_*x$ and $P^1{}_*\beta x$ vanish, so $P^1{}_*\mathrm{ad}_3{}^{p-1}(x)(\beta x) = P^1{}_*L'(x, \ldots, x, \beta x)$ vanishes as required. This completes the proof.

Suppose now that $BU_p{}^{\otimes}$ is a fourth loop space, so that $U_p$ is a fifth loop space. Recall $H_*(U_p)$ from 2.11. We see that in $H_*(U_p)$

$$Q^1 a_1' = a_1'^p \neq 0.$$

By (5.1),

$$P^1{}_*Q^2 a_1' = Q^1 a_1' \neq 0,$$

so $Q^2 a_1' \neq 0$. Since $Q^2 a_1'$ is primitive, $Q^2 a_1'$ is a non-zero multiple of $a_{2p-1}'$. Since $Q^2$ commutes with suspension, (3.4) shows that in $H_*(BU_p{}^{\otimes})$

(5.3) $\quad Q^2 x_1''$ is a non-zero multiple of $x_{2p-1}''$.

Now consider $\xi_3 x_1''$. It is primitive, so must be a multiple of $x_{3p-2}''$ by 3.3. From (3.7) we deduce that

$$P^1{}_*\beta\xi_3 x_1'' = \beta P^1{}_*\xi_3 x_1''.$$

By 5.2, this gives $3\beta Q^2 x_1'' - Q^2\beta x_1'' = 2\beta Q^2 x_1''$; that is,

$$Q^2\beta x_1'' = \beta Q^2 x_1''.$$

By (5.3), $Q^2\beta x_1''$ is therefore a non-zero multiple of $\beta x_{2p-1}''$; use of (4.5) then shows that $Q^2 g_1$ is indecomposable.

Henceforth we shall work in $QH_*(BU_p{}^{\otimes})$; we shall use (4.3), (4.4) and (5.1) repeatedly. So far we have $Q^2 g_1 \neq 0$. Therefore

$$P^p{}_*Q^{p+1}g_p = Q^2 g_1 \neq 0, \quad Q^{p+1}g_p \neq 0, \quad P^{p^2}{}_*Q^{p^2+1}g_{p^2} = Q^{p+1}g_p \neq 0, \quad Q^{p^2+1}g_{p^2} \neq 0,$$

$Q^{p^2+1}g_{p^2}$ is a non-zero multiple of $f_{p^3+p-1}$.

Now

$$P^{p^2-p}{}_*f_{p^3+p-1} = f_{2p^2-1},$$

so

$$P^{p^2-p}{}_*Q^{p^2+1}g_{p^2} \neq 0.$$

That is,

$$0 \neq P^{p^2-p}{}_*Q^{p^2+1}g_{p^2} = Q^{p+1}g_{p^2}.$$

However $2(p+1) - \deg(g_{p^2}) = 2(p+1) - 2p^2$ is negative, so $Q^{p+1}g_{p^2} = 0$. This contradiction completes the proof.

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