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CURVATURE, GEODESICS AND THE BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD II

EXPLOSION PROPERTIES

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§1. Introduction

Let M be an n-dimensional, complete, connected and non compact Riemannian manifold and g be its metric. Δ_M denotes the Laplacian on M.

The Brownian motion on the Riemannian manifold M is defined to be the unique minimal diffusion process $(X_t, \zeta, P_x, x \in M)$ associated with the Laplacian Δ_M where $\zeta(\omega)$ is the explosion time of $X_t(\omega)$ i.e. if $\zeta(\omega) < +\infty$, then $\lim_{t \to \zeta(\omega)} X_t(\omega) = \infty$.

In the previous paper [3], the author has discussed recurrence and transience of the Brownian motion X on M. This paper may be considered to be a continuation, in which the relation between explosions of the Brownian motion X and geodesics, curvature of the Riemannian manifold M will be investigated. It should be remarked that Yau [7] has given a sufficient condition for no explosion of the Brownian motion in terms of the Ricci curvature.

Let us begin with the Brownian motion $X^0 = (X_t^0, \zeta^0, P_x^0, x \in M_0)$ on a model (M_0, g_0) where the model (M_0, g_0) is defined to be a Riemannian manifold $R^n = [0, +\infty) \times S^{n-1}$ given a metric $dr^2 + g_0(r)^2 d\theta^2$, $(r, \theta) \in (0, +\infty) \times S^{n-1}$. See Ichihara [3] for the precise definition. Then by the same reasoning as in Ichihara [3] Section 1, we obtain from Fellers tests for explosions, Mckean [5],

PROPOSITION 1.1. *It holds whether*

$$P_x^0\{\zeta^0 = +\infty\} = 1 \qquad \text{on } M$$

or

$$P_x^0\{\zeta^0 = +\infty\} = 0 \qquad \text{on } M$$

according as

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$$\int^{+\infty} g_0(r)^{-n+1}dr \int^r g_0(s)^{n-1}ds = +\infty \quad or \quad < +\infty .$$

§ 2. Tests for explosions of the Brownian motion on a Riemannian manifold M

Let normal, minimal geodesics be defined as in Ichihara [3]. Ric_M and K_M denote the Ricci, and sectional curvatures respectively. $K_0(r)$, $r \geqq 0$ is the radial sectional curvature of a model (M_0, g_0) defined in Ichihara [3].

Our main theorems are stated as follows.

THEOREM 2.1. *If for some $p \in M$ there exists a model (M_0, g_0) satisfying the following two conditions* (i) *and* (ii), *then no explosion for the Brownian motion X is possible. i.e.*

$$P_x\{\zeta = +\infty\} = 1 \qquad on\ M .$$

(i) *For every minimal geodesic $m(r) : [0, \ell(m)) \to M$, $m(0) = p$,*

$$\mathrm{Ric}_M(\dot{m}(r)) \geqq (n-1)K_0(r) \qquad on\ [0, \ell(m)) .$$

(ii) $\displaystyle\int^{+\infty} g_0(r)^{-n+1}dr \int^r g_0(s)^{n-1}ds = +\infty .$

THEOREM 2.2. *Let M be simply connected. If for some $p \in M$ there exists a model (M_0, g_0) satisfying the following two conditions* (i) *and* (ii), *then explosion for the Brownian motion X is sure. i.e.*

$$P_x\{\zeta < +\infty\} = 1 \qquad on\ M .$$

(i) *For every normal geodesic $m(r) : [0, +\infty) \to M, m(0) = p$,*

$$K_M(\dot{m}(r), X) \leqq K_0(r)\ for\ every\ unit\ vector\ X \in N(\dot{m}(r)) \quad on\ [0, +\infty)$$

(ii) $\displaystyle\int^{+\infty} g_0(r)^{-n+1}dr \int^r g_0(s)^{n-1}ds < +\infty .$

In order to prove the above theorems, we shall introduce the following notations.

$$\sigma_\rho(\omega) = \inf\{t > 0 \mid d(p, X_t(\omega)) \geqq \rho\} , \qquad \rho > 0$$
$$u_\rho(x) = E_x\{e^{-\sigma_\rho}\} , \qquad \Sigma_\rho = \{x \in M \mid d(p, x) < \rho\}$$

where $d(x, y)$ is the distance induced by the Riemannian metric. σ_ρ^0, u_ρ^0 and Σ_ρ^0 denote the corresponding ones of the Brownian motion on a model

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(M_0, g_0) centered at $p =$ the origin 0.

The following proposition will be proved in a similar way to that of Ichihara [2].

PROPOSITION 2.1. *For each* $\rho \in (0, +\infty)$, $u_\rho \in C^\infty(\Sigma_\rho)$ *and* $\Delta_M u_\rho - u_\rho = 0$ in Σ_ρ. Furthermore in case of a model (M_0, g_0)

$$\lim_{\substack{y \to x \\ y \in \Sigma_\rho^0}} u_\rho^0(y) = 1$$

for each $x \in \partial(\Sigma_\rho^0)$, the boundary of Σ_ρ^0.

Proof of Theorem 2.1. Since M_0 is rotationally symmetric about 0, $u_\rho^0(x)$ is a radial function. i.e.

$$u_\rho^0(x) = u_\rho^0(r) \qquad \text{for } x = (r, \theta) \in M_0 .$$

Thus $u_\rho^0 \in C^\infty([0, \rho))$ satisfies

$$\frac{d^2 u_\rho^0(r)}{dr^2} + \frac{(n-1)}{g_0(r)} \frac{dg_0(r)}{dr} \frac{du_\rho^0(r)}{dr} = u_\rho^0(r)$$

on $(0, \rho)$. Note that $u_\rho^0(r)$ is, by definition, an increasing function of r. Set $\tilde{u}_\rho(x) = u_\rho^0(d(p, x))$. Therefore following an argument similar to Yau [6], Appendix, we can obtain under the assumption (i) that

$$\Delta_M \tilde{u}_\rho(x) \leqq \Delta_{M_0} u_\rho^0(r)$$

for $r = d(p, x) < \rho$, in the distribution sense. Consequently

$$\Delta_M \tilde{u}_\rho - \tilde{u}_\rho \leqq \Delta_{M_0} u_\rho^0 - u_\rho^0 = 0 \qquad \text{in } \Sigma_\rho .$$

Set

$$\Phi_\rho(x) = u_\rho(x) - \tilde{u}_\rho(x) ,$$

then it holds that

$$\Delta_M \Phi_\rho - \Phi_\rho = (\Delta_M u_\rho - u_\rho) - (\Delta_M \tilde{u}_\rho - \tilde{u}_\rho) \geqq 0 .$$

i.e.

$$\Delta_M \Phi_\rho \geqq \Phi_\rho \qquad \text{in } \Sigma_\rho$$

in the distribution sense.

We shall show that for each $\rho > 0$

$$\Phi_\rho(x) \leqq 0 \qquad \text{in } \Sigma_\rho .$$

Suppose on the contrary that with some $\rho_0 > 0$

$$\sup_{x \in \Sigma_{\rho_0}} \Phi_{\rho_0}(x) > 0 .$$

Since (*) Φ_{ρ_0} is continuous in Σ_{ρ_0} and

(**) $$\varlimsup_{\substack{y \to x \\ y \in \Sigma_{\rho_0}}} \Phi_{\rho_0}(y) \leqq 0 \text{ for each } x \in \partial\Sigma_{\rho_0}$$

from Proposition 2.1, there exists a point $x_0 \in \Sigma_{\rho_0}$ such that

$$\Phi_{\rho_0}(x_0) = \sup_{x \in \Sigma_{\rho_0}} \Phi_{\rho_0}(x) > 0 .$$

Set

$$C = \{x \in \Sigma_{\rho_0} | \Phi_{\rho_0}(x) > 0\} .$$

Denote by C_{x_0} the connected component containing the point x_0 of the set C. Then from the facts (*) and (**),

$$\varlimsup_{\substack{y \to x \\ y \in C_{x_0}}} \Phi_{\rho_0}(y) \leqq 0 \qquad \text{for each } x \in \partial C_{x_0} .$$

Since Φ_{ρ_0} is weakly Δ_M-subharmonic in C_{x_0}, applying the strong maximum principle in Littman [4] we obtain

$$\Phi_{\rho_0}(x) = \Phi_{\rho_0}(x_0) \qquad \text{for each } x \in C_{x_0} ,$$

which is a contradiction. Thus we have shown that for each $\rho > 0$,

$$\Phi_\rho(x) \leqq 0 \qquad \text{in } \Sigma_\rho .$$

i.e. $$u_\rho(x) \leqq \tilde{u}_\rho(x) \qquad \text{for every } x \in \Sigma_\rho .$$

Under the assumption (ii) in Theorem 2.1, the Brownian motion X^0 on the model (M_0, g_0) is conservative. (See Proposition 1.1.)

i.e. $$P_x^0\{\zeta^0 = +\infty\} = 1 \qquad \text{on } M_0 .$$

Moreover

$$u_\rho^0(r) = u_\rho^0(x) = E_x^0\{e^{-\sigma_\rho^0}\}$$

converges to

$$E_x^0\{e^{-\zeta^0}\}$$

for each $x = (r, \theta) \in M_0$ because $\sigma_\rho^0 \to \zeta^0$ as $\rho \to +\infty$. Thus we see that

$$\lim_{\rho\to+\infty} u_\rho^0(r) = 0 \qquad \text{for every } r \geqq 0 .$$

Hence it follows from the inequality proved above that

$$\lim_{\rho\to+\infty} u_\rho(x) = 0 \qquad \text{for every } x \in M .$$

Since $\sigma_\rho \to \zeta$ as $\rho \to +\infty$, we see that

$$0 = \lim_{\rho\to+\infty} u_\rho(x) = E_x\{e^{-\zeta}\} \qquad \text{for every } x \in M .$$

Thus we can conclude

$$P_x\{\zeta = +\infty\} = 1$$

on M. q.e.d.

Proof of Theorem 2.2. We first note that under the assumptions $\exp_p$ maps $T_p(M)$ diffeomorphically onto M as shown in Ichihara [3]. Thus we have geodesic polar coordinates $(r, \theta) \in (0, +\infty) \times S^{n-1}$ centered at p.

Now define $v = v(r)$, $r \geqq 1$ to be the positive increasing solution:

$$v = \sum_{m=0}^{\infty} v_m \qquad v_0 = 1$$

$$v_m(r) = \int_1^r g_0(s)^{-n+1} ds \int_0^s g_0(t)^{n-1} v_{m-1}(t) dt , \qquad m \geqq 1$$

of

$$\frac{1}{g_0(r)^{n-1}} \frac{d}{dr}\left(g_0(r)^{n-1} \frac{dv(r)}{dr}\right) = v(r) , \qquad r \geqq 1.$$

Then it can be easily seen that

$$v(r) \leqq \exp\{v_1(r)\}$$

for every $r \geqq 1$ and so $v(r)$ is bounded above from the assumption (ii) of Theorem 2.2.

Set $\tilde{v}(x) = v(d(p, x))$. Then with the geodesic polar coordinates (r, θ) and $G(r, \theta) = \sqrt{\det(g_{ij})}(r, \theta)$ where $g = g_{ij} dx_i dx_j$, we have

$$\Delta_M \tilde{v}(x) = \frac{d^2 v(r)}{dr^2} + \frac{1}{G(r, \theta)} \frac{\partial G(r, \theta)}{\partial r} \frac{dv(r)}{dr}\bigg|_{r=d(p,x)} .$$

By virtue of Hessian comparison theorem, Greene and Wu [1]

$$\geqq \frac{d^2 v(r)}{dr^2} + \frac{(n-1)}{g_0(r)} \frac{dg_0(r)}{dr} \frac{dv(r)}{dr}\bigg|_{r=d(p,x)} = v(d(p, x)) = \tilde{v}(x) .$$

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Now applying Itô's formula to the function $e^{-t}\tilde{v}(x)$, we obtain from the above inequality that

$$v(\rho)E_x\{e^{-\sigma_\rho}, \rho_\rho \leqq \tau_1\} + E_x\{e^{-\tau_1}, \sigma_\rho > \tau_1\} \geqq \tilde{v}(x)$$

for each $x \in \Sigma_\rho - \Sigma_1$ where $\tau_1(\omega) = \inf\{t > 0 | d(p, X_t(\omega)) \leqq 1\}$. Letting $\rho \to +\infty$, we have

$$v(\infty)E_x\{e^{-\zeta}, \zeta < \tau_1\} + E_x\{e^{-\tau_1}, \zeta > \tau_1\} \geqq \tilde{v}(x).$$

because $\sigma_\rho \to \zeta$ as $\rho \to +\infty$.

We shall show

(∗) $\quad E_x\{e^{-\tau_1}, \tau_1 < \zeta\} \leqq P_x\{\tau_1 < \zeta\} \longrightarrow 0 \quad$ *as* $d(p, x) \to +\infty$.

Set

$$\psi_\rho(r) = \frac{\int_r^\rho g_0(s)^{-n+1}ds}{\int_1^\rho g_0(s)^{-n+1}ds}, \qquad \Psi_\rho(x) = \psi_\rho(d(p, x))$$

and

$$\phi_\rho(x) = P_x\{\tau_1 < \sigma_\rho\} \qquad \text{for each } \rho > 1.$$

Then it is easy to see that

$$\Delta_M\phi_\rho = 0 \qquad \text{in } \Sigma_\rho - \bar{\Sigma}_1$$

$$\phi_\rho(x) = \begin{cases} 1 & \text{if } d(p, x) = 1 \\ 0 & \text{if } d(p, x) = \rho \end{cases}$$

and

$$\Psi_\rho(x) = \begin{cases} 1 & \text{if } d(p, x) = 1 \\ 0 & \text{if } d(p, x) = \rho. \end{cases}$$

Furthermore Hessian comparison theorem [1] gives that

$$\Delta_M\Psi_\rho \leqq 0 \qquad \text{in } \Sigma_\rho - \bar{\Sigma}_1.$$

Consequently we can deduce by virtue of the maximum principle,

$$\phi_\rho(x) \leqq \Psi_\rho(x) \qquad x \in \Sigma_\rho - \bar{\Sigma}_1.$$

i.e. $\quad P_x\{\tau_1 < \sigma_\rho\} \leqq \psi_\rho(d(p, x)).$

Since $\sigma_\rho \to \zeta$ as $\rho \to +\infty$, we get

$$P_x\{\tau_1 < \zeta\} \leqq \frac{\int_{d(p,x)}^{\infty} g_0(r)^{-n+1}dr}{\int_1^{\infty} g_0(r)^{-n+1}dr} \qquad \text{for } d(p, x) > 1\,,$$

which gives the desired result (*). Thus we obtain from (*)

$$\varliminf_{x\to\infty} E_x\{e^{-\zeta}, \zeta < \tau_1\} \geqq 1$$

and so

$$\varliminf_{x\to\infty} P_x\{\zeta < \infty\} \geqq \varliminf_{x\to\infty} E_x\{e^{-\zeta}\} \geqq 1\,.$$

By the strong Markov property

$$P_x\{\zeta < +\infty\} = E_x\{P_{X_{\sigma_\rho}}\{\zeta < \infty\}\}$$

for every $\rho > d(p, x)$ and hence

$$= \varliminf_{\rho\to+\infty} E_x\{P_{X_{\sigma_\rho}}\{\zeta < +\infty\}\} \geqq E_x\{\varliminf_{\rho\to+\infty} P_{X_{\sigma_\rho}}\{\zeta < +\infty\}\} \geqq 1\,.$$

This completes the proof. q.e.d.

§ 3. Some examples

In [7], Yau has shown that no explosion for the Brownian motion is possible if the Ricci curvature of M is bounded from below by a constant. We shall extend this result as follows.

1. If for a fixed $p \in M$ and every minimal geodesic $m(r) : [0, \ell(m)) \to M$, $m(0) = p$,

$$\mathrm{Ric}_M(\dot m(r)) \geqq -C_1r^2 - C_2 \qquad \text{on } [0, \ell(m))$$

with positive constants C_i $i = 1, 2$, then no explosion for the Brownian motion X is possible.

Proof. In order to prove this, it is enough to show the existence of a model (M_0, g_0) which satisfies the conditions (i) and (ii) in Theorem 2.1.

Set $K_0(r) = -C_1r^2 - C_2$, $r \in [0, +\infty)$ and let $g_0(r) \in C([0, +\infty))$ be the unique solution of the following Jacobi equation.

$$\frac{d^2g_0(r)}{dr^2} = -K_0(r)g_0(r) \qquad g_0(0) = 0, \qquad \frac{dg_0}{dr}(0) = 1\,.$$

Then the Sturm comparison theorem asserts that $g_0(r) > r$ for every $r > 0$.

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Thus we have obtained a model (M_0, g_0) satisfying (i) in Theorem 2.1.

It remains to verify the condition (ii). In order to do it, we shall introduce the function

$$g_1(r) = \exp\{kr^2\}$$

with a positive constant k. Define

$$K_1(r) = -\frac{1}{g_1(r)}\frac{d^2g_1(r)}{dr^2} = -4k^2r^2 - 2k\,.$$

For a fixed positive number r_0, it is easily seen that with a sufficiently large k

$$(*) \qquad K_1(r) \leqq K_0(r) \qquad \text{for every } r \geqq r_0$$

and

$$(**) \qquad \frac{1}{g_1(r_0)}\frac{dg_1}{dr}(r_0) \geqq \frac{1}{g_0(r_0)}\frac{dg_0}{dr}(r_0)\,.$$

From the equations $(d^2g_i(r)/dr^2) = -K_i(r)g_i(r)$, $i = 0, 1$, we have, for every $r \geqq r_0$,

$$\begin{aligned} 0 &= g_1(r)\frac{d^2g_0(r)}{dr^2} - \frac{d^2g_1(r)}{dr^2}g_0(r) + (K_0(r) - K_1(r))g_1(r)g_0(r) \\ &= \frac{d}{dr}\left(g_1(r)\frac{dg_0(r)}{dr}\right) - \frac{d}{dr}\left(g_0(r)\frac{dg_1(r)}{dr}\right) + (K_0(r) - K_1(r))g_1(r)g_0(r)\,. \end{aligned}$$

Hence we see from $(*)$

$$\left[g_1(s)\frac{dg_0(s)}{ds} - g_0(s)\frac{dg_1(s)}{ds}\right]_{r_0}^{r} = \int_{r_0}^{r}(K_1(s) - K_0(s))g_0(s)g_1(s)ds \leqq 0\,.$$

Therefore it follows from $(**)$ that

$$g_1(r)\frac{dg_0(r)}{dr} - g_0(r)\frac{dg_1(r)}{dr} \leqq 0\,.$$

i.e.

$$\frac{1}{g_1(r)}\frac{dg_1(r)}{dr} \geqq \frac{1}{g_0(r)}\frac{dg_0(r)}{dr}$$

for every $r \geqq r_0$.

Set

$$G_i(r) = \int_{r_0}^{r} g_i(u)^{-n+1}du \int_{r_0}^{u} g_i(v)^{n-1}dv \qquad i = 0, 1\,.$$

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Then these functions satisfy

$$\begin{cases} \dfrac{d^2G_i(r)}{dr^2} + B_i(r)\dfrac{dG_i(r)}{dr} = 1 & \text{on } [r_0, +\infty) \\ G_i(r_0) = \dfrac{dG_i}{dr}(r_0) = 0 \end{cases}$$

where

$$B_i(r) = \frac{1}{g_i(r)} \frac{dg_i(r)}{dr}.$$

Since $B_1(r) \geqq B_0(r)$ on $[r_0, +\infty)$ and G_1 is an increasing function, we have

$$1 = \frac{d^2G_1(r)}{dr} + B_1(r)\frac{dG_1(r)}{dr} \geqq \frac{d^2G_1(r)}{dr^2} + B_0(r)\frac{dG_1(r)}{dr}.$$

Solving this differential inequality, we can easily see that

$$G_0(r) \geqq G_1(r)$$

for every $r \geqq r_0$.

Thus in order to verify the condition (ii), it suffices to show

$$G_1(+\infty) = +\infty .$$

We now compute

$$\begin{aligned} G_1(+\infty) &= \int_{r_0}^{+\infty} dr \int_{r_0}^{r} \exp\{-(n-1)kr^2 + (n-1)kt^2\}dt \\ &= \int_{r_0}^{+\infty} dr \int_{r}^{+\infty} \exp\{(n-1)kr^2\}\cdot\exp\{-(n-1)kt^2\}dt . \end{aligned}$$

Using the following inequality

$$\begin{aligned} &\int_{r}^{+\infty} \exp\{-(n-1)kt^2\}dt \\ &\geqq \frac{1}{\sqrt{(n-1)k}}\left(\sqrt{(n-1)k}\,r + \frac{1}{\sqrt{(n-1)k}\,r}\right)^{-1} \exp\{-(n-1)kr^2\}, \end{aligned}$$

we have

$$\geqq \int_{r_0}^{+\infty} ((n-1)kr + 1)^{-1}dr = +\infty .$$

This completes the proof. q.e.d.

The next example will be shown in a way similar to the proof of Example 1.

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2. Suppose M is simply connected and negatively curved. If for a fixed $p \in M$ and every normal geodesic $m(r) : [0, +\infty) \to M$, $m(0) = p$

$$K_M(\dot{m}(r), \cdot) \leqq -C_1 r^{2+\delta} \qquad \text{for every } r \geqq C_2$$

with positive constants C_i, $i = 1, 2$ and δ, then explosion for the Brownian motion X on M is sure.

3. Let S_n be an embeded hypersurface in R^{n+1} defined by

$$x_{n+1} = f(x_1, \cdots, x_n).$$

Suppose f is a radial function, then the Brownian motion X on S_n is conservative

i.e. $$P_x\{\zeta = +\infty\} = 1 \qquad \text{on } S_n.$$

Proof. Since f is a radial function, using polar coordinates (r, θ) of R^n, we have

$$\begin{aligned} dx_1^2 + \cdots + dx_n^2 + dx_{n+1}^2 &= dr^2 + r^2 d\theta^2 + f_r^2 dr^2 \\ &= (1 + f_r^2)dr^2 + r^2 d\theta^2 . \end{aligned}$$

As in Example 4 [3], we can obtain the geodesic polar coordinates (s, θ) with

$$dx_1^2 + \cdots + dx_n^2 + dx_{n+1}^2 = ds^2 + g_0(s)^2 d\theta^2$$

where

$$p(r) = \int_0^r \sqrt{1 + f_u^2}\, du , \qquad r \geqq 0$$
$$s = p(r)$$

and $g_0(r)$ is the inverse function of p. i.e. $s = p(g_0(s))$.

Notice that

$$B_0(s) = \frac{1}{g_0(s)} \frac{dg_0(s)}{ds} = \frac{1}{r} \frac{1}{\sqrt{1 + f_r^2}}$$

is convergent to zero as $s \to +\infty$. Set $g_1(s) = e^s$, then we have

$$B_1(s) = \frac{1}{g_1(s)} \frac{dg_1(s)}{ds} = 1 .$$

Consequently it holds that for some $r_0 > 0$,

$$B_1(s) \geqq B_0(s) \qquad \text{on } [r_0, +\infty) .$$

Now applying the comparison argument in page 123 we get that

$$(***) \qquad \int_{r_0}^{r} g_0(u)^{-n+1}du \int_{r_0}^{u} g_0(v)^{n-1}dv \leqq \int_{r_0}^{r} g_1(u)^{-n+1}du \int_{r_0}^{u} g_1(v)^{n-1}dv\,.$$

It is easy to see that the right hand of the above inequality (***) is divergent to $+\infty$ when r tends to $+\infty$. Thus we have

$$\int_{r_0}^{+\infty} g_0(r)^{-n+1}dr \int_{r_0}^{r} g_0(s)^{n-1}ds = +\infty$$

which implies $P_x\{\zeta = +\infty\} = 1$ on S_n.

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