proved again using the eigenvector map, thereby emphasizing once more the importance of algebraic curves. For the four-dimensional situation the regular level sets are determined and an indication is given of the way in which the bifurcations of the Liouville tori may be obtained. The final chapter departs from the world of tops in order to show how the theory works even for incomplete flows (for rigid bodies the Hamiltonian is a proper map and so the level sets are always compact). The example given is a version of the periodic Toda lattice.

The author's style is precise, yet manages to be friendly, witty and informal as well. The mathematical exposition is complemented by a comprehensive bibliography and index and the text is illustrated by several clear diagrams (I admired in particular the pretty Lagrange top decorating the title page!). I noticed only a few small typographical errors: in line 24 on page 18 the first Γ should be $\Gamma + \epsilon M$; there is a time derivative dot omitted from the LHS of the equation in the statement of Proposition 2.1.1; on page 52 just after equation (8) $sp(4)(\mathrm{R})$ should be $sp(4, \mathrm{R})$; c should be C on line 21 of page 60; in the definition of $\mathrm{Jac}(X)$ on page 115 the RHS should be quotiented by the lattice Λ.

In conclusion I can thoroughly recommend Michèle Audin's *Spinning Tops*, which is an elegant and original contribution to the literature of geometric mechanics. It will be of interest to all workers in the area, especially those appreciating a sophisticated algebraic geometric flavour.

S. T. SWIFT

STUART, A. M. and HUMPHRIES, A. R. *Dynamical systems and numerical analysis* (Cambridge Monographs on Applied and Computational Mathematics No. 2, Cambridge University Press, 1996), xxii+685 pp., 0 521 49672 1, (hardback) £40 (US$59.95).

This book is concerned with the modern dynamical systems approach to analysing the numerical approximation of initial-value autonomous ordinary differential equations, i.e.

$$\dot{u} = f(u) \quad u(0) = U_0 \qquad f : \mathfrak{R}^p \mapsto \mathfrak{R}^p. \tag{1}$$

This way of looking at (1) has grown rapidly in popularity during the last ten years and should be contrasted with the traditional numerical analysis of initial-value O.D.Es, e.g. in [1]. The fundamental idea of the new approach is that conditions are imposed on the vector field f so that (1) defines a dynamical system and then attention is focused on the *evolution semi-group operator* $S(t) : \mathfrak{R}^p \mapsto \mathfrak{R}^p$ defined by

$$S(t)u(0) = u(t). \tag{2}$$

When one moves on to consider a discretisation of (1) from this viewpoint, the first task is to set the discrete problem in a similar framework. Thus, if (1) is approximated using a one-step method with fixed time step Δt, then the discrete equation is either

$$U_{n+1} = F_{\Delta t}(U_n) \tag{3}$$

for an explicit method or

$$G_{\Delta t}(U_{n+1}, U_n) = 0 \tag{4}$$

for an implicit method. This book specifically considers Runge-Kutta methods and derives conditions on f plus restrictions on Δt so that (3) or (4) constitute dynamical systems. Then one may consider the discrete *evolution semi-group operator* $S_{\Delta t} : \mathfrak{R}^p \mapsto \mathfrak{R}^p$ defined by

$$S_{\Delta t} U_0 = U_1. \tag{5}$$

On the other hand, if (1) is approximated using a k-step method with fixed time step Δt, then the discrete equation is either

$$U_{n+k} = F_{\Delta t}(U_{n+k-1}, U_{n+k-2}, \ldots, U_n) \tag{6}$$

for an explicit method or

$$G_{\Delta t}(U_{n+k}, U_{n+k-1}, \ldots, U_n) = 0 \tag{7}$$

for an implicit method. This book specifically considers linear multistep methods and derives conditions on f plus restrictions on Δt so that (6) or (7) constitutes a dynamical system on $\mathfrak{R}^{kp}$ through the transformation

$$\hat{U}_n \equiv (U_n^T, U_{n+1}^T, \ldots, U_{n+k-1}^T)^T.$$

It is then shown that *zero-stable* linear multistep methods contain an embedded one-step method on $\mathfrak{R}^p$ and so may be subsumed within the abstract framework implied by (5). Once this framework has been set up, it is used throughout this book and many approximation results can be derived by considering the error between $S(\Delta t)$ in (2) and $S_{\Delta t}$ in (5). In general the latter would be

$$\|S(\Delta t)U - S_{\Delta t}U\| = O(\Delta t^{r+1})$$

for methods of order r, but for Runge-Kutta methods one has the important strengthening

$$\|S(\Delta t)U - S_{\Delta t}U\| = \|f(U)\|O(\Delta t^{r+1}).$$

Most of the theorems described here are proved within this abstract framework and thus apply to any numerical method for which (5) can be set up.

To understand a dynamical system it is perhaps most important to find its *invariant sets*. Thus, when considering (1) from this viewpoint, it is important to be able to guarantee that a numerical method will faithfully inherit approximations to the invariant sets of (1) and this book considers several such results. The simplest invariant sets are the *fixed points* and conditions are given so that

- fixed points of (1) are approximated to order r,
- *spurious* fixed points of a numerical method must tend to ∞ as $\Delta t \to 0$.

Less simple are *periodic orbits* of (1), but it is shown how these may be approximated by *invariant circles* of a numerical method. The case of a general attractor $\mathcal{A}$ of (1) is complicated because, although there will often be a nearby approximate attractor $\mathcal{A}_{\Delta t}$, this need only converge to a subset of $\mathcal{A}$ as $\Delta t \to 0$, i.e. *upper semi-continuity*. Stronger results do not hold in general for attractors and so it is necessary to consider larger sets, e.g. *inertial manifolds* or *exponential attractors* or, in this book, *uniformly asymptotically stable sets*. For these larger objects one can show approximation of the whole set.

One of the main original reasons for employing the dynamical systems approach was so that features of (1) involving trajectories over long or infinite time intervals could be analysed. Classical error results for approximating (1) have usually been restricted to a single trajectory over a finite time interval $[0, T]$ and have typically taken the form

$$\|U_j - u(j\Delta t)\| \leq C_1 e^{C_2 T}[\Delta t]^r \qquad 0 < j\Delta t \leq T$$

https://doi.org/10.1017/S001309150001957X Published online by Cambridge University Press

for some $C_1, C_2 > 0$: the key point here is that the error constant is *not* uniform, but grows exponentially with the interval length. Using dynamical systems arguments, it is sometimes possible to obtain uniform error constants. The simple example used by the authors to illustrate this key point is the approximation of a trajectory approaching an exponentially attracting fixed point of (1). Less simple is the result that the stable and unstable manifolds of a hyperbolic fixed point of (1), and indeed the whole local phase portrait, may be approximated to order r.

The linear decay problem, i.e.

$$\dot{u} = Au \tag{8}$$

where A is a $p \times p$ matrix whose eigenvalues have strictly negative real part, has been an important test problem in the classical numerical analysis of (1). The dynamics of (8) are, of course, that all trajectories approach the fixed point at the origin but, for the numerical analyst, the important question is: for what range of Δt will a particular numerical method reproduce these dynamics? If the answer is '$\forall \Delta t > 0$' then the numerical method is said to be *A-stable*. A large part of this book is devoted to generalising this approach to the four important classes of *nonlinear* problem below, i.e. determining conditions under which a numerical method will reproduce the dynamics of the continuous problem.

1. *Contractive Systems.* These are systems for which trajectories approach one another as t increases, or at least do not diverge, and a typical assumption is

$$\exists \mu > 0 \text{ s.t. } \langle f(u) - f(v), u - v \rangle \leq -\mu \|u - v\|^2 \qquad \forall u, v \in \mathfrak{R}^p,$$

for which all trajectories converge to a unique fixed point. It is shown that A-stable linear multistep methods reproduce these dynamics, but a similar result for Runge-Kutta methods only holds under the stronger assumption of *algebraic stability*. For other methods within the general framework of (5) the dynamics are reproduced if Δt satisfies a bound depending on the starting value U_0.

2. *Dissipative Systems.* These are systems for which there exists a bounded set in $\mathfrak{R}^p$ which absorbs each trajectory in finite time, and a typical assumption is

$$\exists \alpha \geq 0, \beta > 0 \text{ s.t. } \langle f(u), u \rangle \leq \alpha - \beta \|u\|^2 \qquad \forall u \in \mathfrak{R}^p.$$

It is shown that this behaviour is reproduced by linear multistep, Runge-Kutta and general methods, under similar conditions as for contractive systems.

3. *Gradient Systems.* These are systems for which every trajectory converges to a fixed point, but there may be several of these, and a typical assumption is

$$\exists F : \mathfrak{R}^p \mapsto \mathfrak{R} \text{ s.t. } f(u) = -\nabla F(u) \qquad \forall u \in \mathfrak{R}^p$$

with

$$F(u) \geq 0 \quad \forall u \in \mathfrak{R}^p \quad \text{and} \quad F(u) \mapsto \infty \text{ as } \|u\| \to \infty.$$

Many open questions remain in the approximation of this class of problem, and only a few numerical methods have been shown to reproduce the dynamics for a range of Δt independent of the initial data. General methods within the framework of (5), however, preserve the gradient structure on compact sets provided that Δt satisfies a bound dependent on the set.

4. *Hamiltonian Systems.* These are even-dimensional problems for which the vector field is derived from a scalar function $H : \mathfrak{R}^{2N} \mapsto \mathfrak{R}$, i.e.

$$f(u) \equiv J \nabla H(u)$$

where J is the $2N \times 2N$ skew-symmetric matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Two key properties of such systems are that the *Hamiltonian* H is constant along trajectories and that $S(t)$ in (2) is a *symplectic mapping* for each value of t. Unfortunately, it is impossible for a numerical method to inherit both of these properties without providing the exact solution. Hence, after a discussion of their relative importance, several different techniques for preserving either property are given.

In the above paragraphs I have tried to tease out the main strands in the material of this large [685 pages!] book. Most of the key results which have appeared in the literature are presented, usually with proofs. Although the authors restrict themselves to fixed time-step methods and to ordinary differential equations, there are several references to the currently active research areas of variable time-stepping and the analysis of parabolic partial differential equations as dynamical systems. In my opinion this book is a very fine achievement. It is essential reading for all those interested in computational dynamical systems and there is no easier way of getting close to some of the current research frontiers. It is necessary to point out, however, that there may be dissenting voices. Traditionally, the community of initial value O.D.E. numerical analysts has been predominantly software-oriented, often with little interest in abstract mathematics. Hence the recent interest in the dynamical systems viewpoint has provoked some controversy. Two negative opinions which have been heard are 'computational O.D.Es is an *art* in which heuristics plays a key role' or even 'difficulties can be avoided by designing appropriate variable time-stepping algorithms'. This reviewer's opinion is that the people making such statements and the authors of this book are asking/answering different questions.

Finally, I would like to make a few comments about the suitability of this book for teaching purposes. Two plus factors are the numerous exercises and, especially, the useful final section of each chapter; the latter both extends the material covered and describes the key papers through which it has developed historically. It would be possible to base an undergraduate course on the earlier chapters, which are fairly elementary and contain plenty of simple examples. In the later chapters, however, most important theorems have lengthy and technically complicated proofs, although often they are easy to state, and the material is really postgraduate/research.

G. MOORE

REFERENCE

1. H. J. STETTER, *Analysis of discretisation methods for ordinary differential equations* (Springer-Verlag, 1973).