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SUBNORMAL SUBGROUPS IN DIRECT PRODUCTS OF GROUPS

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Abstract

A group G is called *normally* (*subnormally*) *detectable* if the only normal (subnormal) subgroups in any direct product $G_1 \times \cdots \times G_n$ of copies of G are just the direct factors G_i. We give an internal characterization of finite subnormally detectable groups and obtain analogous results for associative rings and for Lie algebras. The main part of the paper deals with a study of normally detectable groups, where we verify a conjecture of T. O. Hawkes in a number of special cases.

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1. Introduction

During the Warwick Symposium on Soluble Groups in 1977, T. O. Hawkes asked the following question: under which conditions is a finite group "normally detectable"? Here a group G is called *normally detectable* if in any direct product $G_1 \times \cdots \times G_n$, where $G_i \cong G$ for $i = 1, \ldots, n$, the direct factors $G_1, \ldots, G_n$ are the only normal subgroups isomorphic to G. Hawkes conjectured that this is the case if and only if G is directly indecomposable and $|G/G'|$ and $|Z(G)|$ are coprime. It is easy to see that these two properties are necessary for G to be normally detectable. Even more, a well-known result of Remak [6] states that exactly under these conditions on G the groups G_i are the only direct factors of

$G_1 \times \cdots \times G_n$ which are isomophic to G. Thus the validity of Hawkes' conjecture would mean a significant generalization of Remak's theorem. However, the motivation for studying this problem stems from the theory of finite soluble groups; Hawkes was led to the concept of a normally detectable group in connection with certain Fitting class constructions. For details the reader is referred to the forthcoming book of Doerk and Hawkes [3].

In this paper we shall deduce a series of partial results on Hawkes' conjecture. A final answer, however, is not obtained. We shall show that a finite group G is normally detectable if either one of the conditions stated above is somewhat strengthened. For instance, if the coprimeness condition on $|Z(G)|$ and $|G/G'|$ is required not only for the directly indecomposable group G but also for certain non-nilpotent factor groups of G, then G is normally detectable (Theorem 6.5).

On the other hand, if, apart from direct decompositions, some specific factorizations of the form $G = NS$, where N is a non-nilpotent normal subgroup of G and S is a non-nilpotent subnormal complement for N in G, are also excluded, then a group G with $(|Z(G)|, |G/G'|) = 1$ is normally detectable (Theorem 6.2). We also show that Hawkes' conjecture is valid for groups which satisfy certain conditions on the structure of the automorphism group or on the size of the socle, etc. (Theorem 6.9). In the course of proving these results, we not only obtain information about the structure but also about the kind of embedding of a possible counterexample to Hawkes' conjecture (Theorem 6.8).

In contrast to the case of normally detectable groups, *subnormally detectable groups* (which are defined in the obvious way) are much easier to handle. It turns out that a finite group G is subnormally detectable if and only if G is directly indecomposable and $(|G/G'|, |\mathrm{Fit}(G)|) = 1$ (Theorem 4.2).

It is the transitivity of the subnormality relation which makes our method (iterated embeddings) work smoothly. Therefore, it is not surprising that similar arguments provide a short proof of the fact that a finite group $G \neq 1$ can never be characteristic in $G_1 \times \cdots \times G_n$ ($G_i \cong G$ for $i = 1, \ldots, n$) in case $n \geqslant 2$ (Proposition 7.1).

Finally, the result on subnormally detectable groups carries over to Lie algebras and associative rings; this is shown in Theorems 7.3 and 7.5.

The groups considered in this paper are not assumed to be finite; however, we have to impose certain finiteness conditions. Usually, the minimal condition on subnormal subgroups is needed. For most of the results on normally detectable groups we even have to assume that the group in question possesses a finite composition series.

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2. Notation

Iterated commutators are assumed to be normed from the left. The notation $S \trianglelefteq\trianglelefteq G$ means that S is a subnormal subgroup of G, and the defect of a subnormal subgroup S of G is defined to be the smallest integer d such that there exists a chain $S = S_0 \triangleleft S_1 \triangleleft \cdots \triangleleft S_d = G$. For a subgroup S of G, S^G denotes the smallest normal subgroup of G containing S and $\mathrm{Core}_G(S)$ denotes the largest normal subgroup of G contained in S. $\mathrm{Fit}(G)$ stands for the Fitting subgroup of a group G and $\mathrm{Soc}(G)$ for the socle of G, the subgroup generated by all minimal normal subgroups of G.

For a direct product $G_1 \times \cdots \times G_n$, the natural projection onto G_i is always denoted by π_i. Quite frequently we shall use k-tuples $J_k = (j_1, \ldots, j_k)$ of positive integers as indices for the factors in a direct product. Correspondingly, the projections are denoted by π_{J_k}. If m and k are positive integers, $\mathbf{m}^k$ stands for the cartesian product $\{1, \ldots, m\} \times \cdots \times \{1, \ldots, m\}$ (k times). For $J = (j_1, \ldots, j_k) \in \mathbf{m}^k$ and $I = (i_1, \ldots, i_t) \in \mathbf{m}^t$, we write (J, I) for the $(k + t)$-tuple $(j_1, \ldots, j_k, i_1, \ldots, i_t) \in \mathbf{m}^{k+t}$. When k-tuples $J = (j_1, \ldots, j_k)$ are used as indices for groups (or maps), G_J, $G_{j_1, \ldots, j_k}$, and G_{J_0, j_k} denote the same object (where $J_0 = (j_1, \ldots, j_{k-1})$). If σ is a permutation on $\{1, \ldots, k\}$, and if $J = (j_1, \ldots, j_k) \in \mathbf{m}^k$, then $J\sigma$ is the k-tuple $(j_{1\sigma}, \ldots, j_{k\sigma})$.

Finally, (a, b) is used as the notation for the greatest common divisor of the positive integers a and b.

Group theoretical notations not explained here are consistent with those used by Huppert [4].

3. Preparatory lemmas

Groups considered in this paper are usually assumed to satisfy certain finiteness conditions. The minimal condition on subnormal subgroups is needed frequently, often together with the maximal condition on subnormal subgroups. Clearly, these are just the groups possessing a finite composition series. The purpose of this section is to collect some basic results on nilpotent groups with finiteness conditions and to prove some elementary statements about certain subgroups in direct products.

The following lemma is well known (see e.g. [8, Section 12]).

3.1. LEMMA. (a) *If G is a group satisfying the maximal or minimal condition on subnormal subgroups, then* $\mathrm{Fit}(G)$ *is nilpotent and every nilpotent subnormal subgroup of G is contained in* $\mathrm{Fit}(G)$.

(b) *If G is a group with finite composition series, then every nilpotent factor group of a subnormal subgroup of G is finite. In particular,* $\mathrm{Fit}(G)$ *is finite.*

3.2. LEMMA. *Let G be a finitely generated nilpotent group and N a non-trivial central subgroup of G. Then there exists a non-trivial homomorphism from G into N.*

PROOF. Let T be the torsion subgroup of G. If $G \neq T$, then G/T has an infinite cyclic factor group. If $G = T$, then G is finite. In both cases the assertion is clear.

The following lemma contains some basic facts on direct products which will be used frequently in the sequel.

3.3. LEMMA. *Let G be a subgroup of $H_1 \times \cdots \times H_n$ and let I be a subset and i an element of $\{1, \dots, n\}$.*

(a) *If G is subnormal in $G\pi_1 \times \cdots \times G\pi_n$ of defect d, then $G\pi_i/(G \cap H_i)$ is nilpotent of class at most d.*

(b) *If G is normal in $H_1 \times \cdots \times H_n$, then $G\pi_i/(G \cap H_i) \leqslant Z(H_i/(G \cap H_i))$.*

(c) *If G is characteristic in $H_1 \times \cdots \times H_n$, then $G\pi_i/(G \cap H_i)$ is centralized by every automorphism of H_i.*

In parts (d)–(f) *let $H_j \cong G$ for all $j = 1, \dots, n$.*

(d) *Let G satisfy the maximal or minimal condition on subnormal subgroups. If G is subnormal in $H_1 \times \cdots \times H_n$, and if there exists no non-trivial homomorphism from G into* $\mathrm{Fit}(G)$, *then $G(\sum_{j \in I} \pi_j) = 1$ if and only if $G \cap \times_{j \in I} H_j = 1$.*

(e) *Let G satisfy the maximal condition on subnormal subgroups. If G is subnormal in $H_1 \times \cdots \times H_n$, and if there exists no non-trivial homomorphism from G into $Z(G)$, then $G(\sum_{j \in I} \pi_j) \leqslant \mathrm{Fit}(\times_{j \in I} H_j)$ if and only if $G \cap \times_{j \in I} H_j = 1$.*

(f) *If G is normal in $H_1 \times \cdots \times H_n$, and if there exists no nontrivial homomorphism from G into $Z(G)$, then $G(\sum_{j \in I} \pi_j) = 1$ if and only if $G \cap \times_{j \in I} H_j = 1$.*

PROOF. Clearly, $G \cap H_i \trianglelefteq G\pi_i$.

(a) Let $h_{i0}, h_{i1}, \dots, h_{id} \in G\pi_i$ be arbitrary. There exist $h_{jk} \in G\pi_j$, $j = 1, \dots, n$, $j \neq i$, $k = 1, \dots, d$, such that $g_k = h_{ik}\prod_{j \neq i} h_{jk} \in G$ for $k = 1, \dots, d$. Since G is subnormal in $G\pi_1 \times \cdots \times G\pi_n$ of defect d, we conclude that $[h_{i0}, h_{i1}, \dots, h_{id}] = [h_{i0}, g_1, \dots, g_d] \in G \cap H_i$.

(b) and (c) Let $h_i \in G\pi_i$ and $\alpha \in \mathrm{Inn}(H_i)$ or $\alpha \in \mathrm{Aut}(H_i)$, respectively. Then α induces an automorphism $\bar{\alpha}$ on $H_1 \times \cdots \times H_n$ in the obvious way (and $\bar{\alpha}$ is inner if α is inner). Choose $h_j \in H_j$, $j = 1, \ldots, n$, $j \neq i$, such that $g = \prod_{j=1}^{n} h_j \in G$. Then $[h_i, \alpha] = [g, \bar{\alpha}] \in G \cap H_i$.

(d) and (e) If $G \cap \times_{j \in I} H_j = 1$, then $G\pi_j \leqslant \mathrm{Fit}(H_j)$ for all $j \in I$ by (a) and 3.1(a). This proves (d) and one part of (e). For the second part of (e), suppose that $G(\sum_{j \in I} \pi_j)$ is nilpotent but that $G \cap \times_{j \in I} H_j \neq 1$. $G \cap \times_{j \in I} H_j \trianglelefteq G(\sum_{j \in I} \pi_j)$ implies that $G \cap Z(G(\sum_{j \in I} \pi_j)) \neq 1$. As an epimorphic image of G, the nilpotent group $G(\sum_{j \in I} \pi_j)$ is finitely generated. By 3.2, there exists a non-trivial homomorphism from $G(\sum_{j \in I} \pi_j)$ into $G \cap Z(G(\sum_{j \in I} \pi_j)) \leqslant Z(G)$, which is a contradiction.

(f) This follows from (b).

3.4. DEFINITION. Let G be a subgroup of $H_1 \times \cdots \times H_n$. Then G is called *trivially embedded* in $H_1 \times \cdots \times H_n$ if $G \cap \times_{j \neq i} H_j = 1$ for some $i \in \{1, \ldots, n\}$.

The following observation is immediate from 3.3(d) and (f).

3.5. REMARK. Let G be a subnormal subgroup of $G_1 \times \cdots \times G_n$, where $G_i \cong G$ for $i = 1, \ldots, n$. Assume further that one of the following conditions is satisfied:

(a) G satisfies the maximal or minimal condition on subnormal subgroups, and there exists no non-trivial homomorphism from G into $\mathrm{Fit}(G)$;

(b) G is normal in $G_1 \times \cdots \times G_n$, and there exists no non-trivial homomorphism from G into $Z(G)$. Then the following statements are equivalent in pairs.

(i) G is trivially embedded.

(ii) There exists $i \in \{1, \ldots, n\}$ such that $G \cap G_j = 1$ for all $j \neq i$.

(iii) There exists $i \in \{1, \ldots, n\}$ such that $G \trianglelefteq\trianglelefteq G_i$.

Trivial subnormal embeddings are easy to describe.

3.6. LEMMA. *Let $H_1, \ldots, H_n$ be groups which satisfy the maximal or minimal condition on subnormal subgroups.*

(a) *If G is trivially subnormally embedded in $H_1 \times \cdots \times H_n$, i.e. $G \cap \times_{j \neq i} H_j = 1$ for some $i \in \{1, \ldots, n\}$, then $G\pi_i \cong G$, and there exist homomorphisms $\alpha_j \colon G\pi_i \to \mathrm{Fit}(H_j)$ for all $j \neq i$ such that $G = \{g_i \cdot \prod_{j \neq i} g_i\alpha_j | g_i \in G\pi_i\}$.*

(b) *For some $i \in \{1, \ldots, n\}$, let G_i be a subgroup of H_i. For $j \neq i$, let $\alpha_j \colon G_i \to \mathrm{Fit}(H_j)$ be homomorphisms. Then the subgroup $G = \{g_i \cdot \prod_{j \neq i} g_i\alpha_j | g_i \in G_i\}$ of $H_1 \times \cdots \times H_n$ is isomorphic to G_i, and $G \cap \times_{j \neq i} H_j = 1$. Moreover, if G_i is subnormal in H_i, then G is subnormal in $H_1 \times \cdots \times H_n$.*

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PROOF. The straightforward proof is left to the reader.

We conclude this section with a description of those subgroups of a direct product of isomorphic groups which are isomorphic to one of the direct factors. This result is due to Remak [7, Satz 3].

3.7. LEMMA (Remak). *Let G be a group with $G \cong G_i$ via isomorphisms φ_i: $G \to G_i$, $i = 1, \ldots, n$. If α_i: $G \to G$, $i = 1, \ldots, n$, are homomorphisms with $\bigcap_{i=1}^n \ker \alpha_i = 1$, then the subgroup $G_0 = \{\prod_{i=1}^n g\alpha_i\varphi_i | g \in G\}$ of $G_1 \times \cdots \times G_n$ is isomorphic to G. Conversely, every subgroup of $G_1 \times \cdots \times G_n$ which is isomorphic to G is obtained in this manner.*

4. Subnormally detectable groups

4.1. DEFINITION. A group G is called *subnormally detectable* if the following holds: whenever G_0 is subnormal in $G_1 \times \cdots \times G_n$ for some positive integer n, where $G_j \cong G$ for $j = 0, 1, \ldots, n$, then $G_0 = G_i$ for some $i \in \{1, \ldots, n\}$.

The following theorem characterizes those subnormally detectable groups that satisfy the minimal condition on subnormal subgroups.

4.2. THEOREM. *Let G be a group satisfying the minimal condition on subnormal subgroups. Then the following statements are equivalent:*

(i) *G is subnormally detectable;*

(ii) *G is directly indecomposable, and there exists no non-trivial homomorphism from G into* $\mathrm{Fit}(G)$.

PROOF. We show first that (i) implies (ii). If $G = A \times B$, $A \neq 1 \neq B$, then let $A_i \cong A$, $B_i \cong B$, and $G_i = A_i \times B_i$ for $i = 1, 2$. Since $G \cong A_1 \times B_2 \trianglelefteq G_1 \times G_2$, G is not subnormally detectable. If α: $G \to \mathrm{Fit}(G)$ is a non-trivial homomorphism, let $G_0 = \{(g, g\alpha) | g \in G\} \leqslant G \times G$. Then $G_0 \cong G$ and $\ker \alpha \times 1 \leqslant G_0 \leqslant G \times \mathrm{Fit}(G)$. By 3.1(a), $(G \times \mathrm{Fit}(G))/(\ker \alpha \times 1)$ is nilpotent. Hence G_0 is subnormal in $G \times G$. Since $G\alpha \neq 1$, G is not subnormally detectable.

Now assume (ii) and suppose that G is not subnormally detectable. Then there exist groups $G_0, G_1, \ldots, G_n$ ($n \geqslant 2$) isomorphic to G such that $G_0 \trianglelefteq\trianglelefteq G_1 \times \cdots \times G_n$

and such that $G_0\pi_i \neq 1$ for $i = 1, \ldots, n$. Among all such subnormal embeddings choose one with n maximal. This is possible: by 3.3(d), $G_0\pi_j \neq 1$ and $G_0 \cap G_j \neq 1$ are equivalent; because of the minimal condition on normal subgroups, each $G_0 \cap G_j$ contains a minimal normal subgroup of G_0; finally, by the minimal condition on normal subgroups again, $\mathrm{Soc}(G_0)$ is a direct product of finitely many minimal normal subgroups. We now define groups $G_{ij} \cong G$, $i, j = 1, \ldots, n$, such that each G_i is subnormally embedded in $G_{i1} \times \cdots \times G_{in}$ in the same way as G_0 is embedded in $G_1 \times \cdots \times G_n$. Clearly then, G_0 is subnormal in $(G_{11} \times \cdots \times G_{1n}) \times \cdots \times (G_{n1} \times \cdots \times G_{nn})$. By the maximal choice of n, at least $n^2 - n$ of the projections $G_0\pi_{ij}$ are trivial. But then $1 \neq G_0\pi_i \leqslant G_0\pi_{i1} \times \cdots \times G_0\pi_{in}$ implies that for each i there exists exactly one index $j(i)$ such that $G_0\pi_{i,j(i)} \neq 1$. It follows that

$$(*) \qquad G_0 \trianglelefteq\trianglelefteq \left(G_1 \cap G_{1,j(1)}\right) \times \cdots \times \left(G_n \cap G_{n,j(n)}\right).$$

Note that $G_i \cap G_{i,j(i)} \neq 1$ for all i (3.3(d)) and that $G_0\pi_i$ satisfies the minimal condition on subnormal subgroups (since $G_0\pi_i \trianglelefteq\trianglelefteq G_i$). Consequently, there exists some $m \in \{1, \ldots, n\}$ such that $G_0\pi_m$ is not isomorphic to a proper subnormal subgroup of any $G_0\pi_i$, $i = 1, \ldots, n$. We note further that $G_0\pi_i \cong G_j\pi_{ji}$ for all $i, j = 1, \ldots, n$: this is immediate from the fact that G_0 is embedded in $G_1 \times \cdots \times G_n$ in the same way as G_j is embedded in $G_{j1} \times \cdots \times G_{jn}$. Employing $(*)$, we now obtain $G_0\pi_m \trianglelefteq\trianglelefteq G_m \cap G_{m,j(m)} \trianglelefteq G_m\pi_{m,j(m)} \cong G_0\pi_{j(m)}$. By the choice of m, $G_0\pi_m$ is not isomorphic to a proper subnormal subgroup of $G_0\pi_{j(m)}$. Hence, $G_m \cap G_{m,j(m)} = G_m\pi_{m,j(m)}$. But this implies that $G_m = (G_m \cap G_{m,j(m)}) \times (G_m \cap \times_{j \neq j(m)} G_{mj})$, contradicting the indecomposability of G_m.

If we use the fact that for finite nilpotent groups G the prime divisors of $|G|$ coincide with those of $|G/G'|$, then the following corollary is an immediate consequence of Theorem 4.2.

4.3. Corollary. *Let G be a finite group. Then the following statements are equivalent*:

(i) *G is subnormally detectable*;

(ii) *G is directly indecomposable and* $(|G/G'|, |\mathrm{Fit}(G)|) = 1$.

Theorem 4.2 also yields the following somewhat more general statement.

4.4. Corollary. *Let G be a group satisfying the minimal condition on subnormal subgroups. Assume that G is directly indecomposable and that there exists no* *non-trivial homomorphism from G into* $\mathrm{Fit}(G)$. *If G is subnormal in $H_1 \times \cdots \times H_n$, where each H_j is isomorphic to a subnormal subgroup of G, then there exists $i \in \{1, \dots, n\}$ such that $G = H_i$.*

5. Subnormal embeddings

For our approach to Hawkes' problem on normally detectable groups it is necessary to obtain some general information about subnormal embeddings of a group G (with finite composition series) in a direct product of groups isomorphic to G. The aim of this section is to provide a proof of the following result.

5.1. THEOREM. *Let G be a group with finite composition series. Assume further that G_0 is subnormal in $G_1 \times \cdots \times G_n$, where $G_j \cong G$ for $j = 0, 1, \dots, n$. Then one of the following holds*:

(a) *G is nilpotent*;

(b) *$G_0 \cap G_i \neq 1$ and $G_0 \cap \times_{j \neq i} G_j = 1$ for some $i \in \{1, \dots, n\}$*;

(c) *G_0 is non-trivially subnormally embedded* (*in the sense of* 3.4); *in this case, $G = NS$, where $N \trianglelefteq G$, $S \trianglelefteq\trianglelefteq G$, $N \cap S = 1$, $N \neq 1$, S is not nilpotent, and $S^G/\mathrm{Core}_G(S)$ is nilpotent*; *in particular*, $[N, S] \leqslant \mathrm{Fit}(G)$.

Since in nilpotent groups all subgroups are subnormal, the embeddings considered in Theorem 5.1 are classified for nilpotent groups by Lemma 3.7.

The trivial embeddings of 5.1(b) are described in Lemma 3.6. In view of Theorem 4.2 one might be led to conjecture that all subnormal embeddings for directly indecomposable groups arise like those in 3.6. However, in 5.12 we give an example of a finite group G (due to John Cossey) which shows that this is not the case (even if $Z(G) = 1$); hence case (c) of Theorem 5.1 actually occurs.

On the other hand, it is easy to construct examples of groups G (with trivial center) that satisfy the structural properties given in 5.1(c) without admitting a non-trivial subnormal embedding in a direct product of groups isomorphic to G. We note further that Theorem 5.1 also yields a proof of Theorem 4.2 for groups with finite composition series. However, this approach is more elaborate than the direct argument in Section 4.

To prove Theorem 5.1 we observe that $G_0 \cap G_j = 1$ for all $j = 1, \dots, n$ implies that G is nilpotent (3.3(a)). Therefore, we can assume that the non-nilpotent group G_0 is non-trivially subnormally embedded in $G_1 \times \cdots \times G_n$ and have to show that G is factorized in the form given in 5.1(c). This will be accomplished in Theorem 5.10, where also a concrete description of N and S can be found. Actually, it is this description which is most essential for the investigations on normally detectable groups in Section 6.

First of all, however, we have to introduce some notation. We assume in the following that all groups considered possess a finite composition series, although the full strength of this assumption is not always needed.

5.2. NOTATION. Let G be a non-nilpotent group with finite composition series. Suppose that $G_0 \trianglelefteq\trianglelefteq G_1 \times \cdots \times G_n$, where $G_j \cong G$ for $j = 0, 1, \ldots, n$. These hypotheses are tacitly assumed throughout 5.2 to 5.9.

(a) With $G_1, \ldots, G_n$ given, we define groups $G_{J_l} \cong G$, $l \in \mathbb{N}$, $J_l \in \mathbf{n}^l$, in such a way that G_{J_l} is subnormally embedded in $G_{J_l,1} \times \cdots \times G_{J_l,n}$ in the same way as G_0 is embedded in $G_1 \times \cdots \times G_n$. Hence, for all $l \in \mathbb{N}$, G_0 is subnormal in $\times_{J_l \in \mathbf{n}^l} G_{J_l}$, where we use the lexicographical ordering in $\mathbf{n}^l$.

(b) For $l \in \mathbb{N}$, define $\mathscr{B}_l \subseteq \mathbf{n}^l$ by $\mathscr{B}_l = \{J_l \in \mathbf{n}^l | G_0\pi_{J_l} \nleqslant \operatorname{Fit}(G_{J_l})\}$ and set $b_l = |\mathscr{B}_l|$. (π_{J_l} denotes the projection onto G_{J_l}; cf. Section 2.) By 3.3(a), $B_l \subseteq \{J_l \in \mathbf{n}^l | G_0 \cap G_{J_l} \neq 1\}$. Since G is not nilpotent, $b_l \geqslant 1$ for all l. Now $G_0\pi_{J_l} \leqslant G_0\pi_{J_l,1} \times \cdots \times G_0\pi_{J_l,n}$ implies that $b_l \leqslant b_{l+1}$ for all l. Hence there exists some positive integer s_1 such that $b_{s_1} = b_j$ for all $j \geqslant s_1$, because G_0 is a group with finite composition series. (Actually, it can easily be shown that $b_{s_1} = b_j$ for all $j \geqslant s_1$ is equivalent to $b_{s_1} = b_{s_1+1}$.) We choose s_1 minimal with respect to this property and set $b = b_{s_1}$. We call b the *branching number* of the iterated embedding. (This notation is justified by the following fact: as mentioned above, $\mathscr{B}_l \subseteq \{J_l \in \mathbf{n}^l | G_0 \cap G_{J_l} \neq 1\}$. If G does not admit a nontrivial homomorphism into its center (the case we are mainly interested in), then we actually have $\mathscr{B}_l = \{J_l \in \mathbf{n}^l | G_0 \cap G_{J_l} \neq 1\}$ by 3.3(e).)

(c) For $k \geqslant s_1$, let $\mathscr{B}_k = \{J_k^1, \ldots, J_k^b\}$. For $i = 1, \ldots, b$, let $J_{s_1}^i = (j_1^i, \ldots, j_{s_1}^i)$. Since $b_{s_1} = b_{s_1+1}$, and since $G_0\pi_{J_{s_1}^i} \leqslant G_0\pi_{J_{s_1,1}^i} \times \cdots \times G_0\pi_{J_{s_1,n}^i}$, there exists exactly one index $j_{s_1+1}^i \in \{1, \ldots, n\}$ such that $J_{s_1+1}^i = (j_1^i, \ldots, j_{s_1+1}^i)$. Analogously, $j_{s_1+2}^i$, $j_{s_1+3}^i, \ldots$ are defined for $i = 1, \ldots, b$. For $i = 1, \ldots, b$, we denote by J^i the sequence $(j_k^i | k \in \mathbb{N})$. Let $\mathscr{B}_\infty = \{J^i | i = 1, \ldots, b\}$. Finally, for $1 \leqslant l \leqslant k$ and $i = 1, \ldots, b$, set $J_{[l,k]}^i = (j_l^i, \ldots, j_k^i)$; thus $J_k^i = J_{[1,k]}^i$.

(d) Let $k \in \mathbb{N}$ and let $i \in \{1, \ldots, b\}$. Then $G_0\pi_{J_{k+1}^i} = (G_0\pi_{J_k^i})\pi_{J_{k+1}^i}$ is a homomorphic image of $G_0\pi_{J_k^i}$. Application of the maximal condition on normal subgroups to the ascending chain $\ker \pi_{J_k^i} \cap G_0$ yields the existence of some (smallest possible) index $s(i)$ with $G_0\pi_{J_{s(i)}^i} \cong G_0\pi_{J_j^i}$ for all $j \geqslant s(i)$. We set $s = \max\{s(i) | i = 1, \ldots, b\}$ and call s the *stationary level* of the iterated embedding. We show in 5.3 that $s \geqslant s_1$.

(e) For $k \geqslant s$, define $\mathscr{B}_k^* = \{J_k^i \in \mathscr{B}_k | G_0\pi_{J_k^i} \cong G_0\pi_{J_{[k+1,k+s]}^i}\}$ and $\mathscr{B}_k^{**} = \{J_k^i \in \mathscr{B}_k | G_0\pi_{J_k^i}$ is not isomorphic to a proper subnormal subgroup of any $G_0\pi_{J_k^j}$, $j = 1, \ldots, b\}$. It will be shown in 5.7 that $J_k^i \in \mathscr{B}_k^{**}$ for some $k \geqslant s$ if and only if $J_l^i \in \mathscr{B}_l^{**}$ for all $l \geqslant s$. Therefore, it makes sense to define $\mathscr{B}_\infty^{**} = \{J^i \in \mathscr{B}_\infty | J_k^i \in \mathscr{B}_k^{**}$ for all $k \geqslant s\}$. We have been unable to decide whether an analogous result holds for $\mathscr{B}_k^*$.

(f) For $k \geqslant s$ and for $i \in \{1, \ldots, b\}$, define the normal subgroup N_k^i of $G_{J_k^i}$ by $N_k^i = \ker \pi_{J_{k+s}^i} \cap G_{J_k^i} = G_{J_k^i} \cap \times \{G_{J_k^i, J} \mid J \in \mathbf{n}^s,\ J \neq J^i_{[k+1, k+s]}\}$. It follows from (d) that $N_k^i = \ker \pi_{J_{k+l}^i} \cap G_{J_k^i} = G_{J_k^i} \cap \times \{G_{J_k^i, J} \mid J \in \mathbf{n}^l,\ J \neq J^i_{[k+1, k+l]}\}$ for all $l \geqslant s$.

5.3. LEMMA. $s \geqslant s_1$.

PROOF. Suppose that the assertion is false. Then, by definition of s and s_1, there exist $h, i \in \{1, \ldots, b\}$, $h \neq i$, such that $J_k^i = J_k^h$ for some $k \geqslant s$, but such that $j_{k+1}^i \neq j_{k+1}^h$. Since $G_0\pi_{J_k^i} \cong G_0\pi_{J_{k+1}^i}$, it follows from the maximal condition on normal subgroups in $G_0\pi_{J_k^i}$ that $G_0\pi_{J_k^i} \cap \times_{j \neq j_{k+1}^i} G_{J_k^i, j} = 1$. Analogously, $G_0\pi_{J_k^i} \cap \times_{j \neq j_{k+1}^h} G_{J_k^i, j} = 1$. A fortiori, $G_0 \cap G_{J_k^i, j} = 1$ for all $j = 1, \ldots, n$. By 3.3(e), this implies that $G_0\pi_{J_{k+1}^i}$ and $G_0\pi_{J_{k+1}^h}$ are nilpotent, contradicting the fact that $J_{k+1}^i, J_{k+1}^h \in \mathscr{B}_{k+1}$.

5.4. LEMMA. *For $i \in \{1, \ldots, b\}$ and for $m \geqslant 1$, the sequence $(j_l^i \mid l \geqslant m)$ is a member of $\mathscr{B}_\infty$. Moreover, $G_0\pi_{J_{s(i)}^i}$ is isomorphic to a subnormal subgroup of $G_0\pi_{J^i_{[m, m+r]}}$ for any $r \geqslant s(i) - 1$.*

PROOF. Let $r \geqslant s(i) - 1$. We have $G_0\pi_{J_{s(i)}^i} \cong G_0\pi_{J_{m+r}^i}$ as $m + r \geqslant s(i)$. Since $J_{s(i)}^i \in \mathscr{B}_{s(i)}$ and $G_0\pi_{J_{m+r}^i} \trianglelefteq\trianglelefteq G_{J_{m-1}^i}\pi_{J_{m+r}^i} \cong G_0\pi_{J^i_{[m, m+r]}}$, we conclude that $G_0\pi_{J^i_{[m, m+r]}}$ is not nilpotent. The assertion follows.

5.5. LEMMA. *Let $i \in \{1, \ldots, b\}$ and let $k \geqslant s$.*

(a) *The following statements are equivalent*:

(i) $J_k^i \in \mathscr{B}_k^*$;

(ii) $G_0\pi_{J_k^i} \cong G_0\pi_{J^i_{[k+1, k+l]}}$ *for some $l \geqslant s$*;

(iii) $G_0\pi_{J_k^i} \cong G_0\pi_{J^i_{[k+1, k+l]}}$ *for all $l \geqslant s$*;

(iv) $J_k^i \in \mathscr{B}_k$, *and N_k^i is complemented by $G_0\pi_{J_k^i}$ in $G_{J_k^i}$.*

(b) *Let $1 \leqslant m \leqslant k + 1$ and let $s \leqslant l \leqslant k$. If $J_k^i \in \mathscr{B}_k^*$, then $J_l^i \in \mathscr{B}_l^*$. Moreover, $G_0\pi_{J_s^i} \cong G_0\pi_{J^i_{[m, m+s-1]}}$.*

(c) *Let $J_k^i \in \mathscr{B}_k^*$ be such that $J_{2k}^i = (J_k^i, J_k^i)$. Then $J_{mk}^i = (J_{(m-1)k}^i, J_k^i)$ for all $m \geqslant 2$, and $J_l^i \in \mathscr{B}_l^*$ for all $l \geqslant s$. Moreover, if σ denotes the k-cycle $(1, \ldots, k)$ in the symmetric group S_k, then $J_k^i\sigma^r \in \mathscr{B}_k^*$ and $G_0\pi_{J_k^i} \cong G_0\pi_{J_k^i\sigma^r} \cong G_0\pi_{(J_k^i\sigma^r, J_k^i\sigma^r)}$ for all r.*

PROOF. (a) It follows from 5.4 that (i), (ii), and (iii) are equivalent. Suppose $J_k^i \in \mathscr{B}_k^*$. Then $G_0\pi_{J_k^i}/(N_k^i \cap G_0\pi_{J_k^i}) \cong G_0\pi_{J_{k+s}^i} \cong G_0\pi_{J_k^i}$. Now $G_0\pi_{J_k^i}$, being an epimorphic image of G_0, satisfies the maximal condition on normal subgroups.

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Hence, $N_k^i \cap G_0\pi_{J_k^i} = 1$. Furthermore, $G_{J_k^i}/N_k^i \cong G_{J_k^i}\pi_{J_{k+s}^i} \cong G_0\pi_{J_{[k+1,k+s]}^i} \cong G_0\pi_{J_k^i}$, as $J_k^i \in \mathscr{B}_k^*$. Since $G_{J_k^i}$ satisfies the minimal condition on subnormal subgroups, (iv) follows.

If (iv) holds, then $G_0\pi_{J_k^i} \cong G_{J_k^i}/N_k^i \cong G_{J_k^i}\pi_{J_{k+s}^i} \cong G_0\pi_{J_{[k+1,k+s]}^i}$; thus $J_k^i \in \mathscr{B}_k^*$.

(b) By 5.4, $G_0\pi_{J_{[m,m+s-1]}^i} \cong G_0\pi_{J_{[m,k+s]}^i} \trianglelefteq\trianglelefteq G_{J_{[m,k]}^i}\pi_{J_{[m,k+s]}^i} \cong G_0\pi_{J_{[k+1,k+s]}^i} \cong G_0\pi_{J_k^i} \cong G_0\pi_{J_s^i}$ (note that $m \leqslant k+1$ and $J_k^i \in \mathscr{B}_k^*$). On the other hand, $G_0\pi_{J_s^i}$ is isomorphic to a subnormal subgroup of $G_0\pi_{J_{[m,m+s-1]}^i}$ by 5.4. The minimal condition on subnormal subgroups implies that $G_0\pi_{J_{[m,m+s-1]}^i} \cong G_0\pi_{J_s^i}$. Setting $m = l+1 \leqslant k+1$, we obtain $G_0\pi_{J_{[l+1,l+s]}^i} \cong G_0\pi_{J_s^i} \cong G_0\pi_{J_l^i}$. Hence $J_l^i \in \mathscr{B}_l^*$.

(c) We prove $J_{mk}^i = (J_{(m-1)k}^i, J_k^i)$ by induction on m. The case $m = 2$ holds by hypothesis, and so we assume that $m \geqslant 3$. Let $J_{mk}^i = (J_{(m-1)k}^i, J)$, $J \in \mathbf{n}^k$. By the induction hypothesis, $G_{J_{(m-2)k}^i}\pi_{(J_{(m-1)k}^i, J_k^i)} \cong G_0\pi_{J_{2k}^i}$, and this group is not nilpotent. On the other hand, $G_0\pi_{J_{mk}^i} \trianglelefteq\trianglelefteq G_{J_{(m-2)k}^i}\pi_{J_{mk}^i}$. Hence $G_{J_{(m-2)k}^i}\pi_{J_{mk}^i} = G_{J_{(m-2)k}^i}\pi_{(J_{(m-1)k}^i, J)}$ is not nilpotent. Since $k \geqslant s$, we conclude that $J = J_k^i$.

Let $l \geqslant s$ be arbitrary and choose m such that $mk \geqslant l$. Since $J_{mk}^i = J_{[mk+1,2mk]}^i$, part (a) shows that $J_{mk}^i \in \mathscr{B}_{mk}^*$. By part (b), $J_l^i \in \mathscr{B}_l^*$. That $G_0\pi_{J_k^i} \cong G_0\pi_{J_k^i\sigma^r} \cong G_0\pi_{(J_k^i\sigma^r, J_k^i\sigma^r)}$ holds follows from the preceding arguments, from part (b), and from 5.4. In particular, $J_k^i\sigma^r \in \mathscr{B}_k^*$.

5.6. Lemma. *Let $k \geqslant s$ and $i \in \{1, \ldots, b\}$ be such that $J_k^i \in \mathscr{B}_k^*$. Then*
(a) $(G_0\pi_{J_k^i})^{G_{J_k^i}}/\mathrm{Core}_{G_{J_k^i}}(G_0\pi_{J_k^i})$ is nilpotent,
(b) $[G_0\pi_{J_k^i}, N_k^i] \leqslant \mathrm{Fit}(G_{J_k^i})$.

Proof. Recall that, by 5.5(a), $G_0\pi_{J_k^i}$ is a complement to N_k^i in $G_{J_k^i}$.

(a) Clearly, $G_0 \cap G_{J_{k+s}^i} \trianglelefteq G_0\pi_{J_k^i}$. By 3.3(a), $G_0\pi_{J_k^i}/(G_0 \cap G_{J_{k+s}^i})$ is nilpotent. Moreover, $[N_k^i, G_0 \cap G_{J_{k+s}^i}] \leqslant N_k^i \cap G_{J_{k+s}^i} = 1$ according to the definition of N_k^i. Thus $G_0 \cap G_{J_{k+s}^i} \trianglelefteq G_{J_k^i}$, and $G_0\pi_{J_k^i}/\mathrm{Core}_{G_{J_k^i}}(G_0\pi_{J_k^i})$ is nilpotent. Since $G_0\pi_{J_k^i}$ is subnormal in $G_{J_k^i}$, the conclusion follows from 3.1(a).

(b) This is clear from (a).

5.7. Lemma. *Let $i \in \{1, \ldots, b\}$. If $J_k^i \in \mathscr{B}_k^{**}$ for some $k \geqslant s$, then $J_l^i \in \mathscr{B}_l^{**}$ for all $l \geqslant s$. Moreover, if $m \geqslant 1$, then $G_0\pi_{J_{[m,m+s-1]}^i} \cong G_0\pi_{J_s^i}$; in particular, $J_{[m,m+s-1]}^i \in \mathscr{B}_s^{**}$.*

Proof. This follows immediately from the definition and from 5.4.

5.8. Lemma. *Let $k \geqslant s$. Then $\varnothing \neq \mathscr{B}_k^{**} \subseteq \mathscr{B}_k^*$.*

Proof. Since G is not nilpotent, $\mathscr{B}_k \neq \varnothing$. Therefore, $\mathscr{B}_k^{**} \neq \varnothing$ by virtue of the minimal condition on subnormal subgroups. Now let $J_k^i \in \mathscr{B}_k^{**}$. As in the proof of the implication (i) $\Rightarrow$ (iv) in 5.5(a) it follows that $G_0\pi_{J_k^i} \cap N_k^i = 1$. Hence

$G_0\pi_{J_k^i} \cong (G_0\pi_{J_k^i})N_k^i/N_k^i \trianglelefteq\trianglelefteq G_{J_k^i}/N_k^i \cong G_{J_k^i}\pi_{J_{k+s}^i} \cong G_0\pi_{J^i_{[k+1,k+s]}}$. From $J_k^i \in \mathscr{B}_k^{**}$ we conclude that $G_0\pi_{J_k^i} \cong G_0\pi_{J^i_{[k+1,k+s]}}$, i.e. that $J_k^i \in \mathscr{B}_k^*$.

5.9. LEMMA. *Let $i \in \{1,\dots,b\}$ be such that $J_s^i \in \mathscr{B}_s^{**}$. Then there exists $l \geqslant s$ and $J_l^h \in \mathscr{B}_l^{**}$ (for some $h \in \{1,\dots,b\}$) with $J^h_{[l+1,2l]} = J_l^h$ and $G_0\pi_{J_s^i} \cong G_0\pi_{J_l^h}$.*

PROOF. It follows from 5.7 that $J^i_{[us+1,(u+1)s]} \in \mathscr{B}_s^{**}$ for all positive integers u. Since $\mathscr{B}_s^{**}$ is a finite set, there exist positive integers $a < b$ such that $J^i_{[as+1,(a+1)s]} = J^i_{[bs+1,(b+1)s]}$. Set $l = s(b-a) \geqslant s$ and $J = J^i_{[as+1,bs]} \in \mathbf{n}^l$. Then $G_0\pi_{J_s^i} \cong G_0\pi_{J^i_{as}} \cong G_0\pi_J$ by 5.7. Hence, $J \in \mathscr{B}_l^{**}$, say $J = J_l^h$ for some $h \in \{1,\dots,b\}$. It remains to show that $J_l^h = J^h_{[l+1,2l]}$. Using 5.4, we infer from $J^i_{[as+1,(a+1)s]} = J^i_{[bs+1,(b+1)s]}$ that $G_0\pi_{J_l^h} = G_0\pi_{J^i_{[as+1,bs]}} \cong G_0\pi_{J^i_{[as+1,(b+1)s]}} = G_0\pi_{(J_l^h, J^i_{[as+1,(a+1)s]})}$. This proves the assertion in the case $b = a+1$. So we may assume that $b > a+1$. Now $G_0\pi_{(J^i_{[as+1,(a+1)s]}, I)}$ is nilpotent for all $I \in \mathbf{n}^{(s-1)(b-a)}$, $I \neq J^i_{[(a+1)s+1,bs]}$, because $J^i_{[as+1,(a+1)s]} \in \mathscr{B}_s$ and $G_0\pi_{(J^i_{[as+1,(a+1)s]}, J^i_{[(a+1)s+1,bs]})} = G_0\pi_{J_l^h}$ is not nilpotent. Since $G_0\pi_{(J_l^h, J^i_{[as+1,(a+1)s]}, I)} \leqslant G_{J_l^h}\pi_{(J_l^h, J^i_{[as+1,(a+1)s]}, I)} \cong G_0\pi_{(J^i_{[as+1,(a+1)s]}, I)}$ for all $I \in \mathbf{n}^{(s-1)(b-a)}$, we conclude that $G_0\pi_{(J_l^h, J^i_{[as+1,(a+1)s]}, I)}$ is nilpotent for all $I \in \mathbf{n}^{(s-1)(b-a)}$, $I \neq J^i_{[(a+1)s+1,bs]}$. It follows that $J^h_{[l+1,2l]} = J_l^h$.

The following theorem rests upon the results obtained so far in this section. It exhibits a particular nice factorization and implies the validity of Theorem 5.1 (note the remarks following the statement of 5.1).

5.10. THEOREM. *Let G be a group with finite composition series. Let $G_0 \trianglelefteq\trianglelefteq G_1 \times \cdots \times G_n$ be a non-trivial subnormal embedding (cf. 3.4), where $G_j \cong G$ for all $j = 0,1,\dots,n$. Then either G is nilpotent or the following statements hold (with the notation of 5.2(a)). There exists some positive integer l and $J_l = (j_1,\dots,j_l) \in \mathbf{n}^l$ such that*

(a) *$G_{J_l} = N_{J_l} \cdot G_0\pi_{J_l}$, where $N_{J_l} = G_{J_l} \cap \times\{G_{J_l,J} \mid J \in \mathbf{n}^l,\ J \neq J_l\}$*
(b) *$N_{J_l} \cap G_0\pi_{J_l} = 1$*
(c) *$G_0\pi_{J_l}$ is a non-nilpotent subnormal subgroup of G_{J_l}*
(d) *N_{J_l} is a non-trivial normal subgroup of G_{J_l}*
(e) *$(G_0\pi_{J_l})^{G_{J_l}}/\mathrm{Core}_{G_{J_l}}(G_0\pi_{J_l})$ is nilpotent; in particular, $[N_{J_l}, G_0\pi_{J_l}] \leqslant \mathrm{Fit}(G_{J_l})$*
(f) *$G_0 \cap G_{J_l} = G_{J_l} \cap G_{J_l,J_l} \trianglelefteq G_{J_l}$*
(g) *$G_{J_m} = (G_{J_m} \cap \times\{G_{J_m,J} \mid J \in \mathbf{n}^l,\ J \neq J_l\sigma^m\}) \cdot G_0\pi_{J_m}$ for $1 \leqslant m \leqslant l$, where $J_m = (j_1,\dots,j_m)$ and $\sigma = (1,\dots,l) \in S_l$.*

PROOF. Let G be non-nilpotent. According to 5.2, let $\mathscr{B}_k = \{J_k^1,\dots,J_k^b\}$ for $k \geqslant s$. Now G_0 satisfies the minimal condition on normal subgroups. Since $G_0 \cap G_{J_k^i} \geqslant G_0 \cap G_{J_{k+1}^i}$ for all $k \geqslant s$ and all $i = 1,\dots,b$, we deduce the existence of some integer $t \geqslant s$ such that

$$(*) \qquad G_0 \cap G_{J_k^i} = G_0 \cap G_{J_{k+1}^i} \quad \text{for all } k \geqslant t \text{ and for } i = 1, \ldots, b.$$

Since, by 5.8, $\mathscr{B}_s^{**} \neq \varnothing$, we can choose l and $J_l^h \in \mathscr{B}_l^{**}$ as in 5.9. By 5.5(c), we may replace l by al ($a \in \mathbf{N}$) without violating the statement of 5.9. Hence, without loss of generality, we have $l \geqslant t$.

Set $J_l = J_l^h$. Then (a) and (b) follow immediately from 5.8 and from 5.5(a). Statement (c) is a consequence of $J_l \in \mathscr{B}_l^{**} \subseteq \mathscr{B}_l$. The fact that the embedding of G_{J_l} in $G_{J_l,1} \times \cdots \times G_{J_l,n}$ is non-trivial yields (d). Part (e) is proved in 5.6.

By the choice of J_l, and because of $(*)$, we have $G_0 \cap G_{J_l} = G_0 \cap G_{J_l,J_l}$. Thus $G_0 \cap G_{J_l} \trianglelefteq\trianglelefteq G_{J_l} \cap G_{J_l,J_l} \cong G_0 \cap G_{J_l}$. Hence the minimal condition on subnormal subgroups implies the validity of statement (f).

Finally, let $1 \leqslant m \leqslant l$. Set $N_{J_m} = G_{J_m} \cap \times\{G_{J_m,J} \mid J \in \mathbf{n}^l,\ J \neq J_l\sigma^m\}$. Then $G_0\pi_{J_m}/(G_0\pi_{J_m} \cap N_{J_m}) \cong G_0\pi_{(J_m,J_l\sigma^m)} = G_0\pi_{(J_l,J_m)} \cong G_0\pi_{J_l}$. Furthermore, $G_{J_m}/N_{J_m} \cong G_{J_m}\pi_{(J_m,J_l\sigma^m)} \cong G_0\pi_{J_l\sigma^m} \cong G_0\pi_{J_l}$, where the latter isomorphism is given by 5.5(c). Employing once again the minimal condition on subnormal subgroups in G_{J_m}, we obtain (g).

5.11. Remark. Let G be a finite non-nilpotent group satisfying the hypotheses of Theorem 5.10. Assume further that, for every prime divisor p of $|G/G'|$, the Sylow p-subgroups of G are abelian. Then G is directly decomposable (in a non-trivial way). In view of 5.10, this follows from the following simple fact: if the finite group G has abelian Sylow p-subgroups for every prime divisor p of $|G/G'|$, and if G is of the form $G = NS$, where $N \trianglelefteq G$, $S \trianglelefteq\trianglelefteq G$, $N \cap S = 1$, and $S/\mathrm{Core}_G(S)$ is nilpotent, then $G = N \times \tilde{S}$ for some $\tilde{S} \leqslant G$. This is proved by induction on $|G|$. By standard reductions, we may assume that $\mathrm{Core}_G(S) = 1$ and that $S \leqslant O_p(G)$ for some prime p. ($O_p(G)$ denotes the largest normal p-subgroup of G.) Define $T = [S, N]S \leqslant O_p(G)$. Then T is N-invariant. Since p divides $|G/G'|$, the Sylow p-subgroups of G are abelian. Consequently, T is abelian, and p does not divide $|N/C_N(T)|$. Therefore, $T = [T, N] \times C_T(N)$ [4, III.13.4(b)]. Let $S = \langle s_1, \ldots, s_m \rangle$. There exist $x_i \in [T, N]$ such that $s_i x_i \in C_T(N)$ for $i = 1, \ldots, m$. Define $\tilde{S} = \langle s_1x_1, \ldots, s_mx_m \rangle \leqslant C_T(N)$. Since $[T, N] \leqslant N$, we conclude that $G = NS = N\tilde{S}$. Let $y = \prod_{i=1}^m (s_ix_i)^{a_i} = \prod_{i=1}^m s_i^{a_i} \prod_{i=1}^m x_i^{a_i} \in N \cap \tilde{S}$, where $a_i \in \mathbf{Z}$ (note that T is abelian). Then $\prod_{i=1}^m s_i^{a_i} \in N \cap S = 1$; hence $y = \prod_{i=1}^m x_i^{a_i} \in [T, N] \cap \tilde{S} \leqslant [T, N] \cap C_T(N) = 1$. Consequently, $\tilde{S}$ is a complement for N in G. Clearly, then, $G = N \times \tilde{S}$, as $\tilde{S} \leqslant C_T(N)$.

5.12. Examples. (a) (J. Cossey) The following example exhibits a directly indecomposable finite group G with trivial center which possesses a non-trivial subnormal embedding in $G \times G$ (in the sense of 3.4).

Let G be generated by x_1, x_2, x_3, y_1, y_2 according to the following defining relations:

$$x_1^3 = x_2^3 = x_3^3 = [x_1, x_3]^3 = [x_2, x_3]^3 = y_1^2 = y_2^2 = 1,$$
$$[x_1, x_2] = 1, \quad [x_1, x_3, x_i] = [x_2, x_3, x_i] = 1 \text{ for } i = 1,2,3,$$
$$[x_2, y_1] = [x_3, y_1] = [x_1, y_2] = [x_3, y_2] = [y_1, y_2] = 1,$$
$$[x_1, y_1] = x_1, \quad [x_2, y_2] = x_2.$$

Then $|G| = 2^2 3^5$, $Z(G) = 1$, and G is directly indecomposable. (If V denotes the indecomposable faithful Z_6-module over GF(3) of dimension 2 and H the semidirect product of V with Z_6, then G is the direct product of two copies of H with amalgamated factor group Z_3.)

Let G^* be a copy of G with generators x_i^*, y_j^* that correspond to x_i, y_j. It is easy to check that $G_0 = \langle x_1, x_1^*, x_3x_3^*, y_1, y_1^* \rangle$ is a subnormal subgroup of $G \times G^*$ which is isomorphic to G (x_1 corresponds to x_1, x_1^* to x_2, $x_3x_3^*$ to x_3, y_1 to y_1, and y_1^* to y_2). Clearly, $G_0 \cap G \neq 1 \neq G_0 \cap G^*$.

(b) Groups G with the structural properties described in 5.1(c) need not admit non-trivial subnormal embeddings in direct products of groups isomorphic to G.

Let $P = \langle x, y \mid x^3 = y^3 = [x, y, x] = [x, y, y] = 1 \rangle$ be an extraspecial group of order 27 and exponent 3, and let z denote the automorphism of P which inverts x (and $[x, y]$) and leaves y invariant. Let R be the semidirect product of P and $\langle z \rangle$. Then $N = \langle x, [x, y], z \rangle$ is a normal subgroup of index 3 in R. Moreover, R acts on a cyclic group $\langle t \rangle$ of order 7 with kernel N, and $[t, y] = t$. Finally, let G be the semidirect product of $\langle t \rangle$ and R with respect to this action. Then $Z(G) = 1$. Since $\langle t \rangle$ and $\langle [x, y] \rangle$ are the only minimal normal subgroups of G, it is easy to verify that G does not allow non-trivial subnormal embeddings in direct products of groups isomorphic to G. However, statement 5.1(c) is satisfied with N as above and with $S = \langle y, t \rangle$.

It remains an open question as to how those groups G can be characterized which admit a non-trivial subnormal embedding in a direct product of groups isomorphic to G.

We conclude this section with a result on the subnormal embeddings of a group which does not admit a non-trivial homomorphism into its center. This will be of use in our investigations on normally detectable groups.

5.13. PROPOSITION. *Let G be a group satisfying the maximal condition on subnormal subgroups which does not admit a non-trivial homomorphism into its center. Let $G_0 \trianglelefteq\trianglelefteq G_1 \times \cdots \times G_n$, where $G_j \cong G$ for $j = 0, 1, \ldots, n$. If N is a minimal normal subgroup of G_0, and if $i \in \{1, \ldots, n\}$, then either $N\pi_i = 1$, or $N\pi_i$ is a minimal normal subgroup of G_0.*

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PROOF. Suppose that $N\pi_i \neq 1$. Then, if $1 \neq M_i \leqslant N\pi_i$ is a normal subgroup of $G_0\pi_i$, it follows that $1 \neq M = \{n \in N \mid n\pi_i \in M_i\}$ is a normal subgroup of G_0. Hence $N\pi_i$ is a minimal normal subgroup of $G_0\pi_i$. Aiming for a contradiction, we assume that $N\pi_i$ is not a minimal normal subgroup of G_0. Then $N\pi_i \nleqslant G_0$. Moreover, $N \cap \times_{j=1}^{n}(G_0 \cap G_j) = 1$; for otherwise, $N \leqslant \times_{j=1}^{n}(G_0 \cap G_j)$, and hence $N\pi_i \leqslant G_0 \cap G_i \leqslant G_0$, a contradiction.

Now $[N\pi_i \cap G_0,\ G_0\pi_i] = [N\pi_i \cap G_0,\ G_0] \leqslant N\pi_i \cap G_0$. Therefore, $N\pi_i \cap G_0 \trianglelefteq G_0\pi_i$. Since $N\pi_i \nleqslant G_0$ is a minimal normal subgroup of $G_0\pi_i$, we conclude that $N\pi_i \cap G_0 = 1$. By 3.3(a), $G_0\pi_i/(G_0 \cap G_i)$ is nilpotent, say of class c. Then $[N\pi_i, G_0\pi_i, \ldots, G_0\pi_i] \leqslant N\pi_i \cap G_0 \cap G_i = 1$ (here $G_0\pi_i$ occurs c times). Hence $N\pi_i$ is contained in the hypercenter and, as a minimal normal subgroup, even in the center of $G_0\pi_i$. Consequently, $[N, G_0] \leqslant N \cap \times_{j \neq i} G_j$. Since $N\pi_i \neq 1$ and $N \cap \times_{j \neq i} G_j \trianglelefteq G_0$, N is contained in $Z(G_0)$. By 3.3(a) again, $G_0/\times_{j=1}^{n}(G_0 \cap G_j)$ is nilpotent. Moreover, $N \cong N(\times_{j=1}^{n}(G_0 \cap G_j))/(\times_{j=1}^{n}(G_0 \cap G_j)) \leqslant Z(G_0/\times_{j=1}^{n}(G_0 \cap G_j))$; this holds because $N \cap \times_{j=1}^{n}(G_0 \cap G_j) = 1$ and $N \leqslant Z(G_0)$. As a nilpotent factor group of G_0, $G_0/\times_{j=1}^{n}(G_0 \cap G_j)$ is finitely generated. By means of 3.2 we deduce the existence of a non-trivial homomorphism from $G_0/\times_{j=1}^{n}(G_0 \cap G_j)$ into $N \leqslant Z(G_0)$, the desired contradiction.

6. Normally detectable groups

6.1. DEFINITION. A group G is called *normally detectable* if the following holds: whenever G_0 is normal in $G_1 \times \cdots \times G_n$ for some positive integer n, where $G_j \cong G$ for $j = 0, 1, \ldots, n$, then $G_0 = G_i$ for some $i \in \{1, \ldots, n\}$.

The following conjecture is due to T. O. Hawkes.

CONJECTURE. A finite group G is normally detectable if and only if
(1) G is directly indecomposable, and
(2) $(|G/G'|, |Z(G)|) = 1$.

It is clear that (1) and (2) are necessary for a group to be normally detectable: for (1), the same argument as in the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 4.2 works. For (2), assume that there is a non-trivial homomorphism α: $G \to Z(G)$. Then $G_0 = \{(g, g\alpha) \mid g \in G\}$ is a normal subgroup of $G \times G$, $G_0 \cong G$.

We are not able either to prove or disprove Hawkes' conjecture. This section is devoted to the proofs of several partial results. We show that if one of the conditions (1) or (2) is somewhat strengthened, it follows that the group in question is normally detectable (Theorems 6.2 and 6.5). Moreover, for various types of groups we are able to prove that Hawkes' conjecture holds (Theorem 6.9).

To obtain these results, we not only need information about the structure but also about the type of embedding of a possible counterexample (Theorem 6.8). Theorem 5.1 already shows that a group which is a counterexample to Hawkes' conjecture has to admit a certain factorization. In fact, in dealing with normal embeddings we can describe this factorization more precisely than for general subnormal embeddings.

The following theorem collects this information. Beforehand, we mention that in a group G with finite composition series the center $Z(G)$ and the commutator factor group G/G' are finite. Hence for such groups the condition $(|G/G'|, |Z(G)|) = 1$ is meaningful and equivalent to the non-existence of a non-trivial homomorphism from G into $Z(G)$.

6.2. THEOREM. *Let G be a group with finite composition series. Assume that G is directly indecomposable and that $(|G/G'|, |Z(G)|) = 1$. If G is not normally detectable, then the following hold.*

(a) $G = NS$, $N \cap S = 1$, $N \trianglelefteq G$, $S \trianglelefteq\trianglelefteq G$.

(b) *N and S are not nilpotent.*

(c) *$S^G/\mathrm{Core}_G(S)$ is nilpotent; in particular, $[N, S] \leqslant \mathrm{Fit}(G)$.*

(d) $[N, S] \nleqslant \mathrm{Soc}(G)$.

(e) *$[N, S]$ is not characteristic in G; more precisely, there exists an automorphism α of G such that $\langle \alpha \rangle \cong H/M$ for some $H \leqslant G$, $M \trianglelefteq H$, which does not leave $[N, S]$ invariant.*

PROOF. We note first that $G \neq 1$ since G is not normally detectable. The condition $(|G/G'|, |Z(G)|) = 1$ then implies that G is not nilpotent. By assumption, there is a non-trivial normal embedding $G_0 \trianglelefteq G_1 \times \cdots \times G_n$, where $n \geqslant 2$, where $G_j \cong G$ for $j = 0, 1, \ldots, n$, and where $G_0\pi_i \neq 1$ for $i = 1, \ldots, n$. In the sequel we use the notation from 5.2. In particular, $\mathscr{B}_l = \{J_l^1, \ldots, J_l^b\}$ for every $l \geqslant s$. By 5.7 and 5.8 we may assume that there is some a, $1 \leqslant a \leqslant b$, such that $\mathscr{B}_l^{**} = \{J_l^1, \ldots, J_l^a\}$ for all $l \geqslant s$.

Now the proof is carried out in a number of steps.

(1) For all $l \geqslant s$ and all $J_l^i \in \mathscr{B}_l^{**}$, there exists $J_s^j \in \mathscr{B}_s$ such that $G_{J_l^i} \cap (\mathrm{Soc}(G_{J_l^i}) \cap [N_l^i, G_0\pi_{J_l^i}])\pi_{J_l^i, J_s^j} \neq 1$.

$[N_l^i, G_0\pi_{J_l^i}] \neq 1$ as $G_{J_l^i}$ is directly indecomposable, and $N_l^i \neq 1 \neq G_0\pi_{J_l^i}$. Since $[N_l^i, G_0\pi_{J_l^i}] \trianglelefteq N_l^i \cdot G_0\pi_{J_l^i} = G_{J_l^i}$ (5.8 and 5.5(a)), there exists a minimal normal subgroup N of $G_{J_l^i}$ contained in $[N_l^i, G_0\pi_{J_l^i}]$. Let $J \in \mathbf{n}^s$ be such that $N\pi_{J_l^i, J} \neq 1$. By 5.13, $N\pi_{J_l^i, J} \leqslant G_{J_l^i}$. Hence, according to 3.3(e), $J \in \mathscr{B}_s$, say $J = J_s^j$. This proves (1).

(2) For all $l \geqslant s$ and all $i \in \{1, \dots, a\}$ (i.e. $J_l^i \in \mathscr{B}_l^{**}$), put

$$\mathscr{C}\left(J_l^i\right) = \left\{J_s^j \in \mathscr{B}_s \mid G_{J_l^i} \cap \left(\operatorname{Soc}\left(G_{J_l^i}\right) \cap \left[N_l^i, G_0\pi_{J_l^i}\right]\right)\pi_{J_l^i, J_s^j} \neq 1\right\}$$

and $c(J_l^i) = |\mathscr{C}(J_l^i)|$. By (1), $c(J_l^i) \geqslant 1$ for all $l \geqslant s$ and for all $i \in \{1, \dots, a\}$. We choose $t \geqslant s$ and $i \in \{1, \dots, a\}$ such that $c(J_t^i) \leqslant c(J_m^j)$ for all $m \geqslant s$ and for all $j \in \{1, \dots, a\}$. Without loss of generality, we may assume that $i = 1$.

(3) There exists some $r \geqslant s$ and $J_s^k \in \mathscr{B}_s$ such that

$$G_{J_{r+1}^1} \cap \left(\operatorname{Soc}\left(G_{J_{r+1}^1}\right) \cap \left[N_{r+1}^1, G_0\pi_{J_{r+1}^1}\right]\right)\pi_{J_{r+1}^1, J_s^k} \neq 1$$

and

$$G_{J_r^1} \cap \left(\operatorname{Soc}\left(G_{J_r^1}\right) \cap \left[N_r^1, G_0\pi_{J_r^1}\right]\right)\pi_{J_{r+1}^1, J_s^k} = 1.$$

For $k = 1, \dots, s$, let (α_{t+k}) denote the following statement:

$$(\alpha_{t+k})\begin{cases} \text{If } G_{J_{t+k}^1} \cap \left(\operatorname{Soc}\left(G_{J_{t+k}^1}\right) \cap \left[N_{t+k}^1, G_0\pi_{J_{t+k}^1}\right]\right)\pi_{J_{t+k}^1, J} \neq 1 \text{ for some } J \in \mathbf{n}^s, \\ \text{then } G_{J_{t+k-1}^1} \cap \left(\operatorname{Soc}\left(G_{J_{t+k-1}^1}\right) \cap \left[N_{t+k-1}^1, G_0\pi_{J_{t+k-1}^1}\right]\right)\pi_{J_{t+k}^1, J} \neq 1. \end{cases}$$

Suppose, (α_{t+1}) holds. We know that $J_{[2,t+1]}^1 \in \mathscr{B}_t^{**}$; in fact, $G_0\pi_{J_{t+1}^1} = G_{j_1^1}\pi_{J_{t+1}^1}$, since $G_{j_1^1}\pi_{J_{t+1}^1} \cong G_0\pi_{J_{[2,t+1]}^1} \cong G_0\pi_{J_{t+1}^1}$ (Lemma 5.7). Hence

$$G_{J_{t+1}^1} \cap \left(\operatorname{Soc}\left(G_{J_{t+1}^1}\right) \cap \left[N_{t+1}^1, G_0\pi_{J_{t+1}^1}\right]\right)\pi_{J_{t+1}^1, J} \neq 1$$

if and only if $G_{J_{[2,t+1]}^1} \cap (\operatorname{Soc}(G_{J_{[2,t+1]}^1}) \cap [N_{[2,t+1]}^1, G_0\pi_{J_{[2,t+1]}^1}])\pi_{J_{[2,t+1]}^1} \neq 1$, where $N_{[2,t+1]}^1 = G_{J_{[2,t+1]}^1} \cap \times\{G_{J_{[2,t+1]}^1, I} \mid I \in \mathbf{n}^s,\ I \neq J_{[t+2,t+s+1]}^1\}$. Therefore, (α_{t+1}) implies that $c(J_{[2,t+1]}^1) \leqslant c(J_t^1)$. By the minimal choice of $c(J_t^1)$, we have equality. This means that all $J_s^j \in \mathscr{C}(J_t^1)$ begin with j_{t+1}^1.

Proceeding in the same manner, we see that in presence of $(\alpha_{t+1}), \dots, (\alpha_{t+i})$ ($i \in \{1, \dots, s\}$), all $J_s^j \in \mathscr{C}(J_t^1)$ begin with $(j_{t+1}^1, \dots, j_{t+i}^1)$. In particular, if $(\alpha_{t+1}), \dots, (\alpha_{t+s})$ are fulfilled, then $J \in \mathscr{C}(J_t^1)$ if and only if $J = J_{[t+1,t+s]}^1$. But $[N_t^1, G_0\pi_{J_t^1}] \leqslant N_t^1$, whence $[N_t^1, G_0\pi_{J_t^1}]\pi_{J_t^1, J_{[t+1,t+s]}^1} = 1$ by definition of N_t^1. However, this contradicts the definition of $\mathscr{C}(J_t^1)$. Consequently, there exists some $w \in \{1, \dots, s\}$ for which (α_{t+w}) fails to hold. If we set $r = t + w - 1$, then assertion (3) follows.

(4) Let r and $J_s^k \in \mathscr{B}_s$ be as in (3). Then

$$G_{J_r^1} \cap \left(\operatorname{Soc}\left(G_{J_r^1}\right) \cap \left[N_{r+1}^1, G_0\pi_{J_{r+1}^1}, N_r^1\right]\right)\pi_{J_{r+1}^1, J_s^k} \neq 1.$$

Since $[N_{r+1}^1, G_0\pi_{J_{r+1}^1}] \leqslant [G_{J_{r+1}^1}, G_{J_r^1}\pi_{J_{r+1}^1}] \leqslant G_{J_r^1}$ by 3.3(b), it follows from (3) that there exists a minimal normal subgroup M of $G_{J_r^1}$ such that $M \leqslant [N_{r+1}^1, G_0\pi_{J_{r+1}^1}]$ and $M\pi_{J_{r+1}^1, J_s^k} \neq 1$. By 5.13, $M\pi_{J_{r+1}^1, J_s^k} \leqslant G_{J_r^1}$. Now every prime divisor of $|[N_{r+1}^1, G_0\pi_{J_{r+1}^1}]|$ is also a divisor of $|G_0\pi_{J_{r+1}^1}/\operatorname{Core}_{G_{J_{r+1}^1}}(G_0\pi_{J_{r+1}^1})|$, and hence of $|G/G'|$. Since $(|G/G'|, |Z(G)|) = 1$, it follows that $M\pi_{J_{r+1}^1, J_s^k} \nleqslant Z(G_{J_r^1})$. Using the fact that $G_0\pi_{J_r^1}$ is subnormal in $G_{J_r^1}$ of defect at most r, we conclude that $[M\pi_{J_{r+1}^1,J_s^k}, G_0\pi_{J_r^1},\dots,G_0\pi_{J_r^1}] \leqslant [G_{J_r^1}, G_0\pi_{J_r^1},\dots,G_0\pi_{J_r^1}] \leqslant G_0\pi_{J_r^1}$ (here $G_0\pi_{J_r^1}$ occurs r times). On the other hand, this iterated commutator is contained in $G_{J_{r+1}^1,J_s^k}$. According to 3.3(e), $G_{J_r^1} \cap G_{J_{r+1}^1,J_s^k} \neq 1$ if and only if $(j_{r+1}^1, J_s^k) \in \mathscr{B}_{s+1}$. Hence, if $G_0\pi_{J_r^1} \cap G_{J_{r+1}^1,J_s^k} \neq 1$ then $(J_{r+1}^1, J_s^k) = J_{r+s+1}^1$ (note that $G_0\pi_{J_r^1} \cap N_r^1 = 1$). But this would lead to the contradiction $M\pi_{J_{r+1}^1,J_s^k} = M\pi_{J_{r+s+1}^1} \leqslant N_{r+1}^1\pi_{J_{r+s+1}^1} = 1$. Thus $[M\pi_{J_{r+1}^1,J_s^k}, G_0\pi_{J_r^1},\dots,G_0\pi_{J_r^1}] = 1$. But then $[M, N_r^1]\pi_{J_{r+1}^1,J_s^k} = [M\pi_{J_{r+1}^1,J_s^k}, N_r^1] \neq 1$, for otherwise the minimal normal subgroup $M\pi_{J_{r+1}^1,J_s^k}$ of $G_{J_r^1} = N_r^1 \cdot G_0\pi_{J_r^1}$ would be central. This proves (4).

(5) Let r and $J_s^k \in \mathscr{B}_s$ be as in (3). Then

$$G_{J_r^1} \cap \left(\operatorname{Soc}\left(G_{J_r^1}\right) \cap \left[N_r^1, G_0\pi_{J_r^1}\right]^{G_{J_{r+1}^1}}\right)\pi_{J_{r+1}^1,J_s^k} \neq 1.$$

It follows from the Three Subgroups Lemma [4, III.1.4] that

$$\left[N_{r+1}^1, G_0\pi_{J_{r+1}^1}, N_r^1\right] = \left[N_{r+1}^1, G_0\pi_{J_r^1}, N_r^1\right] \leqslant \left[N_r^1, G_0\pi_{J_r^1}\right]^{G_{J_{r+1}^1}}$$

because N_r^1 is normalized by N_{r+1}^1. The assertion follows from (4).

(6) With $G_{J_r^1}$, N_r^1, $G_0\pi_{J_r^1}$ in place of G, N, S, statements (a)–(e) of the theorem are fulfilled.

Apart from the non-nilpotency of N_r^1, assertions (a), (b), (c) are clear from the considerations in Section 5. That N_r^1 is not nilpotent is a simple consequence of $(|G/G'|, |Z(G)|) = 1$: minimal normal subgroups of $G_{J_r^1}$ contained in $[N_r^1, G_0\pi_{J_r^1}]$ are central in $G_{J_r^1}$ in case N_r^1 (and hence $G_{J_r^1}/\operatorname{Core}_{G_{J_r^1}}(G_0\pi_{J_r^1})$) is nilpotent. By (5), $[N_r^1, G_0\pi_{J_r^1}]\pi_{J_{r+1}^1,J_s^k} \neq 1$. If $[N_r^1, G_0\pi_{J_r^1}]$ is contained in $\operatorname{Soc}(G_{J_r^1})$, then 5.13 yields $G_{J_r^1} \cap (\operatorname{Soc}(G_{J_r^1}) \cap [N_r^1, G_0\pi_{J_r^1}])\pi_{J_{r+1}^1,J_s^k} \neq 1$, contradicting (3). The same contradiction arises directly from (5) if $[N_r^1, G_0\pi_{J_r^1}]$ is assumed to be invariant under $G_{J_{r+1}^1}$ (or even characteristic in $G_{J_r^1}$). This shows that (d) and (e) hold, thus completing the proof.

Let G be a group with finite composition series. Theorem 6.2 shows that leaving condition (2) in Hawkes' conjecture as it stands while strengthening condition (1) in such a way that, apart from direct decompositions, factorizations as in Theorem 6.2 are also not allowed, forces G to be normally detectable. In Theorem 6.5 we prove that it is also possible to strengthen condition (2) in such a way that directly indecomposable groups are then normally detectable.

To this end we introduce the following definition.

6.3. Definition. Let G be a group. A factor group $\overline{G}$ of G is called *essential* if either $\overline{G} = G$, or if the following conditions are satisfied:

(1) $\overline{G}$ is not nilpotent;

(2) $\overline{G}$ is directly decomposable (in a non-trivial way);

(3) $\overline{G}$ is isomorphic to a subnormal subgroup of G.

Theorem 6.5 will be a consequence of the following lemma, where again the notation of 5.2 is used.

6.4. LEMMA. *Let G be a group with finite composition series. Assume that $G_0 \trianglelefteq G_1 \times \cdots \times G_n$, where $G_j \cong G$ for $j = 0, 1, \ldots, n$. Assume further that there exists some $J = J_l = (j_1, \ldots, j_l) \in \mathbf{n}^l$ $(l \geqslant 2)$ such that the following conditions are satisfied*:

(1) *$G_J = N \cdot G_0\pi_J$, where $N \trianglelefteq G_J$ and $N \cap G_0\pi_J = 1$;*

(2) *$G_0\pi_J \trianglelefteq G_{J_i}\pi_J$ for some $i \in \{1, \ldots, l-1\}$, where $J_i = (j_1, \ldots, j_i)$. Then $G_0\pi_J \trianglelefteq G_{J_{i+1}}\pi_J$, or $(|Z(G_{J_i}\pi_J)|, |G_{J_i}\pi_J/(G_{J_i}\pi_J)'|) \neq 1$. Moreover, if $i = l-1$, then $G_0\pi_J \trianglelefteq G_J$, or $(|Z(G)|, |G/G'|) \neq 1$.*

PROOF. Note that $Z(G_{J_i}\pi_J)$ and $G_{J_i}\pi_J/(G_{J_i}\pi_J)'$ are finite. Clearly, $G_{J_i}\pi_J \trianglelefteq G_{J_{i+1}}\pi_J$. Using the modular law we conclude from (1) and (2) that $G_{J_i}\pi_J = (N \cap G_{J_i}\pi_J) \times G_0\pi_J$. Now, $[N \cap G_{J_{i+1}}\pi_J, G_0\pi_J] \leqslant N \cap G_{J_i}\pi_J$. Since, for any $n \in N \cap G_{J_{i+1}}\pi_J$, $G_0\pi_J$ and $(G_0\pi_J)^n$ centralize $N \cap G_{J_i}\pi_J$, it follows that $[N \cap G_{J_{i+1}}\pi_J,\ G_0\pi_J] \leqslant Z(G_{J_i}\pi_J)$. Thus, if $G_0\pi_J$ is not normal in $G_{J_{i+1}}\pi_J = (N \cap G_{J_{i+1}}\pi_J) \cdot G_0\pi_J$, there exists some $n \in N \cap G_{J_{i+1}}\pi_J$ such that $1 \neq [n, G_0\pi_J] \subseteq Z(G_{J_i}\pi_J)$. But then φ: $G_{J_i}\pi_J \to Z(G_{J_i}\pi_J)$, defined by $(mx)\varphi = [n, x]$ for $m \in G_{J_i}\pi_J \cap N$ and for $x \in G_0\pi_J$, is a non-trivial homomorphism. This proves the first part of the lemma.

Now let $i = l-1$. We have shown that $1 \neq [n, G_0\pi_J] \subseteq Z(G_{J_{l-1}}\pi_J)$ for some $n \in N$, unless $G_0\pi_J \trianglelefteq G_J$. If $[n, G_0\pi_J] \subseteq Z(G_J)$, then $\pi_J\varphi$ is a non-trivial homomorphism from $G_{J_{l-1}}$ into $Z(G_J)$, where φ is defined as above. Hence $(|Z(G)|, |G/G'|) \neq 1$. So we may assume that there exists some $y \in G_J$ such that $[G_0\pi_J, n, y] \neq 1$. Since $Z(G_{J_{l-1}}\pi_J) \trianglelefteq G_J$, it follows that $[G_0\pi_J, n, y] \subseteq Z(G_{J_{l-1}}\pi_J) \cap G_{J_{l-1}} \leqslant Z(G_{J_{l-1}})$. Then ρ: $G_{J_{l-1}}\pi_J \to Z(G_{J_{l-1}})$, defined by $(mx)\rho = [x, n, y]$ for $x \in G_0\pi_J$ and for $m \in G_{J_{l-1}}\pi_J \cap N$, is a non-trivial homomorphism; this yields $(|Z(G)|, |G/G'|) \neq 1$ again.

6.5. THEOREM. *Let G be a directly indecomposable group with finite composition series. Assume that $(|Z(\overline{G})|, |\overline{G}/\overline{G}'|) = 1$ for all essential factor groups $\overline{G}$ of G. Then G is normally detectable.*

PROOF. By way of contradiction, we assume that G is not normally detectable. Since $(|Z(G)|, |G/G'|) = 1$, G is not nilpotent. We choose $J_l = (j_1, \ldots, j_l)$ as in Theorem 5.10 and note that (a)–(g) of 5.10 are satisfied. For $1 \leqslant i \leqslant l$, let $J_i = (j_1, \ldots, j_i)$. We now prove by induction on i $(1 \leqslant i \leqslant l)$ that

$$(*) \qquad G_{J_i}\pi_{J_l} = \left(G_{J_i}\pi_{J_l} \cap N_{J_l}\right) \times G_0\pi_{J_l},$$

where $N_{J_l} = G_{J_l} \cap \times\{G_{J_l, J} \mid J \in \mathbf{n}^l,\ J \neq J_l\}$. The case $i = l$ then yields a contradiction to the assumption that G_{J_l} is directly indecomposable (note 5.10(c), (d)). Clearly, by the modular law and by parts (a) and (b) of 5.10, $G_{J_i}\pi_{J_l}$ is a

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semidirect product of the normal subgroup $G_{J_i}\pi_{J_l} \cap N_{J_l}$ and the subnormal subgroup $G_0\pi_{J_l}$. Since $G_0\pi_{J_l} \trianglelefteq G_{j_1}\pi_{J_l}$, (*) holds for $i = 1$. Let $1 < i \leqslant l$ and suppose that (*) holds for $i - 1$. Employing 6.4, we see that either (*) holds for i, or that $(|G_{J_{i-1}}\pi_{J_l}/(G_{J_{i-1}}\pi_{J_l})'|, |Z(G_{J_{i-1}}\pi_{J_l})|) \neq 1$. But the latter alternative is impossible since, by induction hypothesis, $G_{J_{i-1}}\pi_{J_l}$ is an essential factor group of $G_{J_{i-1}} \cong G$. This completes the induction argument.

In the sequel we are going to enlarge our knowledge of the type of embedding of a possible counterexample to Hawkes' conjecture. For the proof of Theorem 6.8, which contains the relevant facts, two lemmas are needed.

General hypothesis for 6.6 *and* 6.7: G denotes a non-trivial directly indecomposable group with finite composition series, where we assume that $(|G/G'|, |Z(G)|) = 1$. In particular, G is not nilpotent. Let $G_0 \trianglelefteq G_1 \times \cdots \times G_n$ be a non-trivial embedding, where $n \geqslant 2$, where $G_j \cong G$ for $j = 0, 1, \ldots, n$, and where $G_0\pi_i \neq 1$ for $i = 1, \ldots, n$. Moreover, we use the notation from 5.2.

6.6. LEMMA. *Let r be a positive integer and let $J_r = (j_1, \ldots, j_r) \in \mathbf{n}^r$. Suppose that $\{I_{l(i)}\}_{i=1}^t$ is a (lexicographically ordered) set of $l(i)$-tuples, $l(i) \in \mathbb{N}$, $i = 1, \ldots, t$, $t \in \mathbb{N}$, such that $G_0 \leqslant \times_{i=1}^t G_{I_{l(i)}}$. For those $i \in \{1, \ldots, t\}$ for which $G_0\pi_{(j_2, \ldots, j_r, I_{l(i)})}$ is not nilpotent, assume that there exists $m(i) \in \mathbb{N}$ and $K_{m(i)} \in \mathbf{n}^{m(i)}$ such that $G_0\pi_{(j_2, \ldots, j_r, I_{l(i)}, K_{m(i)})} \cong G_0\pi_{(J_r, I_{l(i)})} \cong G_0\pi_{(J_r, I_{l(i)}, K_{m(i)})}$. (If $r = 1$, then $G_0\pi_{(j_2, \ldots, j_r, I)}$ stands for $G_0\pi_I$.) Then $Z_\infty(G_0\pi_{J_r}) \leqslant Z_\infty(G_{j_1}\pi_{J_r})$.*

PROOF. If $G_0\pi_{(j_2, \ldots, j_r, I_{l(i)})}$ is not nilpotent, then, putting $N_{r,l(i)} = G_{J_r, I_{l(i)}} \cap \times\{G_{(J_r, I_{l(i)}, K)} \mid K \in \mathbf{n}^{m(i)}, K \neq K_{m(i)}\}$, we have $G_{j_1}\pi_{(J_r, I_{l(i)})}/(G_{j_1}\pi_{(J_r, I_{l(i)})} \cap N_{r,l(i)}) \cong G_{j_1}\pi_{(J_r, I_{l(i)}, K_{m(i)})} \cong G_0\pi_{(J_r, I_{l(i)})}$ by the assumed properties of $K_{m(i)}$. Similarly, $G_0\pi_{(J_r, I_{l(i)})}/(G_0\pi_{(J_r, I_{l(i)})} \cap N_{r,l(i)}) \cong G_0\pi_{(J_r, I_{l(i)}, K_{m(i)})} \cong G_0\pi_{(J_r, I_{l(i)})}$. It follows that $G_0\pi_{(J_r, I_{l(i)})} \cap N_{r,l(i)} = 1$ and that $G_{j_1}\pi_{(J_r, I_{l(i)})} = (G_{j_1}\pi_{(J_r, I_{l(i)})} \cap N_{r,l(i)}) \times G_0\pi_{(J_r, I_{l(i)})}$. This implies that $Z_\infty(G_0\pi_{(J_r, I_{l(i)})}) \leqslant Z_\infty(G_{j_1}\pi_{(J_r, I_{l(i)})})$. The same statement holds trivially if $G_{j_1}\pi_{(J_r, I_{l(i)})}$ is nilpotent. We conclude that $Z_\infty(G_0\pi_{J_r}) \leqslant (\times_{i=1}^t Z_\infty(G_0\pi_{(J_r, I_{l(i)})})) \cap G_{j_1}\pi_{J_r} \leqslant (\times_{i=1}^t Z_\infty(G_{j_1}\pi_{(J_r, I_{l(i)})})) \cap G_{j_1}\pi_{J_r} \leqslant Z_\infty(G_{j_1}\pi_{J_r})$.

6.7. LEMMA. *Let $J^i = (j_1^i, j_2^i, \ldots) \in \mathscr{B}_\infty$ for some $i \in \{1, \ldots, b\}$. Suppose that $j = j_1^i \neq j_1^h$ for all $h \in \{1, \ldots, b\}$, $h \neq i$. Then the following statements hold.*

(a) *$j_l^i \neq j$ for all $l > 1$.*

(b) *$s(i) = 1$, i.e. $G_0\pi_{J_1^i} \cong G_0\pi_{J_l^i}$ for all $l \geqslant 1$.*

PROOF. (a) Suppose the assertion is false. If $r \geqslant 1$ is an integer such that $j_{r+1}^i = j$, then it follows from the hypothesis and from 5.4 that $j_k^i = j_{k+mr}^i$ for all positive integers k and m. Hence we may choose such an integer r with $r \geqslant s$

(where s is the stationary level of the iterated embedding of G_0; cf. 5.2(d)). Let $1 \leqslant k \leqslant r-1$ be arbitrary. If, for some $I \in \mathbf{n}^k$, $G_0\pi_{(j^i_{k+2},\ldots,j^i_{r+1},I)}$ is not nilpotent, then, by 5.4, $G_0\pi_{(j^i_{r+1},I)}$ is not nilpotent (note that $k+2 \leqslant r+1$). Since $j^i_{r+1}=j$, we deduce from the hypothesis that $I=(j^i_2,\ldots,j^i_{k+1})=(j^i_{r+2},\ldots,j^i_{r+k+1})$. Let σ denote the r-cycle $(1,\ldots,r)$ in the symmetric group S_r. Then 5.5(c) yields $G_0\pi_{(j^i_{k+2},\ldots,j^i_{r+1},I,j^i_{r+k+2})}=G_0\pi_{(J^i_r\sigma_1^{k+1},j^i_{k+2})}\cong G_0\pi_{J^i_r}$, $G_0\pi_{(j^i_{k+1},\ldots,j^i_{r+1},I)}=G_0\pi_{(J^i_r\sigma^k,j^i_{k+1})}$ $\cong G_0\pi_{J^i_r}$, and $G_0\pi_{(j^i_{k+1},\ldots,j^i_{r+1},I,j^i_{r+k+2})}=G_0\pi_{(J^i_r\sigma^k,j^i_{k+1},j^i_{k+2})}\cong G_0\pi_{J^i_r}$. Hence Lemma 6.6 is applicable. We conclude that

$$(*)\qquad Z_\infty\left(G_{J^i_k}\pi_{J^i_{r+1}}\right) \leqslant Z_\infty\left(G_{J^i_r}\pi_{J^i_{r+1}}\right)\quad\text{for all } 1 \leqslant k \leqslant r.$$

Moreover, a similar argument as above shows that if $(j^i_t,\ldots,j^i_{t+u},J)\in\mathscr{B}_{u+r+1}$, where $t \geqslant 1$, $u \geqslant r-1$, and $J \in \mathbf{n}^r$, then $J=(j^i_{t+u+1},\ldots,j^i_{t+u+r})$ and $G_0\pi_{(j^i_t,\ldots,j^i_{t+u})}\cong G_0\pi_{(j^i_t,\ldots,j^i_{t+r-1})}\cong G_0\pi_J$ (5.5(c)). In particular, $J^i_{r+1}\in\mathscr{B}^*_{r+1}$. It follows from 5.5(a) that $G_{J^i_{r+1}}$ is a semidirect product of N^i_{r+1} and $G_0\pi_{J^i_{r+1}}$.

We now aim to apply Lemma 6.4. Clearly, $G_0\pi_{J^i_{r+1}}\trianglelefteq G_{j^i_1}\pi_{J^i_{r+1}}$. Let $1 \leqslant k \leqslant r$. If $(|G_{J^i_k}\pi_{J^i_{r+1}}/(G_{J^i_k}\pi_{J^i_{r+1}})'|,\ |Z(G_{J^i_k}\pi_{J^i_{r+1}})|)\neq 1$, then, by means of $(*)$ and the fact that the prime divisors of $|Z_\infty(G_{J^i_r}\pi_{J^i_{r+1}})|$ and those of $|Z(G_{J^i_r}\pi_{J^i_{r+1}})|$ coincide, there exists a non-trivial homomorphism φ from $G_{J^i_k}$ into $Z(G_{J^i_r}\pi_{J^i_{r+1}})$. Now, $[G_{J^i_k}\varphi, G_{J^i_{r+1}}]\leqslant Z(G_{J^i_r}\pi_{J^i_{r+1}})\cap G_{J^i_r}\leqslant Z(G_{J^i_r})$. Hence, if $[G_{J^i_k}\varphi, g]\neq 1$ for some $g\in G_{J^i_{r+1}}$, then $\rho\colon G_{J^i_k}\varphi\to Z(G_{J^i_r})$, defined by $y\rho=[y,g]$ for all $y\in G_{J^i_k}\varphi$, is a non-trivial homomorphism, contradicting $(|G/G'|,\ |Z(G)|)=1$. Therefore, $G_{J^i_k}\varphi \leqslant Z(G_{J^i_{r+1}})$. But this leads to the same contradiction. Consequently, $(|G_{J^i_k}\pi_{J^i_{r+1}}/(G_{J^i_k}\pi_{J^i_{r+1}})'|,\ |Z(G_{J^i_k}\pi_{J^i_{r+1}})|)=1$ for all $1\leqslant k\leqslant r$. So we apply Lemma 6.4 to deduce that $G_0\pi_{J^i_{r+1}}\trianglelefteq G_{J^i_{r+1}}$. Thus $G_{J^i_{r+1}}=N^i_{r+1}\times G_0\pi_{J^i_{r+1}}$. Since the embedding of G_0 in $G_1\times\cdots\times G_n$ is non-trivial, this contradicts the hypothesis that G is directly indecomposable.

(b) Let $l\geqslant 1$ be arbitrary and set $N_{1,l}=G_{j^i_1}\cap\times\{G_{(j^i_1,J)}\mid J\in\mathbf{n}^{l-1},\ J\neq J^i_{[2,l]}\}$. By hypothesis, $G_0\pi_{j^i_1}/(G_0\pi_{j^i_1}\cap G_{J^i_l})\cong G_0\tilde{\pi}$ is nilpotent, where $\tilde{\pi}=\Sigma\{\pi_{(j^i_1,J)}\mid J\in\mathbf{n}^{l-1},\ J\neq J^i_{[2,l]}\}$. It follows from 3.3(e) that $G_0\cap N_{1,l}=1$. This implies that $G_0\pi_{j^i_1}\cap N_{1,l}$ is central in $G_{j^i_1}$. If this group is non-trivial, then there exists a non-trivial homomorphism from the nilpotent group $G_0\pi_{j^i_1}/(G_0\pi_{j^i_1}\cap G_{J^i_l})$ into $((G_0\pi_{j^i_1}\cap N_{1,l})\times(G_0\pi_{j^i_1}\cap G_{J^i_l}))/(G_0\pi_{j^i_1}\cap G_{J^i_l})\cong G_0\pi_{j^i_1}\cap N_{1,l}\leqslant Z(G_{j^i_1})$ (Lemma 3.2). This contradicts $(|G/G'|,\ |Z(G)|)=1$. Consequently, $G_0\pi_{j^i_1}\cap N_{1,l}=1$, i.e. $G_0\pi_{j^i_1}=G_0\pi_{J^i_l}$.

6.8. **Theorem.** *Let G be a group with finite composition series. Assume that G is directly indecomposable and that $(|G/G'|,\ |Z(G)|)=1$. If G is not normally detectable, and if $G_0\trianglelefteq G_1\times\cdots\times G_n$ (where $n\geqslant 2$, $G_j\cong G$ for $j=0,1,\ldots,n$, and $G_0\pi_i\neq 1$ for $i=1,\ldots,n$) is a non-trivial embedding, then the following hold.*

(a) *The stationary level s of the iterated embedding is at least* 3.

(b) *The branching number b of the iterated embedding is at least $n+1$. In particular, $b\geqslant 3$.*

(c) *If* $b = 3$, *then, up to interchanging the indices* 1 *and* 2, *there is only one possible embedding. This is of the following type*: $\mathscr{B}_\infty = \{J^1, J^2, J^3\}$, *where* $J^1 = (1,1,1,1,\ldots)$, $J^2 = (1,2,1,1,\ldots)$, *and* $J^3 = (2,1,1,1,\ldots)$. *Moreover, in this case* $s(1) > 2$, $s(2) = 2$, $s(3) = 1$ *and* $\mathscr{B}_\infty^{**} = \{J^1\}$; *in fact*, $\mathscr{B}_k^* = \{J_k^1\}$ *for all* $k \geqslant s$.

PROOF. (a) Using 5.8, we choose $J_s^i \in \mathscr{B}_s^*$. Then, by 5.5(a), $G_{J_s^i}$ is a semidirect product of N_s^i and $G_0\pi_{J_s^i}$. Since the embedding is non-trivial, both factors are distinct from 1. The hypothesis that G is directly indecomposable yields $s > 1$. Moreover, $s = 2$ is impossible by the last statement of Lemma 6.4.

(b) Clearly, $b \geqslant n$. By 3.3(e), (f), $G_0\pi_i \neq 1$ if and only if $G_0\pi_i \nleqslant \mathrm{Fit}(G_i)$. Hence, if $b = n$, we may assumes that $J_1^i = (i)$ for $i = 1, \ldots, n$. It follows from 6.7 that $s(i) = 1$ for all i, i.e. that $s = 1$. This contradicts part (a). Thus $b \geqslant n + 1 \geqslant 3$.

(c) If $b = 3$, then $n = 2$ by part (b). By using 5.4 and 6.7, it is easy to see that, up to interchanging the indices 1 and 2, $J^1, J^2, J^3 \in \mathscr{B}_\infty$ have to be of the form given above. By 6.7, $s(3) = 1$. Hence $G_0\pi_2 \cong G_0\pi_{(2,J_l^1)}$ for all $l \geqslant 1$. We conclude that $G_1\pi_{12} \cong G_1\pi_{(1,2,J_l^1)}$. This implies that $G_0\pi_{12} \cong G_0\pi_{(1,2,J_l^1)}$ for all $l \geqslant 1$, i.e. that $s(2) \leqslant 2$. If $s(2) = 1$, then $G_0\pi_1 \cong G_0\pi_{12} \trianglelefteq G_1\pi_{12} \cong G_0\pi_2$, and, as $s(3) = 1$, $G_0\pi_2 \cong G_0\pi_{21} \trianglelefteq G_2\pi_{21} \cong G_0\pi_1$, i.e. $G_0\pi_1 \cong G_0\pi_2 \cong G_0\pi_{21}$. But this leads to a non-trivial decomposition $G_2 = (G_2 \cap G_{22}) \times G_0\pi_2$, which is a contradiction. Hence $s(2) = 2$. Then $s(1) > 2$ by part (a).

By 5.4, $G_0\pi_{J_s^2}$ and $G_0\pi_{J_s^3}$ are isomorphic to subnormal subgroups of $G_0\pi_{J_s^1}$. Therefore, $J_k^1 \in \mathscr{B}_k^*$ for all $k \geqslant s$, according to 5.8. If $J_k^i \in \mathscr{B}_k^*$ for some $i \in \{2,3\}$, and if $k \geqslant s$, then, by definition of $\mathscr{B}_k^*$, $G_0\pi_{J_k^i} \cong G_0\pi_{J_s^1}$. Since $s(2) = 2$ and $s(3) = 1$, it follows that $G_0\pi_{12} \cong G_0\pi_{J_s^1}$ (in case $i = 2$), or that $G_0\pi_2 \cong G_0\pi_{J_s^1}$ (in case $i = 3$). In the first case, let $N = G_{12} \cap \times\{G_{(1,2,J)} \mid J \in \mathbf{n}^s,\ J \neq J_s^1\}$. Then $G_{12}/N \cong G_0\pi_{J_s^1} \cong G_0\pi_{12}$, and $G_0\pi_{12}/(G_0\pi_{12} \cap N) \cong G_0\pi_{(1,2,J_s^1)} \cong G_0\pi_{12}$. It follows that G_{12} is a semidirect product of N and $G_0\pi_{12}$. The last statement of Lemma 6.4 now leads to a contradiction. The case $G_0\pi_2 \cong G_0\pi_{J_s^1}$ is even easier and leads in a similar way to a non-trivial direct decomposition of G_2.

We have been unable to decide whether the situation in 6.8(c) can actually occur.

6.9. THEOREM. *Let G be a group with finite composition series. Assumes that G is directly indecomposable and that* $(|G/G'|, |Z(G)|) = 1$. *Then G is normally detectable, provided that one of the following conditions is satisfied.*

(1) *If* $\alpha \in \mathrm{Aut}(G)$ *is such that* $\langle\alpha\rangle \cong H/M$ *for some* $H \leqslant G$, $M \trianglelefteq H$, *then* α *preserves the conjugacy classes in G.*

(2) *G possesses a unique maximal normal subgroup.*

(3) Soc(G) *is a direct product of at most two minimal normal subgroups of G or of three minimal normal subgroups of G which are not all elementary abelian of the same order.*

(4) Fit(G) $\leqslant$ Soc(G).

(5) *G is finite and, for every prime divisor p of* $|G/G'|$, *the Sylow p-subgroups of G are abelian.*

PROOF. (1) and (2) follow immediately from 6.2(e) and (a), respectively, and 6.2(c) and (d) imply (4). We note in passing that for case (1) a direct argument along the lines of the proof of Theorem 4.2 is also possible. This is because condition (1) implies that G_0 is normal in $\times_{J \in \pi^l} G_J$ for every $l \in \mathbb{N}$.

From Theorem 6.8(b) we see that the first condition in (3) forces G to be normally detectable. Moreover, if G is not normally detectable, and if Soc(G) is a direct product of three minimal normal subgroups of G, then we are in the situation of Theorem 6.8(c). It follows that $G_0 \cap G_2$, $G_0 \cap G_{12}$, and $G_0 \cap G_{11}$ each contain exactly one minimal normal subgroup of G_0. It is a straightforward matter to show that the minimal normal subgroup of G_0 contained in $G_0 \cap G_2$ ($G_0 \cap G_{12}$) coincides with the minimal normal subgroup of G_2 (G_1) contained in $G_2 \cap G_{211}$ ($G_1 \cap G_{12}$). This implies that all minimal normal subgroups of G are isomorphic. Since at least one of them is abelian (Theorem 6.2(d)), we have the situation which is excluded in (3).

Finally, because G is not nilpotent or $G = 1$, (5) follows immediately from 5.11.

7. Concluding remarks

A. *Characteristic embeddings*

The method of iterated embeddings works smoothly for transitive relations such as subnormality. So it is no surprise that this idea provides a short proof for the following result on characteristic embeddings.

7.1. PROPOSITION. *Let G be a group satisfying the minimal or maximal condition on normal subgroups. If* G_0 *is a characteristic subgroup of* $G_1 \times \cdots \times G_n$, *where* $n \geqslant 2$, *and* $G_j \cong G$ *for* $j = 0, 1, \ldots, n$, *then* $G = 1$.

PROOF. If $G_0 \cap G_i = 1$ for some i, then $G_0 \cap G_j = 1$ for all $j = 1, \ldots, n$, since the symmetric group S_n acts in a natural way on $G_1 \times \cdots \times G_n$. By 3.3(c), $G_0\pi_j$ is centralized by every automorphism of G_j. It follows easily that G is an elementary abelian 2-group. But then $\mathrm{Aut}(G_1 \times \cdots \times G_n)$ acts transitively on the set of non-trivial elements of $G_1 \times \cdots \times G_n$. Now any of the assumed finiteness conditions forces G to be trivial.

So assume that $G_0 \cap G_j \neq 1$ for all j. In the notation of 5.2(a), G_0 is embedded as a characteristic subgroup in $\times_{J \in \mathbf{n}^l} G_J$, $G_J \cong G$, for any positive integer l. By the argument given above, $G_0 \cap G_J \neq 1$ for all $J \in \mathbf{n}^l$. Hence G_0 contains direct products of normal subgroups of unbounded length, which contradicts both the minimal and maximal condition on normal subgroups.

B. *Central products*

The following proposition shows that Theorem 4.2 can be extended to the more general case of central products instead of direct products, at least for groups admitting no non-trivial central extensions.

We recall that a group G is a central product of the normal subgroups $N_1, \ldots, N_k$ if $G = N_1 \cdots N_k$ and $[N_i, N_j] = 1$ for all $i, j = 1, \ldots, k$, $i \neq j$.

7.2. PROPOSITION. *Let G be a group without non-trivial central extensions. If G is subnormally (normally) detectable, then the following hold: whenever G_0 is subnormal (normal) in $G_1 \cdots G_n$, where $G_j \cong G$ for $j = 0, 1, \ldots, n$, and where $G_1 \cdots G_n$ is a central product of $G_1, \ldots, G_n$, then $G_0 = G_i$ for some $i \in \{1, \ldots, n\}$.*

PROOF. Let G_0 be subnormal (normal) in $G_1 \cdots G_n$. The map $\varphi\colon G_1 \times \cdots \times G_n \to G_1 \cdots G_n$ defined by $(g_1, \ldots, g_n)\varphi = g_1 \cdots g_n$ is onto, and $\ker\varphi \leqslant Z(G_1 \times \cdots \times G_n)$. Let $G_0\varphi^{-1}$ denote the full preimage of G_0 in $G_1 \times \cdots \times G_n$. By assumption, $G_0\varphi^{-1} = K \times \ker\varphi$, where $K \cong K\varphi = G_0$. Clearly, $G_0\varphi^{-1}$ is subnormal (normal) in $G_1 \times \cdots \times G_n$. If G is subnormally detectable, then $K \trianglelefteq G_0\varphi^{-1} \trianglelefteq\trianglelefteq G_1 \times \cdots \times G_n$ implies $K = G_i$ for some i. Consequently, $G_0 = G_i$. Now assume that G is normally detectable and that $G_0 \trianglelefteq G_1 \cdots G_n$. If K is not normal in $G_1 \times \cdots \times G_n$, there exists $g \in G_1 \times \cdots \times G_n$ such that $[K, g] \neq 1$. For $k \in K$ define $k\rho \in \ker\varphi$ by $[k, g] = k' \cdot k\rho$, $k' \in K$. Then $\rho\colon K \to \ker\varphi$ is a non-trivial homomorphism. From $\ker\varphi \leqslant Z(G_1 \times \cdots \times G_n)$ we deduce the existence of a non-trivial homomorphism from G into $Z(G)$. But this contradicts the fact that G is normally detectable (compare the remark at the beginning of Section 6). Hence $K \trianglelefteq G_1 \times \cdots \times G_n$, and the desired conclusion follows as in the case of subnormally detectable G.

C. *Lie algebras, associative rings*

The argument used to prove Theorem 4.2 is valid not only for groups. Instead of finding the most general version of 4.2 in the framework of universal algebra, we content ourselves with two important cases.

We recall that for a (not necessarily associative) ring R the Fitting radical $\mathrm{Fit}(R)$ is defined to be the subring of R generated by all nilpotent ideals of R. (Here and in the sequel, ideal means 2-sided ideal.)

Subideals are defined analogously to subnormal subgroups. (For associative rings, these are just the meta-ideals of finite index in the language of Baer [2].)

7.3. THEOREM. *Let L be a Lie algebra over R, where R is an associative and commutative ring with* 1. *Assume that L satisfies the minimal condition on subideals. Then the following statements are equivalent.*

(i) *Whenever L_0 is a subideal of $L_1 \oplus \cdots \oplus L_n$, where $L_j \cong L$ for $j = 0, 1, \ldots, n$, then $L_0 = L_i$ for some $i \in \{1, \ldots, n\}$.*

(ii) *L is directly indecomposable (as a direct sum of ideals), and there exists no non-trivial homomorphism from L into a nilpotent subideal of L.*

PROOF. One simply translates the proofs of Lemma 3.3(a) and Theorem 4.2 into the language of Lie algebras.

7.4. COROLLARY. *Let L be a Lie algebra over K, where K is a field. Assume that L satisfies the minimal condition on subideals. Then the following statements are equivalent.*

(i) *Whenever L_0 is a subideal of $L_1 \oplus \cdots \oplus L_n$, where $L_j \cong L$ for $j = 0, 1, \ldots, n$, then $L_0 = L_i$ for some $i \in \{1, \ldots, n\}$.*

(ii) *L is directly indecomposable, and either L is perfect (i.e. $L = [L, L]$) or there exists no non-trivial abelian subideal of L.*

(In characteristic 0, L contains no non-trivial abelian subideals if and only if $\mathrm{Fit}(L) = 0$.)

PROOF. Because of the fact that, for $\mathrm{char}\, K = 0$, the Fitting radical is a nilpotent ideal which contains every nilpotent subideal of L (see Amayo, Stewart [1, Theorem 6.2.1, Lemma 8.1.3]), the conclusion follows from 7.3.

7.5. THEOREM. *Let R be an associative ring which satisfies the minimal condition on subideals. Then the following statements are equivalent.*

(i) *Whenever R_0 is a subideal of $R_1 \oplus \cdots \oplus R_n$, where $R_j \cong R$ for $j = 0, 1, \ldots, n$, then $R_0 = R_i$ for some $i \in \{1, \ldots, n\}$.*

(ii) *R is directly indecomposable, and there exists no non-trivial homomorphism from R into* $\mathrm{Fit}(R)$.

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PROOF. By a result of Baer [2, Corollary 5], every subring of Fit(R) is a subideal of R because R satisfies the minimal condition on ideals. Moreover, it follows from [2, Proposition 8] that every nilpotent subideal of R is contained in Fit(R). Now the argument of 4.2 is adaptable.

Although Hawkes' conjecture on normally detectable groups (and also the corresponding problem for Lie algebras) remains unsettled, the analogous situation in associative rings with unit can easily be handled.

7.6. PROPOSITION. *Let R be an associative ring with unit. Assume that R is indecomposable (as a direct sum of 2-sided ideals). If R_0 is a 2-sided ideal in $R_1 \oplus \cdots \oplus R_n$, where $R_j \cong R$ for $j = 0, 1, \ldots, n$, then $R_0 = R_i$ for some $i \in \{1, \ldots, n\}$.*

PROOF. If e is the unit of R_0, it is easily shown that e is a central idempotent in $R_1 \oplus \cdots \oplus R_n$. Consequently, R_0 is a direct summand of $R_1 \oplus \cdots \oplus R_n$. But then $R_0 = R_i$ for some i (see Lambek [5, 1.4, Proposition 12]).

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