A QUEUEING SYSTEM WITH GENERAL MOVING AVERAGE INPUT AND NEGATIVE EXPONENTIAL SERVICE TIME

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(Received 27 August 1965)

1. Introduction

If we think of the input to a queueing system as arising from some process and depending on the history of that process, we might well expect the duration of inter-arrival intervals to depend mostly on the recent history and to a much smaller extent on that which is more remote.

The simplest model incorporating such behaviour is where the nth inter-arrival interval τ_n is given by

$$\tau_n = u_n,$$

where $\{u_n\}$ is a sequence of identically and independently distributed non-negative random variables. This is the well-known case of a general recurrent input. The next simplest model has

$$\tau_n = u_{n+1} + \beta u_n,$$

where β is a constant. This model has been considered for negative exponential services and a single server by Finch [1] and by Finch and Pearce [2].

This paper extends the results of [2] to the more general case of a moving average of order $p > 1$.

We consider a single server queueing system in which

(i) customers arrive singly at the instants $0 = A_0 < A_1 < A_2 < \cdots$, where the time between the arrivals at A_m and A_{m+1}

$$A_{m+1} - A_m = b_0 U_{m+p} + b_1 U_{m+p-1} + \cdots + b_p U_m, \qquad m \geqq 0, \tag{1.1}$$

where the b_i, $i = 0, 1, \cdots, p$, are non-negative constants with sum unity and $\{U_m\}$ is a sequence of independently and identically distributed non-negative random variables with common distribution function

$$A(x) = P(U_m \leqq x), \qquad m \geqq 0,\ x \geqq 0,$$

such that

$$\int_0^\infty x dA(x) < \infty,$$

and

(ii) the service time of the $(m+1)$th arrival is S_m, where $\{S_m\}$ is a sequence of independently and identically distributed random variables, and

$$P(S_m \leqq x) = 1 - \exp(-\mu x), \qquad x \geqq 0,\ \mu > 0.$$

If we denote by P_j^m, $j \geqq 0$, $m \geqq 0$, the probability that the arrival at A_m finds exactly j customers already in the system, then by the results of Finch [1],

$$P_j = \lim_{m\to\infty} P_j^m, \qquad j \geqq 0,$$

exists. In fact, general formulae are obtained in [1] for the P_j which are applicable for any input process. In an attempt to simplify these for a moving average input of order two, i.e., where

$$A_{m+1} - A_m = U_{m+1} + \beta U_m \tag{1.2}$$

in the notation of (1.1), Finch made use of a heuristic symbolic method which provided very simple expression for the probabilities P_j.

The conjectured form of solution for a moving average input of order two was investigated rigorously in a paper by Finch and Pearce [2]. It was found that whilst the form of the solution obtained in [1] was confirmed, the value found for a constant characterising the solution was incorrect.

This paper generalises the results of [2] to a moving average input of arbitrary order, and also confirms that the limiting queue length probability distribution is, after the first few probabilities, geometric in form, as follows from the heuristic method. The initial probabilities will not in general agree, as those in the heuristic solution depend only on ψ (a function later defined), whereas the present solution involves also derivatives of ψ.

When the U_m have an arbitrary common distribution function $A(x)$, the particular constants involved in the limiting distribution seem to have no simple form and they are not obtained explicitly in this paper, although equations are given sufficient to determine their values. The case of a moving average of order three is considered in detail as an illustration.

Our starting point is the set of recurrence relations expressing the probabilities of the $(n+1)$th arrival finding a given number of customers already in the queue in terms of queue length as found by the preceding arrival. From these we obtain an equation relating the corresponding probability generating functions, but involving unwanted extra terms which we handle by a complex variable argument, working with Laplace-Stieltjes transforms of the quantities concerned. Having found the functional form of the limiting distribution of queue length by these means, we consider the determination of the constants involved from the initial recurrence relations. The procedure of determining the solution fully is illustrated by the case of a moving average of order three.

2. Definitions and preliminaries

We employ similar notation to that of [2]. Capital letters are used to denote random variables and the corresponding lower case letters for particular values taken on by these variables. The $(n+1)$-tuple $(u_0, u_1, \cdots, u_n)$ is represented by $u^{(n)}$ and the corresponding vector random variable $(U_0, U_1, \cdots, U_n)$ by $U^{(n)}$.

$P_j(u^{(n+p-1)})$, $j \geqq 0$, is the conditional probability, given $U^{(n)} = u^{(n)}$, that the arrival at A_n finds exactly j customers already in the system. $EP_j(U^{(n+p-1)})$ is the (unconditional) probability that the $(n+1)$th arrival finds j customers in the system.

The probability, $k_j(x_0, x_1, \cdots, x_p)$, of j departures from the queue during an interval $b_0x_p+b_1x_{p-1}+\cdots+b_px_0$, given that at the beginning of the interval the queue length was at least $j+1$, is given by

$$k_j(x_0, x_1, \cdots, x_p) = [\{\mu(b_0x_p+b_1x_{p-1}+\cdots+b_px_0)\}^j/j!] \times \exp\{-\mu(b_0x_p+b_1x_{p-1}+\cdots+b_px_0)\}, \qquad j \geqq 0,$$

and since

$$\sum_{i=0}^{\infty} k_i(x_0, x_1, \cdots, x_p) = 1,$$

it follows that the probability $K_j(x_0, x_1, \cdots, x_p)$ of j departures during the interval $b_0x_p+b_1x_{p-1}+\cdots+b_px_0$, given that the queue length was j at the beginning of this interval, is given by

$$K_j(x_0, x_1, \cdots, x_p) = \sum_{i=j}^{\infty} k_i(x_0, x_1, \cdots, x_p).$$

The generating function of the k_i's is

$$k(x_0, x_1, \cdots, x_p; z) = \sum_{i=0}^{\infty} k_i(x_0, x_1, \cdots, x_p)z^i = \exp\{-(1-z)\mu(b_0x_p+b_1x_{p-1}+\cdots+b_px_0)\}.$$

We denote by

$$P(u^{(n+p-1)}; z) = \sum_{i=0}^{\infty} P_i(u^{(n+p-1)})z^i, \qquad |z| \leqq 1,$$

the generating function of the $P_i(u^{(n+p-1)})$, and its integral transform by

$$P^*(s^{(p)}; z; n) = E[P(U^{(n+p-1)}; z) \exp(-s_p U_{n+p-1} - s_{p-1} U_{n+p-2} - \cdots - s_1 U_n)],$$
$$|z| \leqq 1, \qquad \operatorname{Re} s_i \geqq 0, \qquad i = 1, \cdots p,$$

$P_i^*(s^{(p)}, n) =$ coefficient of z^i in the power series of $p^*(s^{(p)}; z; n)$. We shall also need

$$c_i(u^{(n+p-1)}) = \sum_{j=0}^{\infty} P_j(u^{(n+p-1)}) k_{j+i+1}(u_n, u_{n+1}, \cdots, u_{n+p}), \qquad i \geqq 0,$$

and its integral transform

$$c_i^*(s^{(p)}; n) = E[c_i(U^{(n+p-1)}) \exp(-s_p U_{n+p-1} - \cdots - s_1 U_m)],$$
$$\operatorname{Re} s_i = 0, \qquad 1 \leqq i \leqq p.$$

By the methods of [1] it can be shown that provided

$$\int_0^{\infty} x dA(x) > \frac{1}{\mu}, \tag{2.1}$$
$$P_j = \lim_{n\to\infty} E[P_j(U^{(n+p-1)}), \qquad j \geqq 0,$$

exists and forms a proper probability distribution, and that

$$P(w_1, w_2, \cdots, w_p; z) = \lim_{n\to\infty} E[P(U_0, U_1, \cdots, U_{n-1}, u_n, u_{n+1}, \cdots, u_{n+p-1}; z)],$$

where the particular values $u_n, n_{n+1}, \cdots, u_{n+p-1}$ are $w_1, w_2, \cdots, w_p$, exists for $|z| \leqq 1$ and is the generating function of a probability distribution.

We write for its integral transform

$$P^*(s^{(p)}; z) = E[P(W^{(p)}; z) \exp(-s_p W_p - s_{p-1} W_{p-1} - \cdots - s_p W_1)], \tag{2.2}$$
$$|z| \leqq i, \qquad \operatorname{Re} s_i \geqq 0, \qquad 1 \leqq i \leqq p,$$

where the W_i, $1 \leqq i \leqq p$, are identically and independently distributed random variables with common distribution function $A(x)$.

We define

$$(2.3) \qquad \begin{aligned} c_i^*(s^{(p)}) &= \lim_{n\to\infty} c_i^*(s^{(p)}; n), \\ c(s^{(p)}; z) &= \sum_{i=0}^{\infty} (1 - z^{-i}) c_i^*(s^{(p)}), \quad |z| \leqq 1, \quad \operatorname{Re} s_i \geqq 0, \quad 1 \leqq i \leqq p. \end{aligned}$$

It follows from (2.1) by use of Rouché's theorem that the equation

$$z = \psi(1-z), \tag{2.4}$$

where

$$\psi(\alpha) = \int_0^{\infty} \exp(-\mu\alpha u) dA(u),$$

has a unique solution inside the unit circle. This we shall denote by T.

By putting $z = 1$ in (2.2) we obtain

(2.5) $$P^*(s^{(p)}; 1) = \psi(s_p/\mu)\psi(s_{p-1}/\mu)\cdots\psi(s_1/\mu), \quad \mathrm{Re}\, s_i \geqq 0, \quad 1 \leqq i \leqq p.$$

3. Fundamental equations

It follows from the departure probabilities given in the last section that

(3.1) $$P_j(u^{(n+p)}) = \sum_{i=0}^{\infty} P_{j+i-1}(u^{(n+p-1)})k_i(u_n, \cdots, u_{n+p}), \qquad n \geqq 0,\ j \geqq 1,$$
$$P_0(u^{(n+p)}) = \sum_{i=0}^{\infty} P_i(u^{(n+p-1)})K_{i+1}(u_n, u_{n+1}, \cdots, u_{n+p}), \qquad n \geqq 0.$$

We note that

(3.2) $$\sum_{i=0}^{\infty} c_i(u^{(n+p)}) = P_0(u^{(n+p)}).$$

Forming the product of the power series $k(u_n, u_{n+1}, \cdots, u_{n+p}; z)$, $P(u^{(n+p-1)}; z)$ and using the equations above, we obtain

$$P(u^{(n+p)}; z) = \sum_{i=0}^{\infty} (1-z^{-i})c_i(u^{(n+p)}) + zP(u^{(n+p-1)}; z)$$
$$\times \exp[-(1-z^{-1})\mu(b_0u_{n+p}+b_1u_{n+p-1}+\cdots+b_pu_n)]$$

for $|z| \leqq 1$, $z \neq 0$. Hence

$$P^*(s^{(p)}; z; n+1) = \sum_{i=0}^{\infty} (1-z^{-i})c_i^*(s^{(p)}; n+1)$$
(3.3) $$+zP^*[(1-z^{-1})\mu b_1+s_{p+1}, \cdots, (1-z^{-1})\mu b_{p-1}+s_1, (1-z^{-1})\mu b_p; z; n]$$
$$\times \psi\{b_0(1-z^{-1})+s_p/\mu\}$$

for $|z| \leqq 1$, $z \neq 0$, $\mathrm{Re}\, s_i \geqq 0$ $(i = 1, 2, \cdots, p)$, $\mathrm{Re}\,[(1-z^{-1})\mu b_j+s_{p-j}] \geqq 0$ $(j = 0, 1, \cdots, p-1)$, $\mathrm{Re}\,[(1-z^{-1})\mu b_p] \geqq 0$. These conditions are satisfied if z lies both in or on the unit circle and outside or on the circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$, with the origin deleted. We denote by R this domain of the z-plane.

Letting $n \to \infty$ in (3.3) gives us

$$P^*(s^{(p)}; z) = c(s^{(p)}; z)+z\psi\{(1-z^{-1})b_0+s_p/\mu\}$$
(3.4) $$\times P^*[(1-z^{-1})\mu b_1+s_{p-1}, \cdots, (1-z^{-1})\mu b_{p-1}+s_1, (1-z^{-1})\mu b_p; z],$$
$$z \in R, \qquad \mathrm{Re}\, s_i \geqq 0, \qquad 1 \leqq i \leqq p.$$

https://doi.org/10.1017/S1446788700004808 Published online by Cambridge University Press

4. Solution for $P^*(s^{(p)}; z)$

Substitution of

$$s_1 = (1-z^{-1})\mu b_p, \qquad s_2 = (1-z^{-1})\mu(b_{p-1}+b_p), \cdots,$$
$$s_p = (1-z^{-1})\mu(b_1+\cdots+b_p)$$

in (3.4) yields

$$\begin{aligned}(4.1)\quad & P^*[(1-z^{-1})\mu(b_1+\cdots+b_p), (1-z^{-1})\mu(b_2+\cdots+b_p), \cdots, (1-z^{-1})\mu b_p; z] \\ & = c^*[(1-z^{-1})\mu(b_1+\cdots+b_p), \cdots, (1-z^{-1})\mu b_p; z] \\ & [1-z\psi(1-z^{-1})]^{-1}, \qquad z \in R.\end{aligned}$$

Also, if we replace $s_p, s_{p-1}, \cdots, s_1$ by

$$(1-z^{-1})\mu b_1+s_{p-1}, \qquad (1-z^{-1})\mu b_2+s_{p-2}, \cdots, (1-z^{-1})\mu b_p$$

respectively and substitute in (3.4), we obtain

$$\begin{aligned} & P^*[(1-z^{-1})\mu b_1+s_{p-1}, \cdots, (1-z^{-1})\mu b_p; z) \\ & = c[(1-z^{-1})\mu b_1+s_{p-1}, \cdots, (1-z^{-1})\mu b_p; z]+z\psi[(1-z^{-1})(b_0+b_1)+s_{p-1}/\mu] \\ & \times P^*[(1-z^{-1})\mu(b_1+b_2)+s_{p-2}, \cdots, (1-z^{-1})\mu(b_{p-1}+b_p), (1-z^{-1})\mu b_p; z], \\ & z \in R, \qquad \operatorname{Re} s_i \geqq 0, \qquad i = 1, 2, \ldots, p.\end{aligned}$$

By making substitutions in this equation analogous to those in (3.4), and proceeding recursively in this manner, we find that

$$\begin{aligned} & P^*(s^{(p)}; z) = c^*(s^{(p)}; z)+z\psi\{(1-z^{-1})b_0+s_p/\mu\} \\ & \times [c^*\{(1-z^{-1})\mu b_1+s_{p-1}, \cdots, (1-z^{-1})\mu b_{p-1}+s_1, (1-z^{-1})\mu b_p; z\} \\ & +z\psi\{(1-z^{-1})(b_0+b_1)+s_{p-1}/\mu\} \\ & \times [c^*\{(1-z^{-1})\mu(b_1+b_2)+s_{p-2}, \cdots, (1-z^{-1})\mu(b_{p-1}+b_p), (1-z^{-1})\mu b_p; z\} \\ (4.2)\quad & +z\psi\{(1-z^{-1})(b_0+b_1+b_2)+s_{p-2}/\mu\} \\ & \cdots \\ & \times (c^*\{(1-z^{-1})\mu(b_1+\cdots+b_p), \cdots, (1-z^{-1})\mu b_p; z\} \\ & \times [1-z\psi(1-z^{-1})]^{-1}]] \cdots], \\ & z \in R, \qquad \operatorname{Re} s_i \geqq 0, \qquad i = 1, 2, \cdots, p,\end{aligned}$$

the last term arising from the use of (4.1).

Writing the right hand side of (4.2) as $D(s^{(p)}; z)$, consider the function $F(s^{(p)}; z)$ defined by

$$F(s^{(p)}; z) = \begin{cases} (1-Tz)P^*(s; z), & |z| \leqq 1, \quad \operatorname{Re} s_i \geqq 0, \quad 1 \leqq i \leqq p, \\ (1-Tz)D(s^{(p)}; z), & |z| \geqq 1, \quad \operatorname{Re} s_i \geqq 0, \quad 1 \leqq i \leqq p. \end{cases}$$

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Since $P^*(s^{(p)}; z)$ is the generating function of a probability distribution, $P^*(s^{(p)}; z)$ and hence $F(s^{(p)}; z)$ must be a regular function of z for $|z| \leqq 1$, $\text{Re}\, s_i \geqq 0$, $i = 1, 2, \cdots, p$.

Also, since the only zero of $1-z\psi(1-z^{-1})$ outside the unit circle is that of $1-Tz$, $F(s^{(p)}; z)$ must be a regular function of z for $|z| \geqq 1$, $\text{Re}\, s_i \geqq 0$, $i = 1, 2, \cdots, p$.

Hence, by analytic continuation, $F(s^{(p)}; z)$ is a regular function of z for all finite z for $\text{Re}\, s_i \geqq 0$, $1 \leqq i \leqq p$.

It can be shown from (2.3) and (3.2) by Abel's theorem that $c(s^{(p)}; z)$ converges to $\sum_{i=1}^{\infty} c_i^*(s)$ as $z \to \infty$, so from (4.2) $\lim_{z\to\infty} F(s^{(p)}; z)/z^p$ exists, in fact being given by

$$\lim_{z\to\infty} \frac{F(s^{(p)}; z)}{z^p} = -T\psi(b_0+s_p/\mu)\psi(b_0+b_1+s_{p-1}/\mu)\cdots\psi(b_0+b_1+\cdots+b_{p-2}+s_2/\mu)$$
$$\times \Big[\sum_{i=1}^{\infty} c_i^*\{\mu(b_1+\cdots+b_{p-1})+s_1, \cdots, \mu b_p\}$$
$$-\psi(b_0+b_1+\cdots+b_{p-1}+s_1/\mu)\Big[\sum_{i=1}^{\infty} c_i^*\{\mu(b_1+\cdots+b_p), \cdots, \mu b_p\}\Big]\times\{\psi(1)\}^{-1}\Big],$$
$$\text{Re}\, s_i \geqq 0, \qquad 1 \leqq i \leqq p.$$

Since a function $\theta(z)$ which is analytic for all finite z and $0(|z^k|)$, k a non-negative integer, as $z \to \infty$ is a polynomial of degree at most k, $F(s^{(p)}; z)$ must be of the form

$$F(s^{(p)}; z) = F_p(s^{(p)})z^p + F_{p-1}(s^{(p)})z^{p-1} + \cdots + F_0(s^{(p)}), \quad \text{Re}\, s_i \geqq 0, \quad 1 \leqq i \leqq p,$$

where the $F_j(s^{(p)})$ are functions of the s_i alone. Thus

$$P^*(s^{(p)}; z) = [F_p(s^{(p)})z^p + \cdots + F_0(s^{(p)})][1-Tz]^{-1}, \quad \text{Re}\, s_i \geqq 0, \quad |z| \leqq 1,$$

or, more conveniently

$$(4.3) \quad P^*(s^{(p)}; z) = B_{p-1}(s^{(p)})z^{p-1} + \cdots + B_0(s^{(p)}) + B(s^{(p)})(1-zT)^{-1}, \qquad \text{Re}\, s_i \geqq 0, \quad |z| \leqq 1.$$

When we put $s_p = s_{p-1} = \cdots = s_1 = 0$, $P^*(s^{(p)}; z)$ becomes the generating function $\sum_{i=0}^{\infty} P_j z^i$ of the limiting distribution of queue size and the functions $B_j(s^{(p)})$ and $B(s^{(p)})$ reduce to constants B_j, B. The generating function of the limiting queue length distribution is thus given by

$$(4.4) \qquad \sum_{i=0}^{\infty} P_i z^i = B_{p-1}z^{p-1} + \cdots + B_0 + B(1-zT)^{-1}.$$

This is a probability distribution which assumes a geometric form from P_p onwards, the common ratio being T.

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5. Determination of the $B_j(s^{(p)})$

From (3.1)

$$P_j(u^{(n+p)}) = \sum_{i=0} \sum_{l_0, l_1, \cdots, l_p = 0, 1, \cdots, i} P_{j+i-1}(u^{(n+p-1)},$$
$$\prod_{k=0,1,\cdots,p} \left[\exp(-\mu b_k u_{n+k}) \frac{(\mu b_k u_{n+k})^{l_k}}{l_k!} \right], \qquad j \geqq 1,$$

where the summation on the l_k is over non-negative integers subject to the restriction $\sum_{k=0}^{p} l_k = i$.

Hence

$$P_j^*(s^{(p)}, \cdots, n+1) = \sum_{i=0}^{\infty} \frac{(-\mu)^i}{i!} \frac{\partial^i}{\partial \sigma^i} [P_{j+i-1}^*(\sigma b_1 + s_{p-1}, \sigma b_2 + s_{p-2}, \cdots, \sigma b_{p-1} + s_1, \sigma b$$
$$\times \psi\{(\sigma b_0 + s_p)/\mu\}], \qquad \operatorname{Re} s_i \geqq 0, \qquad j \geqq$$

Letting $n \to \infty$ and using (4.3), we see that for $j \geqq p+1$

$$\begin{aligned} B(s^{(p)})T^j &= \sum_{i=0}^{\infty} \frac{(-\mu)^i}{i!} \frac{\partial^i}{\partial \sigma^i} [B(\sigma b_1 + s_{p-1}, \sigma b_2 + s_{p-2}, \cdots, \sigma b_p) T^{j+i-1} \\ &\quad \times \psi\{(\sigma b_0 + s_p)/\mu\}]_{\sigma=\mu} \\ &= B\{\mu(1-T)b_1 + s_{p-1}, \mu(1-T)b_2 + s_{p-2}, \cdots, \mu(1-T)b_p\} T^{j-1} \\ &\quad \times \psi\{(1-T)b_0 + s_p/\mu\}, \qquad \operatorname{Re} s_i \geqq 0, \end{aligned} \tag{5.1}$$

whence

$$\begin{aligned} B(s^{(p)}) &= T^{-p} \psi\{1-T)b_0 + s_p/\mu\} \psi\{(1-T)(b_0+b_1) + s_{p-1}/\mu\} \\ &\qquad \psi\{(1-T)(b_0+b_1+\cdots+b_{p-1}) + s_1/\mu\} \\ &\quad \times B\{\mu(1-T)(b_1+\cdots+b_p), \mu(1-T)(b_2+\cdots+b_p), \cdots, \mu(1-T)b_p\}, \\ &\qquad \operatorname{Re} s_i \geqq 0. \end{aligned} \tag{5.2}$$

Working similar to the above for $1 \leqq j \leqq p-1$ yields

$$B_{p-1}(\mu b_1 + s_{p-1}, \mu b_2 + s_{p-2}, \cdots, \mu b_{p-1} + s_1, \mu b_p) \psi(b_0 + s_p/\mu) = 0, \quad \operatorname{Re} s_i \geqq 0,$$
$$B_{p-1}(s^{(p)}) = B_{p-2}(\mu b_1 + s_{p-1}, \cdots, \mu b_p) \psi(b_0 + s_p/\mu)$$
$$+ (-\mu) \frac{\partial}{\partial \sigma} [B_{p-1}(\sigma b_1 + s_{p-1}, \cdots, \sigma b_p) \psi\{(\sigma b_0 + s_p)/\mu\}]_{\sigma=\mu}$$
$$\operatorname{Re} s_i \geqq 0,$$

$$B_{p-2}(s^{(p)}) = [B_{p-3})\sigma b_1+s_{p-1}, \cdots, \sigma b_p)\psi\{(\sigma b_0+s_p)/\mu\}$$
$$+(-\mu)\frac{\partial}{\partial\sigma}[B_{p-2}(\sigma b_1+s_{p-1}, \cdots, \sigma b_p)\psi\{(\sigma b_0+s_p)/\mu\}]$$
$$+\frac{(-\mu)^2}{2!}\frac{\partial^2}{\partial\sigma^2}[B_{p-1}(\sigma b_1+s_{p-1}, \cdots, \sigma b_p)\psi\{(\sigma b_0+s_p)/\mu\}]]_{\sigma=\mu}, \quad \text{Re } s_i \geqq 0$$

$$\cdots$$

$$B_1(s^{(p)}) = [B_0(\sigma b_1+s_{p-1}, \cdots, \sigma b_p)\psi\{(\sigma b_0+s_p)/\mu\}$$
$$+\cdots$$
$$+\frac{(-\mu)^{p-1}}{(p-1)!}\frac{\partial^{p-1}}{\partial\sigma^{p-1}}[B_{p-1}(\sigma b_1+s_{p-1}, \cdots, \sigma b_p)\psi\{(\sigma b_0+s_p)/\mu\}]]_{\sigma=\mu}, \quad \text{Re } s_i \geqq 0.$$

The now familiar recursive substitution procedure when applied to the second of these equations provides an expression for $B_{p-1}(s^{(p)})$ in terms of B_{p-2} and its derivatives, evaluated at various arguments involving the s_i, similar functions of ψ, and B_{p-1} evaluated at a constant argument. If $B_{p-2}(s)$ is known this suffices for the determination of $B_{p-1}(s)$.

Substituting for $B_{p+1}(s^{(p)})$ (known in terms of B_{p-2}, known functions, and a constant) in the third equation gives an expression for $B_{p-2}(s)$ in terms of B_{p-3} and its derivatives, known functions, and a constant.

Proceeding in this fashion expressions are provided for $B_{p-1}(s^{(p)})$, $B_{p-2}(s^{(p)}), \cdots, B_1(s^{(p)})$ in terms of $B_0(s^{(p)})$ and its derivatives, known functions, and a set of constants. Use of these expressions, (5.2) and

$$\text{(5.3)} \qquad \psi(s_p/\mu)\psi(s_{p-1}/\mu)\cdots\psi(s_1/\mu) = \sum_{i=0}^{p-1} B_i(s^{(p)})+B(s^{(p)})(1-T)^{-1}, \quad \text{Re } s_i \geqq 0,$$

an equation which results directly from (4.3) and (2.5), leads to a solution for the $B_i(s^{(p)})$ and $B(s^{(p)})$.

Putting $s_i = 0$, $i = 1, 2, \cdots, p$ in (4.3) then gives directly the limiting queue length distribution as found by customers entering the system. There does not seem to be a simple general form of solution, but it can be seen that the solution will normally involve the derivatives of ψ as well as ψ itself.

6. Moving average of order three

In this case $P^*(s^{(2)}; z)$ is of the form

$$P^*(s^{(2)}; z) = B_1(s^{(2)})z+B_0(s^{(2)})+B(s^{(2)})(1-Tz)^{-1}, \qquad \text{Re } s_i \geqq 0,$$

where T is the (unique) root within the unit circle of

$$T = \psi(1-T).$$

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The equations determining the solution become

$$(6.1)\quad B_1(\mu b_1+s_1,\, b_2) = 0,$$

$$(6.2)\quad B_1(s^{(2)}) = \psi(b_0+s_2/\mu)\left[B_0(\sigma b_1+s_1,\, \sigma b_2)+(-\mu)\frac{\partial}{\partial\sigma}B_1(\sigma b_1+s_1,\, \sigma b_2)\right]_{\sigma=\mu}. \qquad \operatorname{Re} s_i \geqq 0,$$

$$(6.3)\quad B_1(s^{(2)})+B_0(s^{(2)})+B(s^{(2)})(1-T)^{-1} = \psi(s_2/\mu)\psi(s_1/\mu), \qquad \operatorname{Re} s_i \geqq 0,$$

$$(6.4)\quad B(s^{(2)}) = T^{-2}\psi\{(1-T)b_0+s_2/\mu\}\psi\{(1-T)(b_0+b_1)+s_1/\mu\} B\{\mu(1-T)(b_1+b_2),\, \mu(1-T)b_2\}, \qquad \operatorname{Re} s_i \geqq 0.$$

From (6.3) and (6.4),

$$(6.5)\quad \begin{aligned} B_1(s^{(2)}) = {} & \psi(s_2/\mu)\psi(s_1/\mu)-B_0(s^{(2)}) \\ & -(1-T)^{-1}T^{-2}\psi\{(1-T)b_0+s_2/\mu\}\psi\{(1-T)(b_0+b_1)+s_1/\mu\} \\ & \times B\{\mu(1-T)(b_1+b_2),\, \mu(1-T)b_2\}, \qquad \operatorname{Re} s_i \geqq 0. \end{aligned}$$

Recursive substitution for B_1 in (6.2) shows us that $B_1(s^{(2)})$ is of the form

$$(6.6)\quad B_1(s^{(2)}) = \psi(1+s_2/\mu)[B_0(\mu b_1+s_1,\, \mu b_2)+a\psi(b_0+b_1+s_1/\mu)],$$

where a is a constant. Substituting $s_1 = \mu b_2$ in (6.5) and making use of (6.1) and (6.6), we find that

$$(6.7)\quad \begin{aligned} B_0(s^{(2)}) = {} & \psi(s_2/\mu)\psi(s_1/\mu)-(1-T)^{-1}T^{-2} \\ & \psi\{(1-T)b_0+s_2/\mu\}\psi\{(1-T)(b_0+b_1)+s_1/\mu\} \\ & \times B\{\mu(1-T)(b_1+b_2),\, \mu(1-T)b_2\}-\psi(b_0+s_2/\mu)a\psi(b_0+b_1+s_1/\mu) \\ & -\psi(b_0+s_2/\mu)[\psi(b_1+s_1/\mu)\psi(b_2)-(1-T)^{-1}T^{-2}\psi\{(1-T)b_0+b_1+s_1/\mu\} \\ & \times \psi\{(1-T)(b_0+b_1)+b_2\}B\{\mu(1-T)(b_1+b_2),\, \mu(1-T)b_2\}], \qquad \operatorname{Re} s_i \geqq 0, \end{aligned}$$

$$(6.8)\quad \begin{aligned} B_1(s^{(2)}) = {} & \psi(b_0+s_2/\mu)[a\psi(b_0+b_1+s_1/\mu)+\psi(b_1+s_1/\mu)\psi(b_2) \\ & +(1-T)^{-1}T^{-2}\psi\{(1-T)b_0+b_1+s_1/\mu\} \\ & \psi\{(1-T)(b_0+b_1)+b_2\}B\{\mu(1-T)(b_1+b_2),\, \mu(1-T)b_2\}], \qquad \operatorname{Re} s_i \geqq 0. \end{aligned}$$

A little algebraic manipulation now enables us to find the two constants a and $B\{\mu(1-T)(b_1+b_2),\, \mu(1-T)b_2\}$ required for our solutions (6.5), (6.7) and (6.8) for $B(s^{(p)})$, $B_0(s^{(p)})$ and $B_1(s^{(p)})$ to be completely in terms of known quantities. From (6.8), (6.1) we see that

$$(6.9)\quad \begin{aligned} a = {} & -[\psi(1)]^{-1}[\psi(b_1+b_2)\psi(b_2)+(1-T)^{-1}T^{-2}\psi\{(1-T)b_0+b_1+b_2\} \\ & \psi\{(1-T)(b_0+b_1)+b_2\}B\{\mu(1-T)(b_1+b_2)\mu(1-T)b_2\}], \end{aligned}$$

and using this expression to simplify the working, we derive from (6.8) that

https://doi.org/10.1017/S1446788700004808 Published online by Cambridge University Press

$$(-\mu)\left[\frac{\partial}{\partial\sigma}-B_1(\sigma b_1+s_1,\sigma b_2)\right]_{\sigma=\mu} = -b_2\psi(b_0+b_1+s_1/\mu)[a\psi'(1)$$
$$+\psi'(b_1+b_2)\psi(b_2)$$
$$+(1-T)^{-1}T^{-2}\psi'\{(1-T)b_0+b_1+b_2\}\psi\{(1-T)(b_0+b_1)+b_2\}$$
$$B\{\mu(1-T)(b_1+b_2)\mu(1-T)b_2\}], \qquad \operatorname{Re} s_1 \geqq 0,$$

so that from (6.6)

$$a = [1+b_2\psi'(1)]^{-1}(-b_2)[\psi'(b_1+b_2)\psi(b_2)$$
$$+(1-T)^{-1}T^{-2}\psi'\{(1-T)b_0+b_1+b_2\}\psi\{(1-T)(b_0+b_1)+b_2\}B\{\mu(1-T)$$
$$(b_1+b_2),\ \mu(1-T)b_2\}].$$

Hence from (6.9) we derive

$$\begin{aligned} a &= [1+b_2\psi'(1)-b_2\psi'\{(1-T)b_0+b_1+b_2\}\psi(1)]^{-1} \\ &\times [-b_2\psi'(b_1+b_2)\psi(b_2)-\psi'\{(1-T)b_0+b_1+b_2\} \\ &\qquad \psi(b_1+b_2)\psi(b_2)\psi\{(1-T)b_0+b_1+b_2\}], \end{aligned} \tag{6.10}$$

$$\begin{aligned} &T^{-2}B\{\mu(1-T)(b_1+b_2),\ \mu(1-T)b_2\} \\ &\quad = (1-T)[\psi'\{(1-T)b_0+b_1+b_2\}\psi\{(1-T)(b_0+b_1)+b_2\}]^{-1} \\ &\quad \times [-\psi'(b_1+b_2)\psi(b_2)-b_2^{-1}\{1+b_2\psi'(1)\} \\ &\quad \times \{1+b_2\psi'(1)-b_2\psi'\{(1-T)b_0+b_1+b_2\}\psi(1)\}^{-1} \\ &\quad \times \{-b_2\psi'(b_1+b_2)\psi(b_2)-\psi'\{(1-T)b_0+b_1+b_2\} \\ &\qquad \psi(b_1+b_2)\psi(b_2)\psi\{(1-T)b_0+b_1+b_2\}]. \end{aligned} \tag{6.11}$$

The limiting queue distribution is thus

$$\begin{aligned} \sum_{i=0}^{\infty} P_i z^i &= 1-(1-T)^{-1}\psi(1-T)\psi\{(1-T)(b_0+b_1)\} \\ &\qquad T^{-2}B\{\mu(1-T)(b_1+b_2),\ \mu(1-T)b_2\} \\ &\quad -a\psi(b_0)\psi(b_0+b_1)-\psi(b_0)[\psi(b_1)\psi(b_2)-(1-T)^{-1}\psi\{(1-T)b_0+b_1\} \\ &\quad \times \psi\{(1-T)(b_0+b_1)+b_2\}T^{-2}B\{\mu(1-T)(b_1+b_2),\ \mu(1-T)b_2\}] \\ &\quad +z\psi(b_0)[a\psi(b_0+b_1)+\psi(b_1)\psi(b_2)+(1-T)^{-1}\psi\{(1-T)b_0+b_1\} \\ &\quad \times \psi\{(1-T)(b_0+b_1)+b_2\}T^{-2}B\{\mu(1-T)(b_1+b_2),\ \mu(1-T)b_2\}] \\ &\quad +(1-Tz)^{-1}\psi(1-T)\psi\{(1-T)(b_0+b_1)\} \\ &\qquad T^{-2}B\{\mu(1-T)(b_1+b_2),\ \mu(1-T)b_2\}, \qquad |z| \leqq 1, \end{aligned}$$

where the constants a and $T^{-2}B\{\mu(1-T)(b_1+b_2),\ \mu(1-T)b_2\}$ are given by (6.10) and (6.11)

7. Waiting time distribution

In [3], Loynes considers the possibility of determining the stationary waiting time distribution of single server queues in which inter-arrival intervals and service times are not necessarily independently distributed, and under mild restrictions finds techniques applicable to a wide class of queueing systems.

In this section we deduce the form of the limiting waiting time distribution for the general moving average queue with negative exponential service and compare this with results in [3].

We denote by S_n, T_n, W_n, respectively the service time of the arrival at A_n, the length of the interval (A_n, A_{n+1}), and the waiting time (excluding service) of the arrival at A_n.

Loynes shows in [4] that under the conditions that $\{S_n - T_n\}$ is a strictly stationary process and

$$E(S_n - T_n) < 0, \tag{7.1}$$

the existence of a unique limiting distribution of waiting time is ensured. In the present problem this condition becomes (2.1), our condition for the existence of a unique limiting distribution of queue length, as one would intuitively expect.

The class of systems dealt with in [3] consists of queues for which: $\mathfrak{H}$. There exists a sequence $\{z_n\}$ of random vectors defined in finite-dimensional Euclidean space with the following properties:

(i) $\{z_n, T_n, S_n\}$ *is a strictly stationary process,*
(ii) S_n, T_n, W_n *are conditionally independent given* z_{n+1}, z_n,
(iii) W_n, z_n *are conditionally independent given* z_{n-1}.

(One can regard the components of the z's as being of the nature of the additional variables introduced in a queueing problem to recover the Markovian property, as in D. G. Kendall [5].)

We introduce

$$\phi(s, z_n) = \int_0^\infty \exp(-sx) d_x \operatorname{pr}(\omega_{n+1} \leqq x | z_n),$$

and similarly $\psi(s, z_n)$, $H(s, z_n, z_{n-1})$, $G(s, z_n, z_{n-1})$ corresponding to $W_n + S_n + T_n - \omega_{n+1}$, S_n, T_n, respectively.

Loynes shows that the Laplace-Stieltjes integral form of the equation here corresponding to the ordinary stationary waiting time integral equation is

$$1 - \psi(s, z_n) = \phi(s, z_n) - E[\phi(s, z_{n-1}) H(s, z_n, z_{n-1}) G(-s, z_n, z_{n-1}) | z_n]. \tag{7.2}$$

This equation is set up only for s on the imaginary axis, but it is often possible to continue H and ϕ analytically into the left half plane. Presuming H can be so continued to give a single valued function analytic everywhere in the left half plane except for isolated singularities, the following theorem is derived:

If (7.2) *has a solution* $\beta(s, z_n)$ *such that*

(i) $\beta(s, z_n)$ *is, for fixed* z_n, *the analytic continuation of* $\phi(s, z_n)$,

(ii) $a(z_n)$, *such that, for fixed* z_n, $\lim_{s\to\infty} \exp(as)\beta(s, z_n)/s$ *exists with value zero* (*in the left half plane*), *and*

(iii) *for fixed* z_n, *the analytic function composed of* $\phi(s, z_n)$ *and* $\beta(s, z_n)$ *is regular everywhere except for poles, then x for* $x \geqq a$, $\operatorname{pr}(w_{n+1} \leqq x | z_n) - 1$ *is a finite sum of terms of the form*

$$\sum_{r=0}^{k-1} g_r(z_n) x^r \exp(-bx), \tag{7.3}$$

where $-b$ *is a pole of* β *of order k. These poles may depend on* z_n, *but in any case* $\operatorname{Re} b \geqq 0$.

It is readily verified that $z_n = (u_{n+p}, u_{n+p-1}, \cdots, u_n)$ suffices for $\mathfrak{H}$ to be satisfied.

With negative exponential service of parameter μ and the above choice of the z's, $H(s, z_n, z_{n-1})$ becomes $\mu(\mu+s)^{-1}$, independent of z_n, z_{n-1}.

A subsidiary result in [3] gives that the conditions (i) and (ii) of the main theorem are satisfied with $a = 0$ when H is a rational function of s and is independent of the z's.

We now derive the form of (unconditional) limiting waiting time distribution directly from (4.4).

If an arrival finds the queue empty, he begins service immediately. If on arriving he finds $j > 0$ customers already in the queue, we have

$$\begin{aligned} \operatorname{pr}(\text{waiting time} \leqq x) &= \operatorname{pr}(j \text{ services completed in time } \leqq x) \\ &= 1 - \exp(-\mu x) \sum_{i=0}^{j-1} (\mu x)/i!, \qquad x \geqq 0. \end{aligned}$$

Hence the (unconditional) waiting time distribution for an arrival is

$$\begin{aligned} \operatorname{pr}(w \leqq x) &= P_0 + \sum_{j=1}^{\infty} P_j[1 - \exp(-\mu x) \sum_{i=0}^{j-1} (\mu x)/i!] \\ &= 1 - \exp(\mu x) \sum_{j=0}^{p-2} (\sum_{j=0}^{p-1} B_i)(\mu x)^j/j! \\ &\quad - BT(1-T)^{-1} \exp\{-\mu x(1-T)\}, \qquad p \geqq 2, \quad x \geqq 0, \end{aligned} \tag{7.4}$$

using (4.4).

This is the sort of expression that would arise from (8.3) on integrating out z_n if $\beta(s, z_n)$ were in fact analytic everywhere except for poles at $-\mu$, $-\mu(1-T)$ of orders $p-1$ and 1 respectively, both independent of z_n. That $-\mu$ should be a pole seems natural from (7.2), since

$$\phi(s, z_{n-1}) = \mu(\mu+s)^{-1}$$

has a pole at $s = -\mu$. The possibility is left as hypothesis.

We observe that when $p = 1$, i.e. when we have a general recurrent input, the terms in (7.4) involving the B_i's do not appear, and the distribution becomes negative exponential together with a weight at the origin, a fact noted by Smith [6].

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