CURVATURE-TORSION ENTROPY FOR TWISTED CURVES UNDER CURVE SHORTENING FLOW

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Abstract

We study curve-shortening flow for twisted curves in $\mathbb{R}^3$ (that is, curves with nowhere vanishing curvature κ and torsion τ) and define a notion of torsion-curvature entropy. Using this functional, we show that either the curve develops an inflection point or the eventual singularity is highly irregular (and likely impossible). In particular, it must be a Type-II singularity which admits sequences along which $\tau/\kappa^2 \to \infty$. This contrasts strongly with Altschuler's planarity theorem, which shows that $\tau/\kappa \to 0$ along any essential blow-up sequence.

1. Introduction

Curve shortening flow is the geometric flow defined by the equation

$$\partial_t \gamma = \kappa N, \tag{1.1}$$

where γ is a smooth immersed curve in $\mathbb{R}^n$, κ is the curvature and N is the unit normal vector. Solutions to this flow consist of a family of curves γ_t for $t \in [0, \omega)$ with γ_0 (which we will often denote as γ) as the initial condition.

This flow was introduced by Gage and Hamilton in 1986 as the L^2 gradient flow for the arc length (that is, the flow which shortens curves the quickest) [5]. Their work established short-time existence and uniqueness for the flow and showed that if one starts with a closed convex curve in the plane, the curve shrinks to a point while becoming asymptotically round. Put succinctly, convex curves shrink to round points. The following year, Grayson [6] proved that any curve which is initially embedded (that is, does not self-intersect) in $\mathbb{R}^2$ eventually becomes convex under the flow, and thus converges to a round point.

In two dimensions, curve shortening flow has two fundamental properties which play a crucial role in its analysis. First, if a curve starts as an embedded curve, it remains embedded and does not self-intersect until it reaches a singularity. More generally, if one considers mean-curvature flow for co-dimension one hypersurfaces, the flow is self-avoiding. In other words, surfaces that do not intersect at the initial time will never intersect in the future. Second, the number of inflection points (that is, points where the curvature changes signs) is nonincreasing under the flow. Both of these facts can be shown by a straightforward application of the parabolic maximum principle but fail for curve shortening flow in higher dimensions.

1.1. Singularity formation. Since curve shortening flow shrinks the length of curves as quickly as possible in the L^2 topology, a closed curve must encounter a singularity at some time ω. For closed and embedded curves in the plane, the enclosed area decreases linearly at the rate of -2π, so it is possible to compute this time explicitly. The singularities are fairly well understood in two dimensions and a natural question is to extend these results to curve shortening flow in dimensions three or higher. To discuss this further, we first introduce some notation.

Curve shortening flow exists so long as the curvature is bounded. So to study singularities, we consider a *blow-up sequence*, which is a sequence of points in space-time (p_m, t_m) such that the curvature at (p_m, t_m) goes to infinity. A blow-up sequence is said to be *essential* if $\kappa^2(p_m, t_m) \geq \rho M_{t_m}$ for some $\rho > 0$ where

$$M_t = \sup_{p \in \gamma} \kappa^2(p, t).$$

One can divide the singularities of curve shortening flow into two broad classes: *Type I* and *Type II*. A singularity is said to be Type I if

$$\limsup_{t \to \omega} M_t \cdot (\omega - t)$$

is bounded and Type II otherwise. Type-I singularities are *global* singularities, in that the entire curve shrinks to a point while converging in C^∞ to a homothetic (that is, self-similar) shrinking curve. For closed curves in two dimensions, the possible models for these singularities were classified by Abresch and Langer [1]. However, Type-II singularities are *local* in that the curvature goes to infinity in a small region while possibly remaining bounded elsewhere. Such singularities appear as kinks in the curve. In particular, they admit an essential blow-up sequence which (after rescaling) converges in C^∞ to the Grim Reaper curve $y = -\log(\cos x)$, which is a translating soliton under the flow.

As we have mentioned, curve shortening flow behaves differently in higher dimensions, which complicates its analysis. However, Altschuler [2] established that under the flow, curves in three-dimensional space become asymptotically *planar* near any singularity. More precisely, along any essential blow-up sequence (p_m, t_m), the torsion τ satisfies

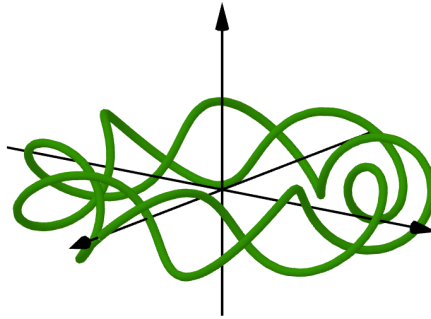$$\lim_{m \to \infty} \frac{\tau}{\kappa}(p_m, t_m) = 0.$$

FIGURE 1. An example of a twisted curve.

This result was then extended to curve shortening flow in $\mathbb{R}^n$ by Yan and Jiao [11]. As such, the singularity models for curve shortening flow in higher dimensions are the same as for curve shortening flow in the plane, although we cannot rule out the appearance of Abresch–Langer solutions (possibly covered multiple times) or Type-II singularities even when the initial curve is embedded.

2. Twisted curves and the curvature-torsion entropy

A curve in $\mathbb{R}^n$ is said to be *twisted* if it has linearly independent derivatives up to order n [4]. In three dimensions, this corresponds to the nowhere vanishing of the curvature and torsion along the curve. As an archetypal example, one can consider a curve which wraps tightly around a torus, as shown in Figure 1.

The main focus of this paper is to study curve shortening flow for twisted curves (that is, curves that are twisted). In particular, we focus on the possible singularities which emerge when a curve is twisted.

THEOREM 2.1. *Suppose γ_t is a solution to curve shortening flow which is twisted up to the time of singularity ω. Then:*

(1) *the singularity is Type II;*
(2) *there exists a sequence (p_m, t_m) so that $t_m \to \omega$ and*

$$\frac{\tau}{\kappa^2}(p_m, t_m) \to \infty.$$

Note that the sequences where $\tau/\kappa^2 \to \infty$ cannot be essential, and may not even be a blow-up sequence at all. To establish this fact, we find a quantity that is (nearly) increasing under the flow. In particular, we will study the following entropy functional, whose behaviour is well controlled under the flow.

DEFINITION 2.2. For a twisted curve in $\mathbb{R}^3$, the curvature-torsion entropy is defined to be the quantity

$$\int_\gamma \kappa \log\left(\frac{\tau}{\kappa^2}\right) ds. \tag{2.1}$$

One could extend this definition to nontwisted curves by squaring the argument of the logarithm. However, this integral will essentially always be $-\infty$ whenever there is a flat point (that is, a point with $\tau = 0$), so we will not consider this generalisation.

3. Nearly monotonic functionals

To prove the main result, we must show that the curvature-torsion entropy is nearly nondecreasing under the flow. Before doing so, we start with a simpler proof that a twisted curve cannot develop a Type-I singularity. This result was previously shown in unpublished work of the author [7]. Since the proof is very short, we include it here.

THEOREM 3.1. *Suppose γ is a twisted curve in $\mathbb{R}^3$. Under curve shortening flow, one of the following two possibilities occurs.*

(1) *There exists a time t_0 where γ_{t_0} has a point with vanishing curvature. Furthermore, after this time, γ_t has a flat point until the singular time.*
(2) *γ_t develops a Type-II singularity.*

PROOF. Consider any curve γ (not necessarily twisted) which develops a Type-I singularity at time ω. After rescaling, γ_t approaches an Abresch–Langer solution [1] with finite winding number in the C^∞ sense [2]. This has two consequences.

(a) All blow-up sequences are essential. In other words, for times close to the singularity, the maximum curvature is a bounded multiple of the minimum curvature. As a result, there exists a time $t_0 \in [0, \omega)$ such that for all times afterward, γ_t has no inflection points, which implies that torsion is defined everywhere on the curve.

(b) Since γ converges to some Abresch–Langer curve, the functional $D(t) = \sup \kappa_t \cdot L_t$ converges to a finite limit as the time goes to ω, where L_t is the length of the curve. In particular, this quantity remains bounded.

Now we consider a solution γ_t which is twisted after t_0. For the following calculations, we parametrise γ_t smoothly by $u \in [0, 2\pi)$ and suppose that the curvature has velocity v. To establish the result, we show that for a twisted curve, the total torsion is increasing:

$$\int_{\gamma_t} \tau \, ds = \int_0^{2\pi} \tau \cdot v \, du.$$

To compute the evolution of this quantity, we use the evolution equations for v, κ and τ (derived in [2, 3]):

$$\begin{aligned}
\partial_t v &= -\kappa^2 v; \\
\partial_t \kappa &= \partial_s^2 \kappa + \kappa^3 - \kappa \tau^2; \\
\partial_t \tau &= 2\kappa^2 \tau + \partial_s\left(\frac{2\tau}{\kappa}\partial_s \kappa\right) + \partial_s^2 \tau.
\end{aligned} \tag{3.1}$$

Using these, we find

$$\partial_t \int_{\gamma_t} \tau \cdot v \, du = \int_0^{2\pi} (\partial_t \tau) \cdot v + (\partial_t v) \cdot \tau \, du$$

$$= \int_0^{2\pi} \left(2\kappa^2 \tau + \partial_s \left(\frac{2\tau}{\kappa} \partial_s \kappa \right) + \partial_s^2 \tau \right) v - \kappa^2 v \tau \, du$$

$$= \int_{\gamma_t} \kappa^2 \tau \, ds + \int_\gamma \partial_s \left(\frac{2\tau}{\kappa} \partial_s \kappa \right) + \partial_s^2 \tau \, ds$$

$$= \int_{\gamma_t} \kappa^2 \tau \, ds.$$

Therefore, the L^1 norm of τ is increasing and approaches a positive (possibly infinite) limit as t goes to ω. However,

$$\sup_{p \in \gamma_t} \tau(p) \cdot L_t \geq \|\tau\|_1(t) > 0.$$

When combined with the fact that $D(t)$ remains bounded, this implies that

$$\limsup_{t \to \omega} \sup_{p \in \gamma_t} \frac{\tau}{\kappa} > 0,$$

which contradicts the planarity theorem. As a result, the curve must develop a Type-II singularity. To complete the proof, we show that if a curve is twisted, it cannot become untwisted unless a point of zero curvature appears.

PROPOSITION 3.2. *Suppose we have a family of curves γ_t in $\mathbb{R}^3$ which satisfy (1.1) and that γ_0 is twisted. Furthermore, suppose that γ_t has no inflection points for $t \in [0, t_0]$. Then for all $t \in [0, t_0]$, γ_t is also twisted.*

This proposition follows from the maximum principle. Suppose that we have a point (p, t) so that $\tau(p, t) = 0$ and this is the first time when τ is ever nonpositive. Since both τ and $\partial_s \tau$ are zero,

$$\partial_t \tau(p) = \partial_s^2 \tau + 2(\partial_s \log \kappa)(\partial_s \tau) + 2\tau(\partial_s^2 \log \kappa + \kappa^2) = \partial_s^2 \tau \geq 0.$$

By the strong parabolic maximum principle, τ must remain strictly positive.   □

This argument fails at inflection points, where torsion is not defined. As such, this result shows that for a Type-I singularity to develop from a twisted curve, an inflection point must emerge and, afterward, a flat point is created where the inflection point occurred. It might seem curious that a single flat point will emerge as we initially expect the torsion to switch signs twice. However, the Frenet–Serret frame bundle can become nontrivial after the emergence of an inflection point.

3.1. The main result. We now turn our attention to proving the main result. To begin, we compute several more time derivatives:

$$\partial_t \int_{\gamma_t} \kappa \, ds = - \int_{\gamma_t} \kappa \tau^2 \, ds$$

$$\partial_t \int_{\gamma_t} \kappa \log \kappa \, ds = \int_{\gamma_t} (\log \kappa + 1)(\partial_s^2 \kappa + \kappa^3 - \kappa \tau^2) - \kappa^3 \log \kappa \, ds$$

$$= \int_{\gamma_t} -\frac{(\partial_s \kappa)^2}{\kappa} - (\kappa \log \kappa)\tau^2 + \kappa^3 - \kappa \tau^2 \, ds \qquad (3.2)$$

$$\partial_t \int_{\gamma_t} \kappa \log \tau \, ds = \int_{\gamma_t} (\partial_s^2 \kappa + \kappa^3 - \kappa \tau^2) \log \tau$$

$$+ \frac{\kappa}{\tau}(2\kappa^2 \tau + \partial_s(2\tau \partial_s \log \kappa) + \partial_s^2 \tau) - \kappa^3 \log \tau \, ds$$

$$= \int_{\gamma_t} -\kappa \tau^2 \log \tau + 2\kappa^3 - 2\frac{(\partial_s \kappa)^2}{\kappa} + \kappa (\partial_s \log \tau)^2 \, ds. \qquad (3.3)$$

Combining (3.2) and (3.3), we find the following evolution equation for the curvature-torsion entropy:

$$\partial_t \int_{\gamma_t} \kappa \log\left(\frac{\tau}{\kappa^2}\right) ds = \int_{\gamma_t} -\kappa \tau^2 \log\left(\frac{\tau}{\kappa^2}\right) + \kappa (\partial_s \log \tau)^2 + 2\kappa \tau^2 \, ds. \qquad (3.4)$$

This immediately provides a second proof that twisted curves cannot develop Type-I singularities. In particular, for a Type-I singularity, τ/κ^2 goes to zero uniformly (by Altschuler's planarity theorem and the fact that all sequences to the singular time are essential). As such, the curvature-torsion entropy must go to negative infinity for a Type-I singularity. However, once $\sup \log(\tau/\kappa^2) < 2$, the curvature-torsion entropy is increasing.

Furthermore, we can also show that $\log(\tau/\kappa^2)$ blows up along some sequence. Suppose that there is a uniform bound

$$\log\left(\frac{\tau}{\kappa^2}\right) < C_1.$$

Then (2.1) must go to negative infinity as t goes to ω. To see this, consider the region $S \subset \gamma_t$ where $\log(\tau/\kappa^2) > 0$. On this set,

$$\int_S \kappa \log\left(\frac{\tau}{\kappa^2}\right) ds \leq C_1 \int_S \kappa \, ds \leq C_1 \int_{\gamma_0} \kappa \, ds,$$

(since the total curvature is decreasing). As such, the positive part of the integral is bounded from above. However, γ_t has a region that converges to a Grim Reaper curve, whose curvature-torsion entropy is $-\infty$.

To reach a contradiction, we can estimate the left-hand side of (3.4) by

$$\partial_t \int_{\gamma_t} \kappa \log\left(\frac{\tau}{\kappa^2}\right) ds \geq \int_{\gamma_t} -(C_1 - 2)\kappa \tau^2 \, ds = (C_1 - 2)\partial_t \int_{\gamma_t} \kappa \, ds.$$

Since the total curvature decreases to a positive limit as t goes to ω, this shows that the difference between the curvature-torsion entropy at the present time and the singular time is bounded from below, and so cannot go to negative infinity.

3.2. Points with large torsion and curvature.   Theorem 2.1 shows that the potential singularities of a twisted curve are highly unusual and we suspect they cannot occur. One strategy would be to try to find an essential sequence so that τ/κ does not go to zero. Although we cannot show this (or even that $\log(\tau/\kappa^2)$ goes to infinity along a blow-up sequence), we can show a weaker result in this direction.

PROPOSITION 3.3. *Suppose γ is a curve in $\mathbb{R}^3$ which develops a twisted singularity. Then one of the following two possibilities occurs.*

(1) *There exists a sequence (p_n, t_n) so that $\kappa(p_n, t_n) \to 0$.*
(2) *There is a sequence (p_n, t_n) so that both*

$$\tau \quad and \quad (2\kappa^2 + 2\partial_s^2 \log \kappa)(\omega - t_n)^\alpha$$

tend to infinity for $\alpha < 1$. Furthermore, this sequence consists of the points whose torsion is maximised on γ_{t_n}.

PROOF. If we assume that κ is bounded away from zero near the singular time, then Theorem 2.1 shows that $\log(\tau)$ must go to infinity, and thus we must show that we can extract a sequence so that $(2\kappa^2 + 2\partial_s^2 \log \kappa)(\omega - t)^\alpha$ blows up. We calculate

$$\partial_t \log(\tau) = \frac{1}{\tau}\left(2\kappa^2\tau + \partial_s\left(\frac{2\tau}{\kappa}\partial_s\kappa\right) + \partial_s^2\tau\right)$$

$$= 2\kappa^2 + 2\partial_s^2 \log \kappa + 2(\partial_s \log \tau)(\partial_s \log \kappa) + \frac{\partial_s^2\tau}{\tau}.$$

At the maximum of $\log(\tau)$, the maximum principle implies that

$$\partial_s \log \tau = 0 \quad and \quad \frac{\partial_s^2\tau}{\tau} \le 0.$$

At the maximum of $\log(\tau)$, we find that

$$\partial_t \log(\tau) \le 2\kappa^2 + 2\partial_s^2 \log \kappa.$$

However, for $\sup \log(\tau)$ to go to infinity in finite time, Hamilton's maximum principle shows we must be able to extract a subsequence where

$$2\kappa^2 + 2\partial_s^2 \log \kappa$$

goes to infinity.

To show the stronger estimate $(2\kappa^2 + 2\partial_s^2 \log \kappa(p_n, t_n))(\omega - t)^\alpha \to \infty$, we suppose that this is not the case. Then, we divide up the time interval (t, ω) into sub-intervals $(t_i, (\omega + t_i)/2)$ and apply the maximum principle on each sub-interval. Doing so, we obtain the estimate

$$\sup_{t \leq t_n} \log \tau(p, t) - \sup \log \tau(p, t_0) \leq \sum_{i=0}^{n} \frac{C_2}{2^{i(1-\alpha)}}.$$

This estimate is uniformly bounded in n, giving a contradiction. $\qquad\square$

A similar argument shows that we can extract sequences so that both $\kappa^\rho \tau$ and $((2 + \rho)\kappa^2 + 2\partial_s^2 \log \kappa)(t - \omega)^\alpha$ go to infinity for any $\rho \in (0, 1]$. However, we cannot conclude that τ must go to infinity on such a sequence, so we do not have a geometric application for this fact.

3.3. A related quantity.   As can be seen from Theorems 2.1 and 3.1, it is possible to control the behaviour of curve shortening flow by finding quantities that are monotone (or nearly monotone) under the flow. There are several other quantities whose evolution is quite simple, and it might be possible to use them to control the singularities that emerge under the flow. For instance, the quantity $\int_{\gamma_t} \tau \log(\tau^2/\kappa^4)\, ds$ evolves as follows:

$$\partial_t \int_{\gamma_t} \tau \log\left(\frac{\tau^2}{\kappa^4}\right) ds = \int_{\gamma_t} \left(\kappa^2 \tau \log\left(\frac{\tau^2}{\kappa^4}\right) + \tau\left((\partial_s \log \kappa^2)^2 - \frac{1}{2}(\partial_s \log \tau^2)^2\right) + 4\tau^3\right) ds.$$

In this expression, the terms in the argument of the logarithm have been squared so that this quantity is well defined for arbitrary space curves without assuming they are twisted. We do not know of a direct geometric application for this quantity. However, the fact that the evolution is so simple suggests that it may be useful for controlling the formation of singularities.

4. A heuristic for the emergence of flat points

It remains an open question to determine the limiting behaviour for generic initial data for spatial curve shortening flow. However, it is reasonable to expect that generic curves converge to round points, the same as for embedded curves in the plane. There are several cases where spatial curve shortening flow is known to converge to a round point. For instance, if the curve is embedded on a standard sphere, we have the following result (a proof can be found in [10]).

THEOREM 4.1. *Given a curve γ_0 embedded on a standard sphere S^2 in $\mathbb{R}^3$, under curve shortening flow, γ_t remains embedded on a shrinking sphere and converges to a round point.*

Note that the total torsion of a spherical curve is zero, which immediately implies that any such curve has flat points. In fact, Sedykh's theorem shows there are at least four such points [9]. Furthermore, in recent work, Litzinger showed that curves whose entropy is at most 2 converge to round points.

THEOREM 4.2 [8]. *Suppose that γ is a smooth curve whose entropy*

$$\lambda(\gamma) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} (4\pi t_0)^{-1/2} \int_{\gamma_t} e^{-|x-x_0|^2/4t_0}\, d\mu$$

satisfies $\lambda(\gamma) \leq 2$. Then under curve shortening flow, γ_t converges to a round point.

Apart from these results, there is another reason why we do not expect initially twisted curves to remain twisted; the reaction terms for the curvature and torsion tend to create points of vanishing curvature. To see this, we ignore the spatial derivative terms in (3.1) describing the evolution of κ and τ and simply consider the system of coupled ordinary differential equations (ODEs)

$$\dot{\kappa} = \kappa^3 - \kappa\tau^2, \quad \dot{\tau} = 2\kappa^2\tau.$$

This system reduces to the homogeneous equation

$$\frac{d\kappa}{d\tau} = \frac{1}{2}\left(\frac{\kappa}{\tau} - \frac{\tau}{\kappa}\right),$$

whose solutions are circular arcs of the form

$$\kappa(\tau) = \sqrt{C\tau - \tau^2}.$$

As such, no matter how large κ is initially, the reaction ODE tends towards a situation where κ vanishes. The full evolution equation for the torsion also includes a complicated term involving its derivatives and those of the curvature, so this analysis does not constitute a complete proof. However, if the initial curve is a helix (a curve of constant curvature and torsion), this calculation shows that the limiting configuration is a straight line, but the limiting value of the torsion is nonzero. The limiting value of the torsion is not the torsion of a straight line, which is undefined. Indeed, the limiting value depends on the initial curvature and torsion of the helix.

Acknowledgements

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