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Twisted Vertex Operators and Unitary Lie Algebras

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Abstract. A representation of the central extension of the unitary Lie algebra coordinated with a skew Laurent polynomial ring is constructed using vertex operators over an integral $\mathbb{Z}_2$-lattice. The irreducible decomposition of the representation is explicitly computed and described. As a by-product, some fundamental representations of affine Kac–Moody Lie algebra of type $A_n^{(2)}$ are recovered by the new method.

1 Introduction

Affine Kac–Moody Lie algebras, or nontrivial central extensions of loop algebras, are a class of infinite dimensional Lie algebras fundamentally important in mathematics and theoretical physics. They were first realized by vertex operators in [LW] and [KKLW] for the principal picture and later in [FK, S] for the homogeneous picture. During the last two decades these constructions have been generalized in several directions. The vertex operator representations of the toroidal Lie algebras have been given in [MRY, Y, EM, T3] (see also [FJW]) in the homogeneous setting and in [B, T2] in the principal setting; vertex representations of quantum affine algebras have been obtained in [FJ] for untwisted cases and in [J] for twisted cases; the vertex representations of the extended affine Lie algebras of type A coordinated by a quantum torus have been provided in [BS, G4] via the principal construction and in [G3] via the homogeneous construction. Finally in [BGT] a unified treatment of vertex representations of affine Lie algebras using mixed bosons and fermions has been studied, and in [T1] the Tits–Kantor–Koecher algebra has also been realized using the vertex operator calculus.

On the other hand, before the development of toroidal Lie algebras, the elementary unitary Lie algebra $\mathfrak{eu}_\nu(\mathcal{R}, \bar{\cdot})$ associated with an involutive associative algebra $\mathcal{R}$ was studied in [AF] as a derived subalgebra of the unitary Lie algebra $\mathfrak{u}_\nu(\mathcal{R}, \bar{\cdot})$. These Lie algebras are generalizations of the usual loop algebras, replacing the commutative coordinate ring with an noncommutative algebra. As is well known, the central extension of Lie algebras is essentially given by A. Connes' cyclic homology [KL]. For the extended affine Lie algebras the central extensions are also described by the dihedral homology [ABG, G1], and the relevant central extensions in the uni-

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tary algebras coordinated with an involutive algebra will be given by a Steinberg Lie algebra structure.

The Steinberg group is the universal central extension of the commutator subgroup of the general linear group. The Steinberg Lie algebra is defined as an associative algebra generated by the generators $x_{ij}(\lambda)$ subject to the Steinberg relations. The extended affine Lie algebras share many common features with loop algebras and have also distinguished central extensions given by dihedral homology. When one further generalizes the algebraic structure by replacing the coordinate rings with the quantum polynomial ring or even an associative algebra, Steinberg unitary Lie algebras will be the key for the universal central extensions. The realization of Steinberg unitary Lie algebras is thus a natural question in this regard.

In this paper, inspired by the work of Wakimoto [W], we will construct a family of twisted vertex operators associated with an integral $\mathbb{Z}_2$-lattice of rank ν to realize the Steinberg relations. More precisely, for any nonzero complex number a, we define vertex operators $X_{ij}(a,z)$ for $1 \leq i, j \leq \nu$ on a Fock space M. For the purpose of computing the commutator relations, two multi-product decompositions of the δ-function and the $D\delta$-function are developed. For any abelian group G with a character $\sigma\colon G \to \mathbb{C}^\times$, we can define a corresponding skew Laurent-polynomial ring $\mathcal{R}_\sigma$ and an anti-involution $\bar{\cdot}$ over $\mathcal{R}_\sigma$. We will show that the vertex operators $X_{ij}(a,z)$ give a representation for the Lie algebra $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot}\,)$ and also provide a nontrivial central extension of the unitary Lie algebra $\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot}\,)$ associated to the pair $(\mathcal{R}_\sigma, \bar{\cdot})$, namely, the twisted vertex representation is actually a module for the central extension $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$.

As was shown in [W], it is interesting to analyze the irreducible decomposition of the twisted module. Based on the observation that the twisted group algebra of the integral $\mathbb{Z}_2$-lattice is isomorphic to a (finite dimensional) Clifford algebra we are able to use the well-known classical representation theory of finite dimensional Clifford algebras to achieve this goal. We remark that this is slightly different from the approach taken by Wakimoto [W]. Let $|\sigma|$ be the order of the group $\sigma(G)$. If $|\sigma| = \infty$ or $|\sigma| \in 2\mathbb{N}$, then each irreducible component of $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module M remains irreducible as an $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module. For the case $|\sigma| \in 2\mathbb{N}+1$, note that in this case all the elements in $\sigma(G)$ are roots of unity. To this end, we introduce a conjugate anti-involution τ on $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ as all the elements of $\sigma(G)$ lie in the unit circle in the complex plane. Then, we prove that the $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module M is unitary with respect to τ, which in turn deduce that M is completely reducible as $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module even if $|\sigma| \in 2\mathbb{N}+1$. As a by-product, when taking $G = \{1\}$ we get a completely reducible module M for the affine Kac–Moody Lie algebra of type $A^{(2)}_{\nu-1}$, which recovers the result in [W].

This paper is organized as follows. In Section 2, we define a family of vertex operators for any $a \in \mathbb{C}^\times$ and then compute their commutation relations. In Section 3, using the derived relations we construct the vertex operator representations for the unitary Lie algebra with nontrivial central extension and then determine the irreducible components. Finally, in Section 4, we construct vertex representations for the elementary unitary Lie algebra with nontrivial central extension. As an example we also obtain a new realization of the affine Kac–Moody algebra of type $A^{(2)}_{\nu-1}$.

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Throughout the paper, we denote the field of complex numbers, the group of non-zero complex numbers, the ring of integers, and the set of non-negative integers by $\mathbb{C}, \mathbb{C}^\times, \mathbb{Z}$, and $\mathbb{N}$, respectively.

2 Fock Space and Vertex Operators

Let $\nu \geq 2$ be a positive integer. Let $\Gamma = \bigoplus_{i=1}^{\nu} \mathbb{Z}\epsilon_i$, $(\epsilon_i, \epsilon_j) = \delta_{ij}$ for $i, j = 1, \ldots, \nu$, and let $\epsilon_i(n)$ be a linear copy of ϵ_i for $n \in 2\mathbb{Z}+1$ and $i = 1, \ldots, \nu$. We define a Lie algebra

$$\mathcal{H} = \mathrm{span}_{\mathbb{C}}\{\epsilon_i(n), 1 | n \in 2\mathbb{Z}+1\}$$

subject to the following Lie algebra relation

$$[\alpha(m), \beta(n)] = \frac{m}{2}(\alpha, \beta)\delta_{m+n,0}$$

for $\alpha, \beta \in \Gamma$, and $m, n \in 2\mathbb{Z}+1$. Let $\overline{\Gamma} = \Gamma/2\Gamma$ be the quotient additive group of Γ factored by the subgroup 2Γ. Let $\mathbb{C}[\overline{\Gamma}] = \bigoplus_{\alpha\in\overline{\Gamma}} \mathbb{C}e^{\alpha}$ be the twisted group over the finite group $\overline{\Gamma}$ with multiplication defined by $e^{\overline{\alpha}}e^{\overline{\beta}} = \varepsilon(\alpha, \beta)e^{\overline{\alpha+\beta}}$ for $\alpha, \beta \in \Gamma$, where the two-cocycle $\varepsilon\colon \Gamma \times \Gamma \to \{\pm 1\}$ is given by

$$\varepsilon(\epsilon_i, \epsilon_j) = \begin{cases} 1 & \text{if } i \leq j, \\ -1 & \text{if } i > j. \end{cases} \tag{2.1}$$

and

$$\varepsilon\Big(\sum m_i\epsilon_i, \sum n_j\epsilon_j\Big) = \prod_{i,j}\big(\varepsilon(\epsilon_i, \epsilon_j)\big)^{m_i n_j}.$$

We note that the multiplication in $\mathbb{C}[\overline{\Gamma}]$ is well defined, as $\varepsilon(\alpha_1, \beta_1) = \varepsilon(\alpha_2, \beta_2)$ if $\alpha_2 - \alpha_1, \beta_2 - \beta_1 \in 2\Gamma$. Let z, w be formal variables and $\alpha \in \Gamma$, set

$$E_{\pm}(\alpha, z) = \exp\Big(-2\sum_{n\in\pm(2\mathbb{N}+1)} \frac{\alpha(n)}{n} z^{-n}\Big),$$

and define the Fock space

$$M = \mathbb{C}[\overline{\Gamma}] \otimes \mathcal{S}(\mathcal{H}^-),$$

where $\mathcal{S}(\mathcal{H}^-)$ is the usual commutative symmetric algebra over the subalgebra $\mathcal{H}^-$ of $\mathcal{H}$ spanned by $\epsilon_i(n)$ with $1 \leq i \leq \nu$ and $n \in -(2\mathbb{N}+1)$.

Lemma 2.1 *For $\alpha, \beta \in \Gamma$, one has*

$$E_{\pm}(\alpha, z) = E_{\pm}(-\alpha, -z),$$

$$E_+(\alpha, z)E_-(\beta, w) = E_-(\beta, w)E_+(\alpha, z)\Big(\frac{1 - w/z}{1 + w/z}\Big)^{(\alpha,\beta)}.$$

Proof The first identity is clear. To prove the second identity, we set

$$A = -2\sum_{n\in(2\mathbb{N}+1)} \frac{\alpha(n)}{n} z^{-n}, \quad B = -2\sum_{n\in-(2\mathbb{N}+1)} \frac{\beta(n)}{n} w^{-n},$$

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and note that

$$\begin{aligned}[A,B] &= 4\sum_{m\in(2\mathbb{N}+1)}\sum_{n\in-(2\mathbb{N}+1)}\frac{z^{-m}w^{-n}}{mn}\big[\alpha(m),\beta(n)\big]\\ &= 4\sum_{m\in(2\mathbb{N}+1)}\frac{z^{-m}w^{m}}{-m^2}(\alpha,\beta)\frac{m}{2}\\ &= -2(\alpha,\beta)\sum_{m\in(2\mathbb{N}+1)}\frac{1}{m}(w/z)^m = \log\Big(\frac{1-w/z}{1+w/z}\Big)^{(\alpha,\beta)}.\end{aligned}$$

Then by applying the formal rule $e^Ae^B = e^{[A,B]}e^Be^A$, if $[A,B]$ commutes with A and B, we get the required identity. ∎

For $a\in\mathbb{C}^\times$ and $1\le i,j\le\nu$, we define the following vertex operators, which act on the Fock space M in the usual way (see [FLM]).

$$X_{ij}(a,z) = \begin{cases}\varepsilon(\epsilon_i,\epsilon_j)e^{\overline{\epsilon_i-\epsilon_j}}E_-(\epsilon_i,z)E_-(-\epsilon_j,az)E_+(\epsilon_i,z)E_+(-\epsilon_j,az) & i\ne j,\\ 4\epsilon_i(z) & i=j,a=1,\\ \frac{1+a}{1-a}\big(E_-(\epsilon_i,z)E_-(-\epsilon_i,az)E_+(\epsilon_i,z)E_+(-\epsilon_i,az)-1\big) & i=j,a\ne1,\end{cases}\tag{2.2}$$

where $\epsilon_i(z)=\sum_{n\in2\mathbb{Z}+1}\epsilon_i(n)z^{-n}$, and $e^{\overline{\epsilon_i-\epsilon_j}}$ is the usual operator acting on the group algebra $\mathbb{C}[\overline{\Gamma}]$ twisted by the two-cocycle ε. Set

$$A_\pm = -2\sum_{n\in\pm(2\mathbb{N}+1)}\frac{\epsilon_i(n)}{n}(az)^{-n},\quad B_\pm = -2\sum_{n\in\pm(2\mathbb{N}+1)}\frac{\epsilon_i(n)}{n}z^{-n}.$$

Then for $a\ne1$, we have

$$\begin{aligned}&\frac{1}{1-a}\big(E_\pm(\epsilon_i,z)-E_\pm(\epsilon_i,az)\big)\\ &= \frac{1}{1-a}\big(e^{B_\pm}-e^{A_\pm}\big) = \frac{1}{1-a}\Big(\sum_{l=1}^{\infty}\frac{1}{l!}B_\pm^l-\sum_{l=1}^{\infty}\frac{1}{l!}A_\pm^l\Big)\\ &= \frac{1}{1-a}\sum_{l=1}^{\infty}\frac{1}{l!}\Big[\Big(2\sum_{n\in\pm(2\mathbb{N}+1)}\frac{\epsilon_i(n)}{n}(az)^{-n}-2\sum_{n\in\pm(2\mathbb{N}+1)}\frac{\epsilon_i(n)}{n}z^{-n}\Big)\sum_{j=0}^{l-1}A_\pm^{l-1-j}B_\pm^j\Big]\\ &= \sum_{l=1}^{\infty}\frac{1}{l!}\Big[2\sum_{n\in\pm(2\mathbb{N}+1)}\frac{\epsilon_i(n)}{n}(z)^{-n}\frac{a^{-n}-1}{1-a}\sum_{j=0}^{l-1}A_\pm^{l-1-j}B_\pm^j\Big]\\ &\longrightarrow 2\sum_{n\in\pm(2\mathbb{N}+1)}\epsilon_i(n)z^{-n}\sum_{l=1}^{\infty}\frac{1}{l!}lB_\pm^{l-1} = 2\sum_{n\in\pm(2\mathbb{N}+1)}\epsilon_i(n)z^{-n}E_\pm(\epsilon_i,z)\end{aligned}$$

as $a\longrightarrow1$.

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From this we obtain that, for $a \neq 1$,

$$\begin{aligned} X_{ii}(a,z) &= \frac{1+a}{1-a}E_-(-\epsilon_i,az)\big(E_-(\epsilon_i,z)E_+(\epsilon_i,z) - E_-(\epsilon_i,az)E_+(\epsilon_i,az)\big)E_+(-\epsilon_i,az) \\ &= (1+a)E_-(-\epsilon_i,az)\Big(\frac{E_-(\epsilon_i,z)-E_-(\epsilon_i,az)}{1-a}E_+(\epsilon_i,z)+ \\ &\qquad E_-(\epsilon_i,az)\frac{E_+(\epsilon_i,z)-E_+(\epsilon_i,az)}{1-a}\Big)E_+(-\epsilon_i,az) \\ &\longrightarrow 4\epsilon_i(z) \quad \text{as } a \longrightarrow 1. \end{aligned}$$

Therefore, we have $\lim_{a\to 1} X_{ii}(a,z) = X_{ii}(1,z)$.

Moreover, from the definition of the vertex operator $X_{ij}(a,z)$, we can easily check the following result.

Lemma 2.2 *For $a \in \mathbb{C}^\times$ and $1 \leq i, j \leq \nu$, one has*

$$X_{ij}(a,z) = -X_{ji}(a^{-1},-az).$$

The vertex operator $X_{ij}(a,z)$ can be formally expanded as follows

$$X_{ij}(a,z) = \sum_{n\in\mathbb{Z}} x_{ij}(a,n)z^{-n},$$

where $x_{ij}(a,n)$ are operators acting on the Fock space M. Then the equation in the previous lemma implies that

$$x_{ji}(a^{-1},n) = -(-1)^n a^n x_{ij}(a,n),$$

for $n \in \mathbb{Z}$, $i, j = 1,\dots,\nu$, $a \in \mathbb{C}^\times$.

Lemma 2.3 *For $\alpha \in \Gamma$, set $\alpha(z) = \sum_{n\in 2\mathbb{Z}+1} \alpha(n)z^{-n}$. Then*

$$\big[\alpha(z), E_\pm(\beta,w)\big] = (\alpha,\beta)E_\pm(\beta,w)\sum_{n\in\pm(2\mathbb{N}+1)}\Big(\frac{z}{w}\Big)^n.$$

Proof The result follows from the formal rule $[A,e^B] = [A,B]e^B$ if $[A,B]$ commutes with B, and the identity

$$[\alpha(z), -2\sum_{n\in\pm(2\mathbb{N}+1)}\frac{\beta(n)}{n}w^{-n}] = (\alpha,\beta)\sum_{n\in\pm(2\mathbb{N}+1)}\Big(\frac{z}{w}\Big)^n. \qquad \blacksquare$$

The following result is well known (see [FLM]) and will be used frequently later on.

Lemma 2.4 *Let $Y(w,z)$ be a formal power series in w,z with coefficient in a vector space such that $\lim_{z\to w} Y(w,z)$ exists (in the sense of [FLM]). Set $D_z = z\frac{\partial}{\partial z}$. Then*

$$\begin{aligned} Y(w,z)\delta\Big(a\frac{w}{z}\Big) &= Y(w,aw)\delta\Big(a\frac{w}{z}\Big), \\ Y(w,z)(D\delta)\Big(a\frac{w}{z}\Big) &= Y(w,aw)(D\delta)\Big(a\frac{w}{z}\Big) + (D_zY)(w,z)\delta\Big(a\frac{w}{z}\Big). \end{aligned}$$

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Now we are going to compute the Lie product $[X_{ij}(a,z), X_{kl}(b,w)]$ for $1 \leq i, j, k, l \leq \nu$, and $a, b \in \mathbb{C}^\times$. For this purpose we need several combinatorial identities.

Lemma 2.5 *Let B and A_i for $i = 0, 1, \ldots, n$ be nonzero distinct complex numbers. Then*

$$\Big(\frac{B}{B-A_0}\Big)^2 \prod_{j=1}^{n} \frac{B}{B-A_j} = \sum_{i=1}^{n} \Big(\frac{A_i}{A_i-A_0}\Big)^2 \Big(\prod_{\substack{1\le j\le n,\\ j\neq i}} \frac{A_i}{A_i-A_j}\Big) \frac{B}{B-A_i} + \Big(\prod_{i=1}^{n} \frac{A_0}{A_0-A_i}\Big) \Big[\Big(\frac{B}{B-A_0}\Big)^2 + \Big(\sum_{l=1}^{n} \frac{A_l}{A_l-A_0}\Big) \frac{B}{B-A_0}\Big],$$

$$\Big(\frac{A_0}{B-A_0}\Big)^2 \prod_{j=1}^{n} \frac{A_j}{B-A_j} = \sum_{i=1}^{n} \Big(\frac{A_0}{A_i-A_0}\Big)^2 \Big(\prod_{\substack{1\le j\le n,\\ j\neq i}} \frac{A_j}{A_i-A_j}\Big) \frac{A_i}{B-A_i} + \Big(\prod_{i=1}^{n} \frac{A_i}{A_0-A_i}\Big) \Big[\Big(\frac{A_0}{B-A_0}\Big)^2 + \Big(\sum_{l=1}^{n} \frac{A_0}{A_l-A_0}\Big) \frac{A_0}{B-A_0}\Big]$$

for $n \geq 1$.

Proof We will prove the first identity by induction on n. The proof of the second one is similar to the first, so we omit it.

It is not difficult to check the result for $n = 1$. For the case $n+1$, we have

$$\begin{aligned}
&\sum_{i=1}^{n+1} \Big(\frac{A_i}{A_i-A_0}\Big)^2 \Big(\prod_{\substack{1\le j\le n+1,\\ j\neq i}} \frac{A_i}{A_i-A_j}\Big) \frac{B}{B-A_i} + \\
&\Big(\prod_{i=1}^{n+1} \frac{A_0}{A_0-A_i}\Big) \Big[\Big(\frac{B}{B-A_0}\Big)^2 + \Big(\sum_{l=1}^{n+1} \frac{A_l}{A_l-A_0}\Big) \frac{B}{B-A_0}\Big] \\
&= \Big(\frac{A_{n+1}}{A_{n+1}-A_0}\Big)^2 \Big(\prod_{j=1}^{n} \frac{A_{n+1}}{A_{n+1}-A_j}\Big) \frac{B}{B-A_{n+1}} \\
&+ \sum_{i=1}^{n} \Big(\frac{A_i}{A_i-A_0}\Big)^2 \Big(\prod_{\substack{1\le j\le n,\\ j\neq i}} \frac{A_i}{A_i-A_j}\Big) \frac{A_i}{A_i-A_{n+1}} \frac{B}{B-A_i} \\
&+ \Big(\prod_{i=1}^{n} \frac{A_0}{A_0-A_i}\Big) \Big[\Big(\frac{A_0}{A_0-A_{n+1}} \frac{B}{B-A_0}\Big) \Big(\frac{B}{B-A_0} - \frac{A_{n+1}}{A_0-A_{n+1}} + \sum_{l=1}^{n} \frac{A_l}{A_l-A_0}\Big)\Big] \\
&= \Big(\frac{A_{n+1}}{A_{n+1}-A_0}\Big)^2 \Big(\prod_{j=1}^{n} \frac{A_{n+1}}{A_{n+1}-A_j}\Big) \frac{B}{B-A_{n+1}} \\
&+ \sum_{i=1}^{n} \Big(\frac{A_i}{A_i-A_0}\Big)^2 \Big(\prod_{\substack{1\le j\le n,\\ j\neq i}} \frac{A_i}{A_i-A_j}\Big) \Big(\frac{A_{n+1}}{A_i-A_{n+1}} + \frac{B}{B-A_i}\Big) \frac{B}{B-A_{n+1}}
\end{aligned}$$

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$$+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[\Big(\frac{B}{B-A_0}\Big)^2-\Big(\frac{A_{n+1}}{A_0-A_{n+1}}\Big)^2\Big]\frac{B}{B-A_{n+1}}$$
$$+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big(\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\Big(\frac{A_{n+1}}{A_0-A_{n+1}}+\frac{B}{B-A_0}\Big)\frac{B}{B-A_{n+1}}$$
$$=\Big(\frac{A_{n+1}}{A_{n+1}-A_0}\Big)^2\Big(\prod_{j=1}^{n}\frac{A_{n+1}}{A_{n+1}-A_j}\Big)\frac{B}{B-A_{n+1}}+F_1+F_2,$$

where

$$F_1=\sum_{i=1}^{n}\Big(\frac{A_i}{A_i-A_0}\Big)^2\Big(\prod_{\substack{1\le j\le n,\\ j\ne i}}\frac{A_i}{A_i-A_j}\Big)\Big(\frac{A_{n+1}}{A_i-A_{n+1}}\Big)\frac{B}{B-A_{n+1}}$$
$$+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[\Big(\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\frac{A_{n+1}}{A_0-A_{n+1}}-\Big(\frac{A_{n+1}}{A_0-A_{n+1}}\Big)^2\Big]\frac{B}{B-A_{n+1}},$$
$$F_2=\sum_{i=1}^{n}\Big(\frac{A_i}{A_i-A_0}\Big)^2\Big(\prod_{\substack{1\le j\le n,\\ j\ne i}}\frac{A_i}{A_i-A_j}\Big)\Big(\frac{B}{B-A_i}\Big)\frac{B}{B-A_{n+1}}$$
$$+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[\Big(\frac{B}{B-A_0}\Big)^2+\Big(\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\frac{B}{B-A_0}\Big]\frac{B}{B-A_{n+1}}.$$

Now, by using induction on n, we find

$$F_1=-\Big(\frac{A_{n+1}}{A_{n+1}-A_0}\Big)^2\Big(\prod_{j=1}^{n}\frac{A_{n+1}}{A_{n+1}-A_j}\Big)\frac{B}{B-A_{n+1}},$$
$$F_2=\Big(\frac{B}{B-A_0}\Big)^2\prod_{j=1}^{n+1}\frac{B}{B-A_j},$$

as required. ∎

Corollary 2.6 *Let $A_i\ne A_j$ for $i\ne j$ be nonzero complex numbers. Then*

$$\Big(\frac{1}{1-A_0x}\Big)^2\prod_{j=1}^{n}\frac{1}{1-A_jx}=\sum_{i=1}^{n}\Big(\frac{A_i}{A_i-A_0}\Big)^2\Big(\prod_{\substack{1\le j\le n,\\ j\ne i}}\frac{A_i}{A_i-A_j}\Big)\frac{1}{1-A_ix}$$
$$+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[\frac{A_0x}{(1-A_0x)^2}+\Big(1+\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\frac{1}{1-A_0x}\Big],$$
$$\Big(\frac{A_0x}{1-A_0x}\Big)^2\prod_{j=1}^{n}\frac{A_jx}{1-A_jx}=\sum_{i=1}^{n}\Big(\frac{A_0}{A_i-A_0}\Big)^2\Big(\prod_{\substack{1\le j\le n,\\ j\ne i}}\frac{A_j}{A_i-A_j}\Big)\frac{A_ix}{1-A_ix}$$
$$+\Big(\prod_{i=1}^{n}\frac{A_i}{A_0-A_i}\Big)\Big[\frac{A_0x}{(1-A_0x)^2}+\Big(\sum_{l=1}^{n}\frac{A_0}{A_l-A_0}-1\Big)\frac{A_0x}{1-A_0x}\Big].$$

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Proposition 2.7 *Let A_i for $0 \le i \le n$ be distinct nonzero complex numbers, then*

$$\begin{aligned}
&\Big(\frac{1}{1-A_0x}\Big)^2\prod_{i=1}^{n}\frac{1}{1-A_ix}-(-1)^n\Big(\frac{A_0^{-1}x^{-1}}{1-A_0^{-1}x^{-1}}\Big)^2\prod_{i=1}^{n}\frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}}\\
&\quad=\sum_{i=1}^{n}\Big[\Big(\frac{A_i}{A_i-A_0}\Big)^2\prod_{1\le j\le n,\,j\ne i}\frac{A_i}{A_i-A_j}\Big]\delta(A_ix)\\
&\qquad+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[(D\delta)(A_0x)+\Big(1+\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\delta(A_0x)\Big]
\end{aligned}$$

Proof By applying the second identity in Corollary 2.6, we have

$$\begin{aligned}
&(-1)^{n-1}\Big(\frac{A_0^{-1}x^{-1}}{1-A_0^{-1}x^{-1}}\Big)^2\prod_{i=1}^{n}\frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}}\\
&\quad=(-1)^{n-1}\sum_{i=1}^{n}\Big(\frac{A_0^{-1}}{A_i^{-1}-A_0^{-1}}\Big)^2\Big(\prod_{\substack{1\le j\le n,\\ j\ne i}}\frac{A_j^{-1}}{A_i^{-1}-A_j^{-1}}\Big)\frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}}\\
&\qquad+(-1)^{n-1}\Big(\prod_{i=1}^{n}\frac{A_i^{-1}}{A_0^{-1}-A_i^{-1}}\Big)\\
&\qquad\quad\cdot\Big[\frac{A_0^{-1}x^{-1}}{(1-A_0^{-1}x^{-1})^2}+\Big(\sum_{l=1}^{n}\frac{A_0^{-1}}{A_l^{-1}-A_0^{-1}}-1\Big)\frac{A_0^{-1}x^{-1}}{1-A_0^{-1}x^{-1}}\Big]\\
&\quad=\sum_{i=1}^{n}\Big(\frac{A_i}{A_i-A_0}\Big)^2\Big(\prod_{\substack{1\le j\le n,\\ j\ne i}}\frac{A_i}{A_i-A_j}\Big)\frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}}\\
&\qquad-\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[\frac{A_0^{-1}x^{-1}}{(1-A_0^{-1}x^{-1})^2}-\Big(1+\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\frac{A_0^{-1}x^{-1}}{1-A_0^{-1}x^{-1}}\Big].
\end{aligned}$$

Therefore, from the first identity in Corollary 2.6, we obtain

$$\begin{aligned}
&\Big(\frac{1}{1-A_0x}\Big)^2\prod_{i=1}^{n}\frac{1}{1-A_ix}-(-1)^n\Big(\frac{A_0^{-1}x^{-1}}{1-A_0^{-1}x^{-1}}\Big)^2\prod_{i=1}^{n}\frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}}\\
&\quad=\sum_{i=1}^{n}\Big(\frac{A_i}{A_i-A_0}\Big)^2\Big(\prod_{1\le j\le n,j\ne i}\frac{A_i}{A_i-A_j}\Big)\Big(\frac{1}{1-A_ix}+\frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}}\Big)\\
&\quad+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big(\frac{A_0x}{(1-A_0x)^2}-\frac{A_0^{-1}x^{-1}}{(1-A_0^{-1}x^{-1})^2}\Big)\\
&\quad+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[\Big(1+\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\Big(\frac{1}{1-A_0x}+\frac{A_0^{-1}x^{-1}}{1-A_0i^{-1}x^{-1}}\Big)\Big]\\
&\quad=\sum_{i=1}^{n}\Big[\Big(\frac{A_i}{A_i-A_0}\Big)^2\prod_{1\le j\le n,j\ne i}\frac{A_i}{A_i-A_j}\Big]\delta(A_ix)\\
&\quad+\Big(\prod_{i=1}^{n}\frac{A_0}{A_0-A_i}\Big)\Big[(D\delta)(A_0x)+\Big(1+\sum_{l=1}^{n}\frac{A_l}{A_l-A_0}\Big)\delta(A_0x)\Big]
\end{aligned}$$

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Proposition 2.8 *Let A_i for $1 \le i \le n$ be distinct nonzero complex numbers. Then*

$$\prod_{i=1}^{n} \frac{1}{1-A_i x} - (-1)^n \prod_{i=1}^{n} \frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}} = \sum_{i=1}^{n} \Big(\prod_{1\le j\le n, j\ne i} \frac{A_i}{A_i - A_j} \Big) \delta(A_i x).$$

Proof By using Proposition 2.7 and the fact that

$$(1-A_0x)^2 \Big(\frac{1}{1-A_0x}\Big)^2 = 1 = (1-A_0x)^2 \Big(\frac{A_0^{-1}x^{-1}}{1-A_0^{-1}x^{-1}}\Big)^2,$$

one may get that,

$$\begin{aligned}
&\prod_{i=1}^{n} \frac{1}{1-A_i x} - (-1)^n \prod_{i=1}^{n} \frac{A_i^{-1}x^{-1}}{1-A_i^{-1}x^{-1}} \\
&\quad = (1-A_0x)^2 \sum_{i=1}^{n} \Big[\Big(\frac{A_i}{A_i - A_0}\Big)^2 \prod_{1\le j\le n, j\ne i} \frac{A_i}{A_i - A_j} \Big] \delta(A_i x) \\
&\qquad + (1-A_0x)^2 \Big(\prod_{i=1}^{n} \frac{A_0}{A_0 - A_i} \Big) \Big[(D\delta)(A_0x) + \Big(1 + \sum_{l=1}^{n} \frac{A_l}{A_l - A_0}\Big) \delta(A_0x) \Big] \\
&\quad = \sum_{i=1}^{n} \Big(\prod_{1\le j\le n, j\ne i} \frac{A_i}{A_i - A_j} \Big) \delta(A_i x),
\end{aligned}$$

where we have used Lemma 2.4 for the second equation. ∎

For convenience, we set

$$\begin{aligned}
:X_{ij}(a,z)X_{kl}(b,w): = e^{\overline{\epsilon_i - \epsilon_j + \epsilon_k - \epsilon_l}} E_-(\epsilon_i, z)E_-(-\epsilon_j, az)E_-(\epsilon_k, w)E_-(-\epsilon_l, bw) \\
\cdot E_+(\epsilon_i, z)E_+(-\epsilon_j, az)E_+(\epsilon_k, w)E_+(-\epsilon_l, bw),
\end{aligned}$$

where $1 \le i, j, k, l \le \nu$, $a, b \in \mathbb{C}^\times$ with the condition that $i \ne j$ if $a = 1$ and $k \ne l$ if $b = 1$. The proof of the following results is straightforward and is omitted.

Lemma 2.9 *For $a, b \in \mathbb{C}^\times$ and $\alpha \in \Gamma$, one has*

$$\begin{aligned}
\big[X_{ij}(a,z), e^{\overline{\alpha}}\big] &= \big((-1)^{(\epsilon_i - \epsilon_j, \alpha)} - 1\big) e^{\overline{\alpha}} X_{ij}(a,z), \\
X_{ij}(a,z)X_{kl}(b,w) &= :X_{ij}(a,z)X_{kl}(b,w): P_{kl}^{ij},
\end{aligned}$$

where $i \ne j$ if $a = 1$ and $k \ne l$ if $b = 1$, and

$$\begin{aligned}
P_{kl}^{ij} = \varepsilon(\epsilon_i, \epsilon_j)\varepsilon(\epsilon_k, \epsilon_l)\varepsilon(\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l) \Big(\frac{1 - w/az}{1 + w/az}\Big)^{-\delta_{jk}} \\
\cdot \Big(\frac{1 - w/z}{1 + w/z}\Big)^{\delta_{ik}} \Big(\frac{1 - bw/az}{1 + bw/az}\Big)^{\delta_{jl}} \Big(\frac{1 - bw/z}{1 + bw/z}\Big)^{-\delta_{il}}.
\end{aligned}$$

By symmetry, from the previous lemma and (2.1), we obtain the following lemma.

Lemma 2.10 *Let $i \ne j$ if $a = 1$ and $k \ne l$ if $b = 1$. Then*

$$\big[X_{ij}(a,z), X_{kl}(b,w)\big] = :X_{ij}(a,z)X_{kl}(b,w): \Delta_{kl}^{ij},$$

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where

$$\Delta_{kl}^{ij} = \left(\frac{1+a}{1-a}\right)^{\delta_{ij}} \left(\frac{1+b}{1-b}\right)^{\delta_{kl}} \varepsilon(\epsilon_i, \epsilon_j)\varepsilon(\epsilon_k, \epsilon_l)\varepsilon(\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l) \cdot \left(\frac{az+w}{az}\right)^{\delta_{jk}} \left(\frac{z-w}{z}\right)^{\delta_{ik}} \left(\frac{az-bw}{az}\right)^{\delta_{jl}} \left(\frac{z+bw}{z}\right)^{\delta_{il}} Q_{kl}^{ij}, \tag{2.3}$$

and

$$Q_{kl}^{ij} = \left(\frac{1}{1+\frac{w}{z}}\right)^{\delta_{ik}} \left(\frac{1}{1-\frac{w}{az}}\right)^{\delta_{jk}} \left(\frac{1}{1-\frac{bw}{z}}\right)^{\delta_{il}} \left(\frac{1}{1+\frac{bw}{az}}\right)^{\delta_{jl}} - \left(\frac{\frac{z}{w}}{1+\frac{z}{w}}\right)^{\delta_{ik}} \left(\frac{-\frac{az}{w}}{1-\frac{az}{w}}\right)^{\delta_{jk}} \left(\frac{-\frac{z}{bw}}{1-\frac{z}{bw}}\right)^{\delta_{il}} \left(\frac{\frac{az}{bw}}{1+\frac{az}{bw}}\right)^{\delta_{jl}}.$$

In the rest of this section, we will compute the Lie product $[X_{ij}(a,z), X_{kl}(b,w)]$ with $1 \le i, j, k, l \le \nu$, and $a, b \in \mathbb{C}^\times$, where the vertex operators are defined by (2.2). For this purpose, we divide the argument into the following three cases:

Case 1 $i \ne j, k \ne l$;
Case 2 $i = j, k \ne l$;
Case 3 $i = j, k = l$.

We first consider Case 1, with $i \ne j$, $k \ne l$.

Proposition 2.11 *Let $i \ne j$, $k \ne l$, and $a, b \in \mathbb{C}^\times$.*

(i) *If i, j, k, l are distinct integers, then $[X_{ij}(a,z), X_{kl}(b,w)] = 0$.*

(ii) *If $j = k$, $i \ne l$, then*

$$\big[X_{ij}(a,z), X_{jl}(b,w)\big] = 2X_{il}(ab,z)\delta\Big(\frac{w}{az}\Big).$$

(iii) *If $j = k$, $i = l$, and $ab \ne 1$, then*

$$\big[X_{ij}(a,z), X_{ji}(b,w)\big] = 2X_{ii}(ab,z)\delta\Big(\frac{w}{az}\Big) - 2X_{jj}(ab,w)\delta\Big(\frac{bw}{z}\Big) + 2\frac{1+ab}{1-ab}\Big(\delta\Big(\frac{w}{az}\Big) - \delta\Big(\frac{bw}{z}\Big)\Big).$$

(iv) *If $j = k$, $i = l$, and $ab = 1$, then*

$$\big[X_{ij}(a,z), X_{ji}(b,w)\big] = 2(X_{ii}(1,z) - X_{jj}(1,w))\delta\Big(\frac{bw}{z}\Big) + 4(D\delta)\Big(\frac{bw}{z}\Big).$$

Proof From (2.3), one can easily see statement (i). For statement (ii) of the proposition, we have

$$\Delta_{jl}^{ij} = \varepsilon(\epsilon_i, \epsilon_l)\Big(\frac{az+w}{az}\Big)\Big[\frac{1}{1-\frac{w}{az}} + \frac{\frac{az}{w}}{1-\frac{az}{w}}\Big] = \varepsilon(\epsilon_i, \epsilon_l)\Big(\frac{az+w}{az}\Big)\delta\Big(\frac{w}{az}\Big).$$

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Thus, by Lemma 2.10, we get

$$\begin{aligned}&\big[X_{ij}(a,z),X_{jl}(b,w)\big]\\&\quad=\varepsilon(\epsilon_i,\epsilon_l)\Big(\frac{az+w}{az}\Big)\delta\Big(\frac{w}{az}\Big)\,{:}X_{ij}(a,z)X_{jl}(b,w){:}\\&\quad=2\varepsilon(\epsilon_i,\epsilon_l)\delta\Big(\frac{w}{az}\Big)\,e^{\overline{\epsilon_i-\epsilon_l}}E_-(\epsilon_i,z)E_-(-\epsilon_l,abz)E_+(\epsilon_i,z)E_+(-\epsilon_l,abz)\\&\quad=2X_{il}(ab,z)\delta\Big(\frac{w}{az}\Big).\end{aligned}$$

This gives statement (ii). Similarly for statement (iii), we have, from (2.3) and Proposition 2.8,

$$\begin{aligned}\Delta_{ji}^{ij}&=\Big(\frac{az+w}{az}\Big)\Big(\frac{z+bw}{z}\Big)\Big[\frac{1}{1-\frac{w}{az}}\frac{1}{1-\frac{bw}{z}}-\frac{\frac{az}{w}}{1-\frac{az}{w}}\frac{\frac{z}{bw}}{1-\frac{z}{bw}}\Big]\\&=\Big(\frac{az+w}{az}\Big)\Big(\frac{z+bw}{z}\Big)\Big(\frac{1}{1-ab}\delta\Big(\frac{w}{az}\Big)+\frac{1}{1-\frac{1}{ab}}\delta\Big(\frac{bw}{z}\Big)\Big).\end{aligned}$$

Thus, from Lemma 2.10, we obtain

$$\begin{aligned}&\big[X_{ij}(a,z),X_{ji}(b,w)\big]\\&=\Big(\frac{az+w}{az}\Big)\Big(\frac{z+bw}{z}\Big)\Big(\frac{1}{1-ab}\delta(\frac{w}{az})+\frac{1}{1-\frac{1}{ab}}\delta\Big(\frac{bw}{z}\Big)\Big)\,{:}X_{ij}(a,z)X_{ji}(b,w){:}\\&=2\frac{1+ab}{1-ab}\delta(\frac{w}{az})E_-(\epsilon_i,z)E_-(-\epsilon_i,abz)E_+(\epsilon_i,z)E_+(-\epsilon_i,abz)\\&\quad+2\frac{ab+1}{ab-1}\delta\Big(\frac{bw}{z}\Big)E_-(\epsilon_j,w)E_-(-\epsilon_j,abw)E_+(\epsilon_j,w)E_+(-\epsilon_j,abw)\\&=2X_{ii}(ab,z)\delta\Big(\frac{w}{az}\Big)-2X_{jj}(ab,w)\delta\Big(\frac{bw}{z}\Big)+2\frac{1+ab}{1-ab}\Big(\delta\Big(\frac{w}{az}\Big)-\delta\Big(\frac{bw}{z}\Big)\Big).\end{aligned}$$

Finally, we prove statement (iv) of the proposition. In this case we have, from (2.3),

$$\begin{aligned}\Delta_{ji}^{ij}&=\Big(\frac{z+bw}{z}\Big)^2\Big[\Big(\frac{1}{1-\frac{bw}{z}}\Big)^2-\Big(\frac{\frac{z}{bw}}{1-\frac{z}{bw}}\Big)^2\Big]\\&=\Big(\frac{z+bw}{z}\Big)^2\Big(\frac{bw}{z}\Big)^{-1}(D\delta)\Big(\frac{bw}{z}\Big).\end{aligned}$$

Thus, from this and Lemma 2.10, we obtain

$$\begin{aligned}&\big[X_{ij}(a,z),X_{ji}(b,w)\big]\\&\quad=\Big(\frac{z+bw}{z}\Big)^2\Big(\frac{bw}{z}\Big)^{-1}(D\delta)\Big(\frac{bw}{z}\Big)\,{:}X_{ij}(a,z)X_{ji}(b,w){:}\\&\quad=4(D\delta)\Big(\frac{bw}{z}\Big)+4\delta\Big(\frac{bw}{z}\Big)\big(2\epsilon_i(z)-2\epsilon_j(az)\big)\\&\quad=2\big(X_{ii}(1,z)-X_{jj}(1,w)\big)\,\delta\Big(\frac{bw}{z}\Big)+4(D\delta)\Big(\frac{bw}{z}\Big).\end{aligned}$$

This completes the proof of the proposition. ∎

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Now we consider Case 2 for $[X_{ij}(a,z), X_{kl}(b,w)]$, with $i = j, k \neq l$. The result will be divided into three subcases.

Proposition 2.12 *Let $1 \leq i,k,j \leq \nu$, and $a,b \in \mathbb{C}^\times$.*

(i) *If i,k,j are distinct, then $[X_{ii}(a,z), X_{kj}(b,w)] = 0$.*

(ii) *If $i \neq j$, and $a = 1$, then*

$$\big[X_{ii}(a,z), X_{ij}(b,w)\big] = 2X_{ij}(b,w)\Big(\delta\Big(\frac{w}{z}\Big) - \delta\Big(-\frac{w}{z}\Big)\Big).$$

(iii) *If $i \neq j$, and $a \neq 1$, then*

$$\big[X_{ii}(a,z), X_{ij}(b,w)\big] = 2X_{ij}(ab,z)\delta\Big(\frac{w}{az}\Big) + 2X_{ji}(ab^{-1},bz)\delta\Big(-\frac{w}{z}\Big).$$

Proof Statement (i) of the proposition is clear. To prove (ii), we have

$$\begin{aligned}
&\big[X_{ii}(a,z), X_{ij}(b,w)\big] \\
&\quad = \big[4\epsilon_i(z), X_{ij}(b,w)\big] \\
&\quad = 4\big[\epsilon_i(z), \varepsilon(\epsilon_i,\epsilon_j)e^{\overline{\epsilon_i-\epsilon_j}}E_-(\epsilon_i,w)E_-(-\epsilon_j,bw)E_+(\epsilon_i,w)E_+(-\epsilon_j,bw)\big] \\
&\quad = 4\varepsilon(\epsilon_i,\epsilon_j)e^{\overline{\epsilon_i-\epsilon_j}}E_-(\epsilon_i,w)E_-(-\epsilon_j,bw)E_+(\epsilon_i,w)E_+(-\epsilon_j,bw)\sum_{n\in 2\mathbb{Z}+1}\Big(\frac{z}{w}\Big)^n \\
&\quad = 2X_{ij}(b,w)\Big(\delta\Big(\frac{w}{z}\Big) - \delta\Big(-\frac{w}{z}\Big)\Big),
\end{aligned}$$

where we have used Lemma 2.3 in the second last identity. We now prove (iii). By (2.3), Proposition 2.8, and Lemma 2.10, we have

$$\begin{aligned}
&[X_{ii}(a,z), X_{ij}(b,w)] \\
&= \frac{1+a}{1-a}\varepsilon(\epsilon_i,\epsilon_j)\Delta_{ji}^{ii}{:}X_{ii}(a,z)X_{ij}(b,w){:} \\
&= \frac{1+a}{1-a}\varepsilon(\epsilon_i,\epsilon_j)\Big(1+\frac{w}{az}\Big)\Big(1-\frac{w}{z}\Big) \\
&\quad \cdot\Big[\frac{1}{1-\frac{w}{az}}\frac{1}{1+\frac{w}{z}} + \frac{\frac{z}{w}}{1+\frac{z}{w}}\frac{\frac{az}{w}}{1-\frac{az}{w}}\Big]{:}X_{ii}(a,z)X_{ij}(b,w){:} \\
&= \frac{1+a}{1-a}\varepsilon(\epsilon_i,\epsilon_j)\Big(1+\frac{w}{az}\Big)\Big(1-\frac{w}{z}\Big) \\
&\quad \cdot\Big[\frac{1}{1+a}\delta\Big(\frac{w}{az}\Big) + \frac{1}{1+\frac{1}{a}}\delta\Big(-\frac{w}{z}\Big)\Big]{:}X_{ii}(a,z)X_{ij}(b,w){:} \\
&= 2\varepsilon(\epsilon_i,\epsilon_j)e^{\overline{\epsilon_i-\epsilon_j}}E_-(\epsilon_i,z)E_-(-\epsilon_j,abz)E_+(\epsilon_i,z)E_+(-\epsilon_j,abz)\delta\Big(\frac{w}{az}\Big) \\
&\quad - 2\varepsilon(\epsilon_i,\epsilon_j)e^{\overline{\epsilon_i-\epsilon_j}}E_-(\epsilon_i,-az)E_-(-\epsilon_j,-bz)E_+(\epsilon_i,-az)E_+(-\epsilon_j,-bz)\delta\Big(-\frac{w}{z}\Big) \\
&= 2X_{ij}(ab,z)\delta\Big(\frac{w}{az}\Big) - 2X_{ij}(ba^{-1},-az)\delta\Big(-\frac{w}{z}\Big) \\
&= 2X_{ij}(ab,z)\delta\Big(\frac{w}{az}\Big) + 2X_{ji}(ab^{-1},bz)\delta\Big(-\frac{w}{z}\Big)
\end{aligned}$$

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Finally we consider Case 3, that is to compute $[X_{ij}(a,z), X_{kl}(b,w)]$ with $i = j = k = l$, and $a, b \in \mathbb{C}^\times$. From the definition of $X_{ii}(a,z), X_{ii}(b,w)$, we may assume that $a \neq -1, b \neq -1$.

Proposition 2.13 *Let $1 \leq i \leq \nu$ and $a, b \in \mathbb{C}^\times$.*

(i) *For $a = b = 1$, we have*

$$\big[X_{ii}(1,z), X_{ii}(1,w)\big] = 4\Big((D\delta)\Big(\frac{w}{z}\Big) - (D\delta)\Big(-\frac{w}{z}\Big)\Big).$$

(ii) *For $a = 1$, $b \neq 1$, then*

$$\begin{aligned}[X_{ii}(1,z), X_{ii}(b,w)] &= 2X_{ii}(b,w)\Big(\delta\Big(\frac{w}{z}\Big) - \delta\Big(-\frac{w}{z}\Big) - \delta\Big(\frac{bw}{z}\Big) + \delta\Big(-\frac{bw}{z}\Big)\Big)\\ &\quad + 2\frac{1+b}{1-b}\Big(\delta\Big(\frac{w}{z}\Big) - \delta\Big(-\frac{w}{z}\Big) - \delta\Big(\frac{bw}{z}\Big) + \delta\Big(-\frac{bw}{z}\Big)\Big).\end{aligned}$$

(iii) *If $a \neq 1$, $b \neq 1$, and $a \neq b, ab \neq 1$, then*

$$\begin{aligned}&\big[X_{ii}(a,z), X_{ii}(b,w)\big]\\ &\quad = 2\Big(X_{ii}(ab,z) + \frac{1+ab}{1-ab}\Big)\delta\Big(\frac{w}{az}\Big) - 2\Big(X_{ii}(ab,b^{-1}z) + \frac{1+ab}{1-ab}\Big)\delta\Big(\frac{bw}{z}\Big)\\ &\qquad + 2\Big(X_{ii}(ab^{-1},bz) + \frac{1+ab^{-1}}{1-ab^{-1}}\Big)\delta\Big(-\frac{w}{z}\Big)\\ &\qquad - 2\Big(X_{ii}(ab^{-1},z) + \frac{1+ab^{-1}}{1-ab^{-1}}\Big)\delta\Big(-\frac{bw}{az}\Big).\end{aligned}$$

(iv) *If $a \neq 1, b \neq 1$, and $ab = 1, a \neq b$, then*

$$\begin{aligned}&\big[X_{ii}(a,z), X_{ii}(a^{-1},w)\big]\\ &\quad = 2\Big(X_{ii}(1,z) - X_{ii}(1,az)\Big)\delta\Big(\frac{w}{az}\Big) + 4(D\delta)\Big(\frac{w}{az}\Big)\\ &\qquad + 2\Big(X_{ii}(a^2,a^{-1}z) + \frac{1+a^2}{1-a^2}\Big)\delta\Big(-\frac{w}{z}\Big) - 2\Big(X_{ii}(a^2,z) + \frac{1+a^2}{1-a^2}\Big)\delta\Big(-\frac{w}{a^2z}\Big).\end{aligned}$$

(v) *If $a \neq 1, b \neq 1$, and $a = b, ab \neq 1$, then*

$$\begin{aligned}&\big[X_{ii}(a,z), X_{ii}(a,w)\big]\\ &\quad = 2\Big(X_{ii}(a^2,z) + \frac{1+a^2}{1-a^2}\Big)\delta\Big(\frac{w}{az}\Big) - 2\Big(X_{ii}(a^2,a^{-1}z) + \frac{1+a^2}{1-a^2}\Big)\delta\Big(\frac{aw}{z}\Big)\\ &\qquad - 2\Big(X_{ii}(1,az) - X_{ii}(1,z)\Big)\delta\Big(-\frac{w}{z}\Big) - 4(D\delta)\Big(-\frac{w}{z}\Big).\end{aligned}$$

Proof Part (i) of the proposition is straightforward. To prove (ii), we have

$$\begin{aligned}&\big[X_{ii}(1,z), X_{ii}(b,w)\big] =\\ &\qquad 4\Big[\epsilon_i(z), \frac{1+b}{1-b}E_-(\epsilon_i,w)E_-(-\epsilon_i,bw)E_+(\epsilon_i,w)E_+(-\epsilon_i,bw)\Big].\end{aligned}$$

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Then by Lemma 2.3, we have

$$\begin{aligned}
&\big[X_{ii}(1,z),X_{ii}(b,w)\big]\\
&\quad=2\frac{1+b}{1-b}E_-(\epsilon_i,w)E_-(-\epsilon_i,bw)E_+(\epsilon_i,w)E_+(-\epsilon_i,bw)\\
&\qquad\cdot\Big(\delta\Big(\frac{w}{z}\Big)-\delta\Big(-\frac{w}{z}\Big)-\delta\Big(\frac{bw}{z}\Big)+\delta\Big(-\frac{bw}{z}\Big)\Big)\\
&\quad=2\Big(X_{ii}(b,w)+\frac{1+b}{1-b}\Big)\Big(\delta(\frac{w}{z})-\delta\Big(-\frac{w}{z}\Big)-\delta\Big(\frac{bw}{z}\Big)+\delta\Big(-\frac{bw}{z}\Big)\Big).
\end{aligned}$$

Next, we prove (iii). Note that in this case, one has $1\neq a^{-1}\neq b\neq a^{-1}b$. Now, from (2.3), Proposition 2.8, and Lemma 2.10, we have

$$\begin{aligned}
&\big[X_{ii}(a,z),X_{ii}(b,w)\big]\\
&=:X_{ii}(a,z)X_{ii}(b,w):\frac{1+a}{1-a}\frac{1+b}{1-b}\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)\Big(\frac{az-bw}{az}\Big)\Big(\frac{z+bw}{z}\Big)\\
&\quad\cdot\Big(\frac{1}{1-\frac{w}{az}}\frac{1}{1+\frac{w}{z}}\frac{1}{1+\frac{bw}{az}}\frac{1}{1-\frac{bw}{z}}-\frac{\frac{az}{w}}{1-\frac{az}{w}}\frac{\frac{z}{w}}{1+\frac{z}{w}}\frac{\frac{az}{bw}}{1+\frac{az}{bw}}\frac{\frac{z}{bw}}{1-\frac{z}{bw}}\Big)\\
&=:X_{ii}(a,z)X_{ii}(b,w):\frac{1+a}{1-a}\frac{1+b}{1-b}\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)\Big(\frac{az-bw}{az}\Big)\Big(\frac{z+bw}{z}\Big)\\
&\quad\cdot\Big[\frac{1}{(1-ba)(1+a)(1+b)}\delta\Big(\frac{w}{az}\Big)+\frac{1}{(1-a^{-1}b^{-1})(1+a^{-1})(1+b^{-1})}\delta\Big(\frac{bw}{z}\Big)\\
&\quad+\frac{1}{(1-ba^{-1})(1+b)(1+a^{-1})}\delta\Big(-\frac{w}{z}\Big)+\frac{1}{(1-b^{-1}a)(1+b^{-1})(1+a)}\delta(-\frac{bw}{az})\Big]\\
&=:X_{ii}(a,z)X_{ii}(b,w):\\
&\quad\cdot\Big[\frac{1+ab}{1-ab}\delta\Big(\Big(\frac{w}{az}\Big)-\delta\Big(\frac{bw}{z}\Big)\Big)+\frac{1+ab^{-1}}{1-ab^{-1}}\Big(\delta(-\frac{w}{z})-\delta\Big(-\frac{bw}{az}\Big)\Big)\Big]\\
&=2\Big(X_{ii}(ab,z)+\frac{1+ab}{1-ab}\Big)\delta(\frac{w}{az})-2\Big(X_{ii}(ab,b^{-1}z)+\frac{1+ab}{1-ab}\Big)\delta(\frac{bw}{z})\\
&\quad+2\Big(X_{ii}(ab^{-1},bz)+\frac{1+ab^{-1}}{1-ab^{-1}}\Big)\delta\Big(-\frac{w}{z}\Big)\\
&\quad-2\Big(X_{ii}(ab^{-1},z)+\frac{1+ab^{-1}}{1-ab^{-1}}\Big)\delta\Big(-\frac{bw}{az}\Big)
\end{aligned}$$

Finally, we will prove part (v) of the proposition. The proof of part (iv) is similar to that of this case, which we omit. From (2.3), Proposition 2.7, and Lemma 2.10, one has

$$\begin{aligned}
&\big[X_{ii}(a,z),X_{ii}(a,w)\big]\\
&\quad=:X_{ii}(a,z)X_{ii}(a,w):\Big(\frac{1+a}{1-a}\Big)^2\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)^2\Big(\frac{z+aw}{z}\Big)\\
&\qquad\cdot\Big[\frac{1}{1-\frac{w}{az}}\Big(\frac{1}{1+\frac{w}{z}}\Big)^2\frac{1}{1-\frac{aw}{z}}-\frac{\frac{az}{w}}{1-\frac{az}{w}}\Big(\frac{\frac{z}{w}}{1+\frac{z}{w}}\Big)^2\frac{\frac{z}{aw}}{1-\frac{z}{aw}}\Big]\\
&\quad=:X_{ii}(a,z)X_{ii}(a,w):\Big(\frac{1+a}{1-a}\Big)^2\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)^2\Big(\frac{z+aw}{z}\Big)
\end{aligned}$$

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$$\begin{aligned}
&\Big[\frac{1}{(1+a)^2(1-a^2)}\delta\Big(\frac{w}{az}\Big)+\frac{1}{(1+a^{-1})^2(1-a^{-2})}\delta\Big(\frac{aw}{z}\Big)\\
&\quad+\frac{1}{(1+a)(1+a^{-1})}\Big[(D\delta)\Big(-\frac{w}{z}\Big)+2\delta\Big(-\frac{w}{z}\Big)\Big]\Big]\\
&=:X_{ii}(a,z)X_{ii}(a,w):\frac{1+a^2}{1-a^2}\delta\Big(\Big(\frac{w}{az}\Big)-\delta\Big(\frac{aw}{z}\Big)\Big)\\
&\quad-2(\epsilon_i(az)-\epsilon_i(z))\delta\Big(-\frac{w}{z}\Big)-4(D\delta)\Big(-\frac{w}{z}\Big)\\
&\quad+:X_{ii}(a,z)X_{ii}(a,w):\frac{a}{(1-a)^2}\Big[2\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)^2\Big(\frac{z+aw}{z}\Big)\\
&\quad-\frac{w}{az}\Big(\frac{z-w}{z}\Big)^2\Big(\frac{z+aw}{z}\Big)-\frac{aw}{z}\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)^2\\
&\quad-2\Big(-\frac{w}{z}\Big)\Big(\frac{az+w}{az}\Big)\Big(\frac{z-w}{z}\Big)\Big(\frac{z+aw}{z}\Big)\Big]\delta\Big(-\frac{w}{z}\Big)\\
&=2\Big(X_{ii}(a^2,z)+\frac{1+a^2}{1-a^2}\Big)\delta\Big(\frac{w}{az}\Big)-2\Big(X_{ii}(a^2,a^{-1}z)+\frac{1+a^2}{1-a^2}\Big)\delta\Big(\frac{aw}{z}\Big)\\
&\quad-2\big(X_{ii}(1,az)-X_{ii}(1,z)\big)\delta\Big(-\frac{w}{z}\Big)-4(D\delta)\Big(-\frac{w}{z}\Big).
\end{aligned}$$

■

Remark 2.14 It is easy to see that

$$\frac{\delta(z)-a^2\delta(az)}{1-a}\to(D\delta)(z)+2\delta(z)\quad\text{as } a\to 1.$$

This gives the following identities

$$\frac{\delta(w/az)-a^2b^2\delta(bw/z)}{1-ab}\to(D\delta)(w/az)+2\delta(w/az)\quad\text{as } ab\to 1,$$

and

$$\frac{a^2\delta(-w/z)-b^2\delta(-bw/az)}{a-b}\to a(D\delta)(-w/z)+2a\delta(-w/z),\quad\text{as } b\to a.$$

In summary, from Propositions 2.11, 2.12, and 2.13, we can get the following commutator relations.

Theorem 2.15 *For $1\le i,j,k,l\le\nu$ and $a,b\in\mathbb{C}^\times$, one has*

$$\begin{aligned}
&\big[X_{ij}(a,z),X_{kl}(b,w)\big]\\
&\quad=2\delta_{jk}\Big(X_{il}(ab,z)+\delta_{il}(1-\delta_{ab,1})\frac{1+ab}{1-ab}\Big)\delta\Big(\frac{w}{az}\Big)\\
&\qquad-2\delta_{il}\Big(X_{kj}(ab,b^{-1}z)+\delta_{jk}(1-\delta_{ab,1})\frac{1+ab}{1-ab}\Big)\delta\Big(\frac{bw}{z}\Big)\\
&\qquad+2\delta_{ik}\Big(X_{lj}(ab^{-1},bz)+\delta_{jl}(1-\delta_{a,b})\frac{1+ab^{-1}}{1-ab^{-1}}\Big)\delta\Big(-\frac{w}{z}\Big)\\
&\qquad-2\delta_{jl}\Big(X_{ik}(ab^{-1},z)+\delta_{ik}(1-\delta_{a,b})\frac{1+ab^{-1}}{1-ab^{-1}}\Big)\delta\Big(-\frac{bw}{az}\Big)\\
&\qquad+4\delta_{il}\delta_{jk}\delta_{ab,1}(D\delta)\Big(\frac{w}{az}\Big)-4\delta_{ik}\delta_{jl}\delta_{a,b}(D\delta)\Big(-\frac{w}{z}\Big).
\end{aligned}$$

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3 Representations of Unitary Lie Algebras

In this section we begin by recalling the unitary Lie algebra $\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ associated with a skew Laurent polynomial ring $\mathcal{R}_\sigma$ and an anti-involution $\bar{\cdot}$, which was first introduced in [AF]. Using the commutator relation among the vertex operators developed in Theorem 2.15, we find that M turns to a representation for a non-trivial central extension of the unitary Lie algebra $\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$. Moreover, we determine the irreducible decomposition of M explicitly.

Let G be an abelian group with a character σ; that is, $\sigma\colon G \to \mathbb{C}^\times$ is a group homomorphism. Extend σ to be an automorphism of the group algebra $\mathcal{R}$ associated with G determined by $\sigma(e^\alpha) = \sigma(\alpha)e^\alpha, \alpha \in G$. Then we can form the skew Laurent polynomial ring $\mathcal{R}_\sigma = \mathcal{R}[t^{\pm 1}, \sigma]$ with basis $t^m e^\alpha, m \in \mathbb{Z}, \alpha \in G$ and multiplication $e^\alpha t^m = (\sigma(\alpha))^m t^m e^\alpha$. For simplicity, we denote $\sigma(\alpha)$ by $\tilde{\alpha}$ in the following.

Let $\bar{\cdot}$ be an anti-involution of $\mathcal{R}_\sigma$ defined by $\bar{t} = -t$, $\overline{e^\alpha} = e^{-\alpha}$. Then $\overline{t^m e^\alpha} = (-\tilde{\alpha})^{-m} t^m e^{-\alpha}$ for $\alpha \in G, m \in \mathbb{Z}$.

Define an operator on the $\nu \times \nu$ matrix algebra $M_\nu(\mathcal{R}_\sigma)$,

$$\begin{aligned} {}^*\colon M_\nu(\mathcal{R}_\sigma) &\to M_\nu(\mathcal{R}_\sigma), \\ X &\mapsto \overline{X}^t \end{aligned}$$

where $X \in M_\nu(\mathcal{R}_\sigma)$ and X^t is the transpose of the matrix X. Then $\theta(X) = -X^*$ induces an involution on the Lie algebra $gl_\nu(\mathcal{R}_\sigma) = M_\mu(\mathcal{R}_\sigma)^-$. The fixed point subalgebra of θ is called a unitary Lie algebra (see [AF, G2]):

$$\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot}) = \{X \in M_\nu(\mathcal{R}_\sigma) | X^* = -X\}.$$

For convenience, we set

$$e_{ij}(m, \alpha) = E_{ij} t^m e^\alpha - E_{ji} \overline{t^m e^\alpha},$$

where $1 \le i, j \le \nu, \alpha \in G, m \in \mathbb{Z}$. Then $e_{ij}(m, \alpha) = -(-\tilde{\alpha})^{-m} e_{ji}(m, -\alpha)$ and $\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ is spanned by $e_{ij}(m, \alpha)$ for all $1 \le i, j \le \nu, m \in \mathbb{Z}$ and $\alpha \in G$.

We define a 1-dimensional central extension of $\widehat{gl}_\nu(\mathcal{R}_\sigma) = gl_\nu(\mathcal{R}_\sigma) \oplus \mathbb{C}c$ of $gl_\nu(\mathcal{R}_\sigma)$ with Lie bracket (see [G3, G4])

$$\begin{aligned} (3.1) \quad [E_{ij} t^m e^\alpha, E_{kl} t^n e^\beta] &= \delta_{jk} \tilde{\alpha}^n E_{il} t^{m+n} e^{\alpha+\beta} \\ &\quad - \delta_{il} \tilde{\beta}^m E_{kj} t^{m+n} e^{\alpha+\beta} + m \delta_{il} \delta_{jk} \delta_{m+n,0} \delta_{\tilde{\alpha}\tilde{\beta},1} \tilde{\alpha}^n c, \end{aligned}$$

where $1 \le i, j, k, l \le \nu, m, n \in \mathbb{Z}$ and $\alpha, \beta \in G$.

Now we can get $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot}) = \mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot}) \oplus \mathbb{C}c$ from a central extension of $\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ with Lie bracket as (3.1). Set $e_{ij}(\alpha, z) = \sum_{n \in \mathbb{Z}} e_{ij}(n, \alpha) z^{-n}$, for $1 \le i, j \le \nu, \alpha \in G$. Then we have the following proposition.

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Proposition 3.1 *In* $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$,

$$\begin{aligned}
&\big[e_{ij}(\alpha,z), e_{kl}(\beta,w)\big]\\
&\quad= \delta_{jk}e_{il}(\alpha+\beta,z)\delta\Big(\frac{w}{\tilde{\alpha}z}\Big) - \delta_{il}e_{kj}(\alpha+\beta,\tilde{\beta}^{-1}z)\delta\Big(\frac{\tilde{\beta}w}{z}\Big)\\
&\qquad+ \delta_{ik}e_{lj}(\alpha-\beta,\tilde{\beta}z)\delta\Big(-\frac{w}{z}\Big) - \delta_{jl}e_{ik}(\alpha-\beta,z)\delta\Big(-\frac{\tilde{\beta}w}{\tilde{\alpha}z}\Big)\\
&\qquad+ 2\delta_{il}\delta_{jk}\delta_{\tilde{\alpha}\tilde{\beta},1}c(D\delta)\Big(\frac{w}{\tilde{\alpha}z}\Big) - 2\delta_{ik}\delta_{jl}\delta_{\tilde{\alpha},\tilde{\beta}}c(D\delta)\Big(-\frac{w}{z}\Big),
\end{aligned}$$

where $1 \le i, j, k, l \le \nu$, $\alpha, \beta \in G$.

Proof We have

$$\begin{aligned}
&\Big[\sum_{m\in\mathbb{Z}}(E_{ij}t^m e^\alpha - E_{ji}\overline{t^m e^\alpha})z^{-m}, \sum_{n\in\mathbb{Z}}(E_{kl}t^n e^\beta - E_{lk}\overline{t^n e^\beta})w^{-n}\Big]\\
&\quad= \sum_{m,n\in\mathbb{Z}}\delta_{jk}(E_{il}t^m e^\alpha t^n e^\beta - E_{li}\overline{t^n e^\beta}\,\overline{t^m e^\alpha})z^{-m}w^{-n}\\
&\qquad- \sum_{m,n\in\mathbb{Z}}\delta_{il}(E_{kj}t^n e^\beta t^m e^\alpha - E_{jk}\overline{t^m e^\alpha}\overline{t^n e^\beta})z^{-m}w^{-n}\\
&\qquad+ \sum_{m,n\in\mathbb{Z}}\delta_{ik}(E_{jl}\overline{t^n e^\beta}t^m e^\alpha - E_{lj}\overline{t^m e^\alpha}t^n e^\beta)z^{-m}w^{-n}\\
&\qquad- \sum_{m,n\in\mathbb{Z}}\delta_{jl}(E_{ik}t^m e^\alpha\overline{t^n e^\beta} - E_{ki}t^n e^\beta\overline{t^m e^\alpha})z^{-m}w^{-n}\\
&\qquad+ \sum_{m\in\mathbb{Z}}2m\delta_{jk}\delta_{il}\delta_{\tilde{\alpha}\tilde{\beta},1}(\tilde{\alpha})^{-m}cz^{-m}w^m - \sum_{m\in\mathbb{Z}}2m\delta_{ik}\delta_{jl}\delta_{\tilde{\alpha},\tilde{\beta}}c(-z)^{-m}w^m\\
&\quad= \sum_{m,n\in\mathbb{Z}}\delta_{jk}e_{il}(m+n,\alpha+\beta)z^{-m-n}(\tilde{\alpha}z)^n w^{-n}\\
&\qquad- \sum_{m,n\in\mathbb{Z}}\delta_{il}e_{kj}(m+n,\alpha+\beta)(\tilde{\beta}^{-1}z)^{-m-n}z^n(\tilde{\beta}w)^{-n}\\
&\qquad+ \sum_{m,n\in\mathbb{Z}}\delta_{ik}e_{lj}(m+n,\alpha-\beta)(\tilde{\beta}z)^{-m-n}z^n(-w)^{-n}\\
&\qquad- \sum_{m,n\in\mathbb{Z}}\delta_{jl}e_{ik}(m+n,\alpha-\beta)z^{-m-n}(\tilde{\alpha}z)^n(-\tilde{\beta}w)^{-n}\\
&\qquad+ 2\delta_{il}\delta_{jk}\delta_{\tilde{\alpha}\tilde{\beta},1}c(D\delta)\Big(\frac{w}{\tilde{\alpha}z}\Big) - 2\delta_{ik}\delta_{jl}\delta_{\tilde{\alpha},\tilde{\beta}}c(D\delta)\Big(-\frac{w}{z}\Big),
\end{aligned}$$

as required. ∎

Comparing Theorem 2.15 and Proposition 3.1, one can see that the following result holds true.

Theorem 3.2 *Let G be an abelian group with a character σ. Then M is a vertex operator representation for the unitary Lie algebra $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ with the action given by*

$$2e_{ij}(m,\alpha) \mapsto x_{ij}(\tilde{\alpha}, m) + \delta_{ij}(1-\delta_{\tilde{\alpha},1})\frac{1+\tilde{\alpha}}{1-\tilde{\alpha}},\ c \mapsto \frac{1}{2},$$

where $1 \le i, j \le \nu$, *and* $\alpha \in G$.

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Next, we are going to determine the irreducible decomposition of $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module M. Let $\mathbb{C}[\overline{Q}]$ be the subalgebra of $\mathbb{C}[\overline{\Gamma}]$ generated by $e^{\overline{\epsilon_j - \epsilon_{j+1}}}, 1 \leq j \leq \nu - 1$ that has dimension $2^{\nu-1}$. Using the standard techniques developed in the vertex representation of affine Lie algebras, it is easy to see that if U is an irreducible $\mathbb{C}[\overline{Q}]$-submodule of $\mathbb{C}[\overline{\Gamma}]$, then $U \otimes \mathcal{S}(\mathcal{H}^-)$ remains irreducible as a $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module. Therefore, we need to determine the irreducible components of $\mathbb{C}[\overline{\Gamma}]$ for detail. As a by-product, we also give the irreducible decomposition of $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module M.

We observe that $e^{\bar{\epsilon}_j} e^{\bar{\epsilon}_k} + e^{\bar{\epsilon}_k} e^{\bar{\epsilon}_j} = 2\delta_{jk}, 1 \leq j, k \leq \nu$. So, consider the Clifford algebra $Cl(\nu)$ with generators $\psi_j, 1 \leq j \leq \nu$ and relation

$$\psi_j \psi_k + \psi_k \psi_j = 2\delta_{jk}, \ 1 \leq j, k \leq \nu.$$

Since $\dim(\mathbb{C}[\overline{\Gamma}]) = \dim(Cl(\nu)) = 2^\nu$, $\mathbb{C}[\overline{\Gamma}]$ is isomorphic to $Cl(\nu)$ as an associative algebra. Then $\mathbb{C}[\overline{Q}]$ is isomorphic to the subspace of $Cl(\nu)$ spanned by products of an even number of elements $\psi_j, j = 1, \ldots, \nu$.

By the classical representation theory for Clifford algebras, we see that $\mathbb{C}[\overline{Q}]$ is semisimple. Moreover, if $\nu = 2d + 1$, then $\mathbb{C}[\overline{Q}]$ has a unique simple module S, which has dimension 2^d. If $\nu = 2d$, then $\mathbb{C}[\overline{Q}]$ has exactly two simple modules $S^\pm$, which have dimension 2^{d-1}.

Remark 3.3 It is well known that the subspace $Cl_2(\nu)$ of $Cl(\nu)$ spanned by products of 2 different element $\psi_i, i = 1, \ldots, \nu$ is closed under the Lie bracket and is isomorphic to the simple orthogonal Lie algebra $\mathfrak{o}(\nu)$. Fix bases in the root systems of types D_d and B_d as follows:

$$\Pi(D_d) = \{h_1 - h_2, \ldots, h_{d-1} - h_d, h_{d-1} - h_d\},$$
$$\Pi(B_d) = \{h_1 - h_2, \ldots, h_{d-1} - h_d, h_d\}.$$

Then S is the spinor representation of $\mathfrak{o}(2d+1)$ with highest weight $\frac{1}{2}(h_1 + \cdots + h_d)$. And $S^\pm$ are the semispinor representation of $\mathfrak{o}(2d)$ with highest weight $\frac{1}{2}(h_1 + \cdots + h_{d-1} \pm h_d)$.

For $\rho = \pm 1$, let $w_{2j-1}(\rho) = 1 + \sqrt{-1}\rho e^{\bar{\epsilon}_{2j-1}}$ and $w_{2j}(\rho) = 1 + \rho e^{\bar{\epsilon}_{2j}}$. For $\rho_1, \ldots, \rho_\nu \in \{\pm 1\}$, we put

$$w(\rho_1, \ldots, \rho_\nu) := \prod_{j=1}^{\nu} w_j(\rho_j).$$

Then the collection of these elements $\{w(\rho_1, \ldots, \rho_\nu) : \rho_1, \ldots, \rho_\nu = \pm 1\}$ form a basis of $\mathbb{C}[\overline{\Gamma}]$. The construction of such basis elements are motivated by the work [W]. It is easy to check the following lemma (or see [W, Lemma 4.1]).

Lemma 3.4 *For $\rho_1, \ldots, \rho_\nu \in \{\pm 1\}$ and $1 \leq j \leq \nu$, we have*

$$e^{\bar{\epsilon}_j} w(\rho_1, \ldots, \rho_\nu) = \begin{cases} \sqrt{-1}\rho_j w(-\rho_1, \ldots, -\rho_j, \rho_{j+1}, \ldots, \rho_\nu) & \text{if } j \text{ is odd}, \\ \rho_j w(-\rho_1, \ldots, -\rho_{j-1}, \rho_j, \ldots, \rho_\nu) & \text{if } j \text{ is even}. \end{cases}$$

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Then for $1 \leq j \leq [\frac{\nu}{2}]$, we have

$$\begin{aligned} e^{\overline{\epsilon_{2j-1}-\epsilon_{2j}}} w(\rho_1,\dots,\rho_\nu) &= -\sqrt{-1}\rho_{2j-1}\rho_{2j} w(\rho_1,\dots,\rho_\nu), \\ e^{\overline{\epsilon_{2j}-\epsilon_{2j+1}}} w(\rho_1,\dots,\rho_\nu) &= \\ &\quad -\sqrt{-1}\rho_{2j}\rho_{2j+1} w(\rho_1,\dots,\rho_{2j-1},-\rho_{2j},-\rho_{2j+1},\rho_{2j+2},\dots,\rho_\nu). \end{aligned} \tag{3.2}$$

From (3.2), we see that the signs $\rho_1, \rho_{2j}\rho_{2j+1}, 1 \leq j \leq [\frac{\nu}{2}]$ and ρ_ν if ν is even are invariant under the action of $\mathbb{C}[\overline{Q}]$.

In the case $\nu = 2d+1$, for any tuple $\gamma = (\gamma_1,\dots,\gamma_{d+1}) \in \mathbb{Z}_2^{d+1}, \mathbb{Z}_2 = \{\pm 1\}$, set

$$V_\nu(\gamma) = \mathrm{span}_{\mathbb{C}}\{w(\rho_1,\dots,\rho_\nu) | \rho_1 = \gamma_1, \rho_{2j}\rho_{2j+1} = \gamma_{j+1}, 1 \leq j \leq d\}. \tag{3.3}$$

Then $V_\nu(\gamma)$ is a $\mathbb{C}[\overline{Q}]$-module with dimension 2^d, hence is simple. So, we have shown that

$$\mathbb{C}[\overline{\Gamma}] = \bigoplus_{\gamma\in\mathbb{Z}_2^{d+1}} V_\nu(\gamma),$$

where each $V_\nu(\gamma)$ is an irreducible $\mathbb{C}[\overline{Q}]$-module.

In the case $\nu = 2d$, for any tuple $\gamma = (\gamma_1,\dots,\gamma_{d+1}) \in \mathbb{Z}_2^{d+1}$, set

$$\begin{aligned} V_\nu(\gamma) = \mathrm{span}_{\mathbb{C}}\{&w(\rho_1,\dots,\rho_\nu) | \rho_1 \\ &= \gamma_1, \rho_{2j}\rho_{2j+1} = \gamma_{j+1}, 1 \leq j \leq d-1, \rho_{2d} = \gamma_{d+1}\}. \end{aligned} \tag{3.4}$$

Then $V_\nu(\gamma)$ is an irreducible $\mathbb{C}[\overline{Q}]$-module with

$$\mathbb{C}[\overline{\Gamma}] = \bigoplus_{\gamma\in\mathbb{Z}_2^{d+1}} V_\nu(\gamma).$$

In summary, we have obtained the following theorem.

Theorem 3.5 *The $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module M is completely reducible. Moreover,*

$$M = \bigoplus_{\gamma\in\mathbb{Z}_2^{[\frac{\nu}{2}]+1}} V_\nu(\gamma) \otimes \mathcal{S}(\mathcal{H}^-),$$

where each submodule $V_\nu(\gamma)\otimes\mathcal{S}(\mathcal{H}^-)$ is irreducible as a $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module, and $V_\nu(\gamma)$ are defined in (3.3) and (3.4).

4 Application to Elementary Unitary Lie Algebras

Let G be an abelian group with a character σ and $|\sigma|$ be the order of $\sigma(G)$. Define the set $\Lambda_\sigma = \{(m,\alpha) \in (\mathbb{Z}, G) | m \in |\sigma|\mathbb{Z}, \alpha \in \ker(\sigma)\}$. Here, we set $|\sigma|\mathbb{Z} = 0$ if $|\sigma| = \infty$. Then we have the following basic result about $\mathcal{R}_\sigma$, whose proof is straightforward.

Lemma 4.1 *The derived algebra $\mathcal{R}'_\sigma$ of $\mathcal{R}_\sigma$ has a basis consisting of monomials $t^m e^\alpha$ for $(m,\alpha) \notin \Lambda_\sigma$, and the center Z_σ of $\mathcal{R}_\sigma$ has basis consisting of monomials $t^m e^\alpha$ for $(m,\alpha) \in \Lambda_\sigma$. Therefore, we have $\mathcal{R}_\sigma = \mathcal{R}'_\sigma \oplus Z_\sigma$.*

The elementary unitary Lie algebra $\mathfrak{eu}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ is the derived algebra of $\mathfrak{u}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$, which is generated by $e_{ij}(m,\alpha)$ for $1 \leq i \neq j \leq \nu, m \in \mathbb{Z}$ and $\alpha \in G$. By applying

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Lemma 4.1, one can see that

$$\mathfrak{u}_\nu(\mathcal{R}_\sigma,\bar{\cdot}) = \mathfrak{eu}_\nu(\mathcal{R}_\sigma,\bar{\cdot}) \oplus \sum_{(n,\beta)\in\Lambda_\sigma} \mathbb{C}\Big(\sum_{i=1}^{\nu} e_{ii}(n,\beta)\Big).$$

In particular, we have that $e_{ii}(m,\alpha) - e_{jj}(m,\alpha), e_{ii}(n,\beta) \in \mathfrak{eu}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ for $1 \le i \ne j \le \nu, m,n \in \mathbb{Z}, \alpha,\beta \in G$, and $(n,\beta) \notin \Lambda_\sigma$. Let $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ be the subalgebra of $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ generated by $e_{ij}(m,\alpha)$ for $1 \le i \ne j \le \nu, m \in \mathbb{Z}, \alpha \in G$. Then $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ is a one-dimensional nontrivial central extension of $\mathfrak{eu}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$.

Remark 4.2 It is known that both $\mathfrak{eu}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ and $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ are BC_d-graded with grading subalgebra $\mathfrak{o}(\nu)$ (see [ABG] for details), where $d = [\frac{\nu}{2}]$.

Next, we consider the unitarizability of M. For our purpose, we may assume that all the elements in $\sigma(G)$ lie in the unity circle in the complex plane; that is, $|a| = 1,\ \forall a \in \sigma(G)$. Define a conjugate linear map $\tau\colon \widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma,\bar{\cdot}) \to \widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$ by letting

$$\tau(e_{ij}(m,\alpha)) = (-1)^{(m+1)} e_{ij}(-m,\alpha),\ \tau(c) = c,$$

where $1 \le i,j \le \nu, m \in \mathbb{Z}$ and $\alpha \in G$. Due to $|\tilde{\alpha}| = 1, \alpha \in G$, one can get that τ is a conjugate anti-involution of $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$.

Define a Hermitian form $\langle\cdot \mid \cdot\rangle$ on $\mathcal{H}^-$ by

$$\langle\alpha(m) \mid \beta(n)\rangle = \frac{m}{2}(\alpha,\beta)\delta_{m,n},\ \forall\, \alpha(m),\beta(n) \in \mathcal{H}^-,$$

and extend this form to $S(\mathcal{H}^-)$ as usual:

$$\langle x_1 \dots x_m \mid y_1 \dots y_n\rangle = \delta_{m,n} \sum_{\sigma\in S_m} \prod_{i=1}^{m} \langle x_i \mid y_{\sigma(i)}\rangle.$$

Next, we define $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}[\overline{\Gamma}]$ by

$$\langle e^{\overline{\alpha}} \mid e^{\overline{\beta}}\rangle = \delta_{\overline{\alpha},\overline{\beta}},\ \forall\, \alpha,\beta \in \Gamma.$$

This gives a positive definite Hermitian form on M (see also [FK, (2.12)–(2.14)] such that

$$\langle \epsilon_i(m).w \mid v\rangle = \langle w \mid \epsilon_i(-m).v\rangle, \quad \langle e^{\bar{\epsilon}_i}.w \mid v\rangle = \langle w | e^{\bar{\epsilon}_i}.v\rangle,$$

for $1 \le i \le \nu, m \in 2\mathbb{Z}+1, w,v \in M$.

Proposition 4.3 *Suppose that all the elements in $\sigma(G)$ lie in the unity circle in the complex plane, then the $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma,\bar{\cdot})$-module M is unitary with respect to τ.*

Proof It is sufficient to show that

$$\langle X_{ij}(a,z).w \mid v\rangle = \langle w \mid -X_{ij}(a,-z^{-1}).v\rangle,$$

for $1 \le i,j \le \nu, a \in \sigma(G), w,v \in M$.

By applying the fact that $\bar{a} = a^{-1}$, one can get that

$$\Big\langle \frac{\epsilon_i(m)}{m}(az)^{-m}.w \,\Big|\, v\Big\rangle = \Big\langle w \,\Big|\, \frac{-\epsilon_i(-m)}{-m}(az^{-1})^m.v\Big\rangle,$$

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and that

$$\left\langle E_{\pm}(\epsilon_i, az)\,.\,w \mid v\right\rangle = \left\langle w \mid E_{\mp}(-\epsilon_i, az^{-1})\,.\,v\right\rangle = \left\langle w \mid E_{\mp}(\epsilon_i, -az^{-1})\,.\,v\right\rangle$$

for $1 \le i \le \nu, a \in \sigma(G), w, v \in M$.

Hence, for $i \ne j$, we find that

$$\begin{aligned}&\langle X_{ij}(a,z)\,.\,w \mid v\rangle\\ &\quad= \left\langle w \mid E_-(-\epsilon_j, -az^{-1})E_-(\epsilon_i, -z^{-1})E_+(-\epsilon_j, -az^{-1})E_+(\epsilon_i, -az^{-1})e^{\overline{\epsilon_j}}e^{\overline{\epsilon_i}}\,.\,v\right\rangle\\ &\quad= \left\langle w \mid -X_{ij}(a, -z^{-1})\,.\,v\right\rangle.\end{aligned}$$

The case for $i = j, a = 1$ is clear. For the case $i = j, a \ne 1$, one only needs to note that $\overline{(\frac{1+a}{1-a})} = -\frac{1+a}{1-a}$. ∎

By taking restriction, we know that each component $V_\nu(\gamma) \otimes S(\mathcal{H}^-), \gamma \in \mathbb{Z}_2^{[\frac{\nu}{2}]+1}$ of M is an $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module.

Theorem 4.4 *Let $V_\nu(\gamma) \otimes S(\mathcal{H}^-)$ be the $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module described in Theorem 3.5.*

(i) *If $|\sigma| = \infty$ or $|\sigma| \in 2\mathbb{N}$, then $V_\nu(\gamma) \otimes S(\mathcal{H}^-)$ is irreducible as a $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module.*

(ii) *If $|\sigma| = N \in 2\mathbb{N}+1$, let $\mathcal{H}_0^-$ be the subspace of $\mathcal{H}^-$ spanned by $\epsilon_i(m) - \epsilon_j(m), \epsilon_i(n)$ for $1 \le i \ne j \le \nu, m, n \in -(2\mathbb{N}+1), n \notin N\mathbb{Z}$. Then $V_\nu(\gamma) \otimes S(\mathcal{H}_0^-)$ is an irreducible $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-submodule of $V_\nu(\gamma) \otimes S(\mathcal{H}^-)$. Moreover, $V_\nu(\gamma) \otimes S(\mathcal{H}^-)$ is completely reducible as an $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module.*

Proof First, we write

$$L = \widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot}), \quad V = V_\nu(\gamma) \otimes S(\mathcal{H}^-), \quad V_0 = V_\nu(\gamma) \otimes S(\mathcal{H}_0^-),$$

for simplicity.

If $|\sigma| = \infty$ or $|\sigma| \in 2\mathbb{N}$, then $e_{ii}(m,0) \in L, 1 \le i \le \nu, m \in 2\mathbb{Z}+1$, and hence V remains irreducible as L-module.

Next, if $|\sigma| = N \in 2\mathbb{N}+1$, we observe that

$$E_-(\epsilon_i, z)E_-(-\epsilon_j, az) \in \mathrm{End}(V_0)[[z]],$$

for $1 \le i, j \le \nu$ and $a \in \sigma(G)$. From the definition of vertex operators and the fact that $e_{ii}(m,0) \notin L, m \in 2\mathbb{Z}+1 \cap N\mathbb{Z}$, we have obtained that the subspace V_0 is invariant under the action of L. For the usual reason, one can easily see that V_0 is irreducible as L-module.

Finally, we will prove that if $|\sigma| \in 2\mathbb{N}+1$, then $V_\nu(\gamma) \otimes S(\mathcal{H}^-)$ is completely reducible as an $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-module. Let d be the degree derivation on $\mathcal{R}_\sigma$ such that $d\,.\,t^m e^\alpha = mt^m e^\alpha$ for $m \in \mathbb{Z}, \alpha \in G$. Let $\widetilde{L}$ be the semi-direct product of the Lie algebra L and the derivation d. Extend τ to a conjugate anti-involution of $\widetilde{L}$ by setting $\tau(d) = d$. Let d_0 be the degree operator on $S(\mathcal{H}^-)$ determined by $d_0.\epsilon_i(m) = m\epsilon_i(m)$ for $1 \le i \le \nu, m \in -(2\mathbb{N}+1)$. Then, by letting the action of d as $1 \otimes d_0$, V can be extended to an $\widetilde{L}$-module and is unitary with respect to τ. One has a natural $\mathbb{Z}$-graded

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structure on $\widetilde{L}$ and V with respect to the action of d:

$$\tilde{L} = \bigoplus_{n\in\mathbb{Z}} \tilde{L}_n, \quad V = \bigoplus_{m\in-\mathbb{N}} V(m).$$

For any submodule V' of V, write $V'(m) = V(m) \cap V'$ for $m \in -\mathbb{N}$ so that $V' = \bigoplus_m V'(m)$. Note that each graded subspace $V(m)$ is finite dimensional. Therefore, each $\tilde{L}$-submodule of V has an orthogonal compliment being unitary (in this case, each $a \in \sigma(G)$ is a root of unity so that V is unitary by Proposition 4.3).

Let W be the sum of all irreducible $\tilde{L}$-submodules (which contain V_0) and let W' be its orthogonal compliment. Let $W' = \bigoplus_{m\le n'} W'(m)$ with $W'(n') \neq 0$ for some $n' \in -\mathbb{N}$. Let U be the $\tilde{L}_0$-module generated by $W'(n')$, then $U \subseteq V(n')$ and is finite dimensional. Notice that $\tau(\tilde{L}_0) = \tilde{L}_0$, then U is a finite dimensional unitary $\tilde{L}_0$-module and hence completely reducible. Let U_0 be an irreducible component of $\tilde{L}_0$-module U and consider the $\tilde{L}$-module U' generated by U_0. Now, U' is an irreducible $\tilde{L}$-submodule of W' as $U'(n') = U_0$ cannot split. But W is supposed to contain all the irreducible $\tilde{L}$-submodules of V, a contradiction. Therefore, $V = W$ and V is completely reducible as $\tilde{L}$-module (also as L-module). ∎

In the rest of this paper, we will pay attention to the trivial case where $G = \{1\}$. In this case, the elementary unitary Lie algebra $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ is isomorphic to the twisted affine Kac–Moody algebra of type $A^{(2)}_{\nu-1}$. Let $\Lambda_j, 0 \le j \le d, d = [\frac{\nu}{2}]$ be the fundamental integral weight of $A^{(2)}_{\nu-1}$. Following from Theorem 4.4 and the techniques developed by Wakimoto (See [W, Theorem 4.2]) , we have the following corollary.

Corollary 4.5 *M is a completely reducible representation for the affine Kac–Moody algebra of type $A^{(2)}_{\nu-1}$ with the action given in Theorem 3.2. If $\nu = 2d$, then $V_{2d}(\gamma) \otimes S(\mathcal{H}_0^-), \gamma \in \mathbb{Z}_2^{d+1}$ are the irreducible $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-modules with highest weight Λ_{d-1} if $\gamma_1 \dots \gamma_{d+1} = -1$ or Λ_d if $\gamma_1 \dots \gamma_{d+1} = 1$. If $\nu = 2d+1$, then $V_{2d+1}(\gamma) \otimes S(\mathcal{H}_0^-), \gamma \in \mathbb{Z}_2^{d+1}$ are the irreducible $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$-modules with highest weight Λ_d.*

Remark 4.6 The vertex operator representations $\mathbb{C}[\overline{Q}] \otimes S(\mathcal{H}_0^-)$ for affine Kac–Moody algebra of type $A^{(2)}_{\nu-1}$ have been studied in [W], where the author used a different two cocycle $\varepsilon(\,\cdot\,,\,\cdot\,)\colon Q \to Q$ from ours. Explicitly, let $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1}, 1 \le i \le \nu-1\}$ be the simple root system of type $A_{\nu-1}$. Then the two-cocycle $\varepsilon(\,\cdot\,,\,\cdot\,)$ is determined by $\varepsilon(\alpha_i, \alpha_j)(1 \le i, j \le \nu-1)$ and its bi-multiplicative property. The Dynkin diagram of Π with orientation corresponds to ε as follows:

$$\varepsilon(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{if } \overset{\alpha_i}{\bigcirc} \longrightarrow \overset{\alpha_j}{\bigcirc} \text{ or } \alpha_i \text{ is not connected with } \alpha_j, \\ -1 & \text{if } i = j \text{ or } \overset{\alpha_i}{\bigcirc} \longleftarrow \overset{\alpha_j}{\bigcirc}. \end{cases}$$

The orientation of the Dynkin diagram used in [W] was

$$\nu = 2d: \quad \overset{\alpha_1}{\bigcirc} \longrightarrow \overset{\alpha_2}{\bigcirc} \longleftarrow \overset{\alpha_3}{\bigcirc} \longrightarrow \overset{\alpha_4}{\bigcirc} \longleftarrow \overset{\alpha_5}{\bigcirc} \quad \dots \quad \overset{\alpha_{2d-3}}{\bigcirc} \longrightarrow \overset{\alpha_{2d-2}}{\bigcirc} \longleftarrow \overset{\alpha_{2d-1}}{\bigcirc}$$

$$\nu = 2d+1: \quad \overset{\alpha_1}{\bigcirc} \longrightarrow \overset{\alpha_2}{\bigcirc} \longleftarrow \overset{\alpha_3}{\bigcirc} \longrightarrow \overset{\alpha_4}{\bigcirc} \longleftarrow \overset{\alpha_5}{\bigcirc} \quad \dots \quad \overset{\alpha_{2d-2}}{\bigcirc} \longleftarrow \overset{\alpha_{2d-1}}{\bigcirc} \longrightarrow \overset{\alpha_{2d}}{\bigcirc}$$

while the orientation used in this paper is as follows:

$$\overset{\alpha_1}{\bigcirc} \longrightarrow \overset{\alpha_2}{\bigcirc} \longrightarrow \overset{\alpha_3}{\bigcirc} \longrightarrow \overset{\alpha_4}{\bigcirc} \longrightarrow \overset{\alpha_5}{\bigcirc} \quad \dots \quad \overset{\alpha_{\nu-3}}{\bigcirc} \longrightarrow \overset{\alpha_{\nu-2}}{\bigcirc} \longrightarrow \overset{\alpha_{\nu-1}}{\bigcirc}$$

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Remark 4.7 Let $G = \mathbb{Z}$ and consider the subalgebra $\mathcal{G}$ of $\widehat{\mathfrak{eu}}_\nu(\mathcal{R}_\sigma, \bar{\cdot})$ generated by the elements $e_{ij}(0, m), 1 \leq i \neq j \leq \nu, m \in \mathbb{Z}$. Then $\mathcal{G}$ is a fixed point subalgebra of affine Kac–Moody algebra $A^{(1)}_{\nu-1}$. But, $\mathcal{G}$ is not isomorphic to the (untwisted or twisted) affine Kac–Moody algebra. The representation theory of $\mathcal{G}$ is little known so far. It is interesting to see that $\tau(x) = -x$ for all $x \in \mathcal{G}$ if $|\sigma(1)| = 1$.

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