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THE FIFTH UNITARY PERFECT NUMBER

BY
CHARLES R. WALL

1. **Introduction.** A divisor d of a positive integer n is a *unitary divisor* if d and n/d are relatively prime. An integer is said to be *unitary perfect* if it equals the sum of its proper unitary divisors. Subbarao and Warren [2] gave the first four unitary perfect numbers: 6, 60, 90 and 87360. In 1969, I reported [3] that

$$146\ 361\ 946\ 186\ 458\ 562\ 560\ 000$$
$$= 2^{18}3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

is also unitary perfect. The purpose of this paper is to show that this last number, which for brevity we denote by W, is indeed the next unitary perfect number after 87360.

If d is a unitary divisor of n, we write $d \| n$; note that this notation is consistent with the standard notation for exact division by prime powers. Let $\sigma^*(n)$ be the sum of all unitary divisors of n:

$$\sigma^*(n) = \sum_{d\|n} d.$$

It is easy to show that σ^* is a multiplicative function, and in fact

$$\sigma^*(p^a q^b \cdots) = (1+p^a)(1+q^b)\cdots,$$

where $p, q, \ldots$ are distinct primes and the exponents are positive. We remark that $\sigma^*(n)$ is odd only for $n=1$ and n any power of 2.

If p and q are distinct primes, and $p \mid n$ but $q \nmid n$, then

$$\sigma^*(pn)/pn < \sigma^*(n)/n < \sigma^*(qn)/qn. \tag{1}$$

Thus the value $\sigma^*(n)/n$ decreases as the primes dividing n are repeated, so if we wish to maximize $\sigma^*(n)/n$ and at the same time minimize n, we must take n squarefree.

The requirement that N be unitary perfect is clearly equivalent to $\sigma^*(N)=2N$. Thus the search for unitary perfect numbers is the search for solutions to the Diophantine equation

$$2 = \frac{x+1}{x} \cdot \frac{y+1}{y} \cdots, \tag{2}$$

with the restriction that $x, y \ldots$ are powers of distinct primes. If N is unitary perfect and $N=2^A k$ with k odd, then as a consequence of (2), the number of distinct prime divisors of k is no more than $A+1$.

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For the remainder of this paper we assume that N is unitary perfect, that $N \leq W$, and that $2^A \| N$.

2. **Elimination methods.** Subbarao [1] has reported the impossibility of having A be 0, 3, 4, 5, 7, 8, 9 or 10, and that

if $A=1$, then $N=6$ or 90;
if $A=2$, then $N=60$;
if $A=6$, then $N=87360$.

Thus we may restrict our attention here to $A \geq 11$. Since $\sigma^*(2^A)=1+2^A$, we may write $N=2^A(1+2^A)d$ with $d \geq 1$. Then $N \leq W$ requires $A \leq 38$, since $W<(3/2)10^{23}$.

The simplest way to eliminate a case is to show:

(3) N has enough known (or assumed) divisors to require that $N>W$.

Our basic procedure is to start with a given value for A; then $\sigma^*(2^A)=1+2^A$ provides us with some known divisors of N. We then sort the known (or assumed) divisors into two categories: known (or assumed) unitary divisors, and other known (or assumed) divisors. We let p be some prime, usually the largest, in the latter category; then use of (3) allows us to obtain an upper bound on how many times p can divide N. Once we have this bound we may consider cases in which $p^e \| N$; then $\sigma^*(p^e)=1+p^e$ in general provides us with other known odd divisors of N, and we repeat the procedure.

We write

$$N = 2^A 3^B 5^C s,$$

with $(s, 30)=1$. If $A \geq 11$, $B \geq 3$ and $C \geq 3$, then

$$2 = \sigma^*(N)/N \leq (2049/2048)(28/27)(126/125)\sigma^*(s)/s,$$

so that $\sigma^*(s)/s>1.91$. If s is the product of the primes from 7 through 59, inclusive, then $\sigma^*(s)/s<1.90$. Thus by the remarks following (1), s can be no smaller than the product of the primes from 7 through 61, but this would imply

$$N \geq 2^{11}3^3 5^3 s > 10^{28}.$$

Since $28/27>126/125>1$, the same lower bound for $\sigma^*(s)/s$ also holds if $B=C=0$, if $B=0$ and $C \geq 3$, or if $B \geq 3$ and $C=0$. However, each of these conditions implies that $N>10^{24}$. Therefore:

(4) For $A \geq 11$ we have $(N, 15)>1$, and if $3^3 \mid N$ then either $5 \| N$ or $5^2 \| N$, and if $5^3 \mid N$ then either $3 \| N$ or $3^2 \| N$.

For brevity we let $f(N)=\sigma^*(N)/N$.

If $p^e \| N$ then we have a contradiction if $f(N)$ has more then e factors p in its numerator. In the table in the next section we refer to this occurrence as "excess p's."

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(5) The known unitary divisors of N determine other divisors of N because of odd primes appearing in the numerator of $f(N)$. Let m be the largest known odd unitary divisor of N, and let n be the largest known odd divisor of N such that $(n, m) = 1$. Suppose $2^a \| \sigma^*(m)$ and that n has b distinct prime divisors. Let j be defined by $a+b+j=A+1$, and let r be the product of the first j primes not dividing $2mn$. Then we have a contradiction if $f(2^A m)f(n)f(r) < 2$.

The remaining remarks in this section refer to the portion of the proof which was done by computer [the steps could just as well be done by hand]. Let A be fixed, let $N=2^A k$ with k odd, and let p be the largest prime dividing $\sigma^*(2^A)=1+2^A$.

For $11\le A\le 38$, $p \| (1+2^A)$ by observation. If we assume $p^e \mid N$, then $p^{e-1} \mid k$, so the unitary divisors of k must include enough odd prime powers to contribute $e-1$ factors p to the numerator of $f(N)$, and the product of these prime powers is at least $2p^{e-1}-1$. Therefore:

(6) We have a contradiction if $p^e \mid N$ and $2^A p^e(2p^{e-1}-1)>W$. [Done by computer.]

If $p \| N$, let q be the largest prime divisor of $\sigma^*(p)=p+1$. Then:

(7) We have a contradiction if $2^A pq > W$. [Done by computer.]

3. **Table of cases.** The computer program immediately eliminates the cases $A=28$, $A=29$ and $31\le A\le 38$ by (7). The following table lists the remaining cases, with the reason for eliminating the case if other than (3). Brackets are used in conjunction with (4) to indicate for clarity the known powers of 3 and 5 which divide N.

As a convenience, we use the following notation: if p^e is a prime power, we let $*p^e$ denote the product of p^e and the largest *odd* unitary divisor of $\sigma^*(p^e)=1+p^e$. For example, since $\sigma^*(59)=2^2 3\cdot 5$, $*59=3\cdot 5\cdot 59$.

4. **Special cases.** In this section we take care of the cases left open in the previous section.

(8) If $A = 11$, the only possible case requires $2^{11}683 \| N$ and $3^3 19 \mid N$. However, $f(2^{11}683) = 3^3 19/2^9 = f(2^9)$. If there were an integer m with $(2\cdot 683, m) = 1$ and $2^{11}683m$ unitary perfect, then $2^9 m$ would be unitary perfect, which cannot occur.

(9) We write $N = 2^{12}241\cdot 11^4 7321\cdot 523\cdot 131\cdot m$, where $3\cdot 7\cdot 17 \mid m$. If $17 \| m$, then $3^3 7\cdot 17 \mid m$; otherwise $3\cdot 7\cdot 17^2 \mid m$. Since $N\le W$, $m\le 20189$. The only possible values for m are 3213, 6069, 16065 and 18207, but each leaves an excess 11 in the denominator of $f(N)$.

(10) The case $2^{12}241\cdot 11^2 61\cdot 31 \| N$ is impossible since

$$f(2^{12}241\cdot 11^2 61\cdot 31) = 17/16 = f(2^4)$$

and $A = 4$ is impossible. The reasoning is similar to that in (8).

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Table 1.

Unitary Divisors	Other Divisors	Elimination if not (3)
2^{11}	$3 \cdot 683^4$	(6)
$2^{11}683^3$	$3^3 19 \cdot (155269^2$ or $*155269)$	
$2^{11}683^2$	$3 \cdot 5 \cdot (46649^3$ or $*46649^2)$	
$2^{11}683^2 46649$	$3^3 5^3[3^2 5^3$ known]	(4)
$2^{11}683^2 46649 \cdot 3^2$	$5^4 \cdot (311^3$ or $*311^2)$	
$2^{11}683^2 46649 \cdot 3^2 311$		excess 3's
$2^{11}683$	$3^3 19$	see (8) in next section
2^{12}	$17 \cdot 241^5$	(6)
$2^{12}241^4$	$17 \cdot 1686701281$	
$2^{12}241^3$	$7 \cdot 17 \cdot (11^3$ or $*11^2) \cdot 8263^2$	
$2^{12}241^3 8263$	$7 \cdot 17 \cdot 11^2 \cdot (1033^2$ or $*1033)$	
$2^{12}241^2$	$17 \cdot 113 \cdot (257^5$ or $*257^4$ or $*257^3/241)$	
$2^{12}241^2 257^2$	$5^2 17 \cdot 113 \cdot (1321^2$ or $*1321)$	
$2^{12}241^2 257$	$3 \cdot 17 \cdot 43 \cdot (113^5$ or $*113^4$ or $*113^3)$	
$2^{12}241^2 257 \cdot 113^2$	$3 \cdot 5 \cdot 17 \cdot 43 \cdot (1277^2$ or $*1277)$	
$2^{12}241^2 257 \cdot 113$	$3^2 17 \cdot 19 \cdot (43^5$ or $*43^4$ or $*43^3)$	
$2^{12}241^2 257 \cdot 113 \cdot 43^2$	$3^2 5^2 17 \cdot 19 \cdot 37^2$	
$2^{12}241^2 257 \cdot 113 \cdot 43^2 37$	$3^2 5^2 17 \cdot 19^2$	
$2^{12}241^2 257 \cdot 113 \cdot 43$	$3^2 11 \cdot 17 \cdot (19^5$ or $*19^4$ or $*19^3)$	
$2^{12}241^2 257 \cdot 113 \cdot 43 \cdot 19^2$	$3^2 11 \cdot 17 \cdot (181^2$ or $*181)$	
$2^{12}241^2 257 \cdot 113 \cdot 43 \cdot 19$	$3^2 5 \cdot 11 \cdot (17^4$ or $*17^3)$	
$2^{12}241^2 257 \cdot 113 \cdot 43 \cdot 19 \cdot 17^2$	$3^2 5^2 11 \cdot (29^2$ or $*29)$	
$2^{12}241^2 257 \cdot 113 \cdot 43 \cdot 19 \cdot 17$	$3^4 5 \cdot 11$	(5)
$2^{12}241$	$(17^2$ or $*17)(11^{15}$ or $*11^e, 9 \leq e \leq 14)$	
$2^{12}241 \cdot 11^8$	$17^2 6304673$	
$2^{12}241 \cdot 11^7$	$3 \cdot 17 \cdot (1623931^2$ or $*1623931)$	
$2^{12}241 \cdot 11^6$	$13 \cdot 17 \cdot (61^2$ or $*61)(1117^2$ or $*1117)$	
$2^{12}241 \cdot 11^5$	$3 \cdot 17 \cdot (13421^3$ or $*13421^2)$	
$2^{12}241 \cdot 11^5 13421$	$3 \cdot 17 \cdot (2237^2$ or $*2237)$	
$2^{12}241 \cdot 11^4$	$(17^2$ or $*17)(7321^3$ or $*7321^2)$	
$2^{12}241 \cdot 11^4 7321$	$7 \cdot (17^2$ or $*17)(523^3$ or $*523^2)$	
$2^{12}241 \cdot 11^4 7321 \cdot 523$	$7 \cdot (17^2$ or $*17) \cdot 131^2$	
$2^{12}241 \cdot 11^4 7321 \cdot 523 \cdot 131$	$3 \cdot 7 \cdot 17$	see (9) in next section
$2^{12}241 \cdot 11^3 5$	$3^3 17 \cdot 37$	(5)
$2^{12}241 \cdot 11^3$	$3^2 17 \cdot 37 \cdot 5^2$ (or N prime to 5)	(5)
$2^{12}241 \cdot 11^2$	$17 \cdot (61^8$ or $*61^7$ or $*61^6$ or $*61^5/11$ or $*61^4)$	
$2^{12}241 \cdot 11^2 61^3$	$7 \cdot (17^2$ or $*17) \cdot 31 \cdot 523^2$	
$2^{12}241 \cdot 11^2 61^3 523$	$7 \cdot 17 \cdot 31 \cdot 131^2$	
$2^{12}241 \cdot 11^2 61^3 523 \cdot 131$		excess 11's
$2^{12}241 \cdot 11^2 61^2$	$(17^2$ or $*17)(1861^3$ or $*1861^2)$	
$2^{12}241 \cdot 11^2 61^2 1861$	$(7^3$ or $*7^2)(17^2$ or $*17)(19^3$ or $*19^2)$	
$2^{12}241 \cdot 11^2 61^2 1861 \cdot 19$	$(5^2$ or $*5)(7^3$ or $*7^2)(17^3$ or $*17^2)$	
$2^{12}241 \cdot 11^2 61^2 1861 \cdot 19 \cdot 17$	$(3^3$ or $*3^2)(5^2$ or $*5)(7^4$ or $*7^3)$	
$2^{12}241 \cdot 11^2 61^2 1861 \cdot 19 \cdot 17 \cdot 7^2$	$3^2 \cdot (5^5$ or $*5^4)$	
$2^{12}241 \cdot 11^2 61^2 1861 \cdot 19 \cdot 17 \cdot 7^2 5^3$	3^4	(4)
$2^{12}241 \cdot 11^2 61$	$(17^2$ or $*17)(31^8$ or $*31^e$ for $4 \leq e \leq 7)$	
$2^{12}241 \cdot 11^2 61 \cdot 31^3$	$7^2 17 \cdot 19$	
$2^{12}241 \cdot 11^2 61 \cdot 31^2$	$17 \cdot (481^4$ or $*481^3/241$ or $*481^2)$	
$2^{12}241 \cdot 11^2 61 \cdot 31^2 481$		excess 241's
$2^{12}241 \cdot 11^2 61 \cdot 31$		see (10) in next section

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Table 1. contd.

Unitary Divisors	Other Divisors	Elimination if not (3)
2^{13}	$3 \cdot 2731^4$	(6)
2^{13}	$3 \cdot 5092195973$	
$2^{13}2731^2$	$3 \cdot (3729181^2$ or $*3729181)$	
$2^{13}2731$	$3 \cdot (683^6$ or $*683^5$ or $*683^4$ or $*683^3)$	
$2^{13}2731 \cdot 683^2$	$3 \cdot 5 \cdot (46649^2$ or $*46649)$	
$2^{13}2731 \cdot 683$	$3^3 19$	see (11) in next section
2^{14}	$5 \cdot 29 \cdot 113^6$	(6)
2^{14}	$5 \cdot (29^2$ or $*29)(*113^5$ or $*113^4)$	
$2^{14}113^3$	$3^2 5 \cdot 19 \cdot 29 \cdot (4219^3$ or $*4219^2)$	
$2^{14}113^3 4219$	$3^2 5^2 19 \cdot (29^2$ or $*29)(211^2$ or $*211)$	
$2^{14}113^2$	$5^2 29 \cdot (1277^4$ or $*1277^3$ or $*1277^2)$	
$2^{14}113^2 1277$	$(3^3$ or $*3^2)(5^3$ or $*5^2)(71^2$ or $*71)$ $\cdot (29^4$ or $*29^3$ or $*29^2)$	
$2^{14}113^2 1277 \cdot 29$	$3^3 5^3$ [known]	(4)
$2^{14}113$	$3 \cdot 5 \cdot 19 \cdot (29^{10}$ or $*29^e$ for $5 \leq e \leq 9)$	
$2^{14}113 \cdot 29^4$	$3 \cdot 5 \cdot 19 \cdot (353641^2$ or $*353641)$	
$2^{14}113 \cdot 29^3$	$3^3 5^3$ $[3^3 5^2$ known$]$	(4)
$2^{14}113 \cdot 29^3 5^2 19$		excess 5's
$2^{14}113 \cdot 29^3 5^2$	$3^3 13 \cdot (19^3$ or $*19^2)(271^2$ or $271 \cdot 17 \cdot 9$ or $271 \cdot 17^2)$	
$2^{14}113 \cdot 29^2$	$3 \cdot 5 \cdot 19 \cdot (421^5$ or $*421^4$ or $*421^3)$	
$2^{14}113 \cdot 29^2 421^2$	$3 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot (401^2$ or $*401)$	
$2^{14}113 \cdot 29^2 421$	$3 \cdot 5 \cdot (19^2$ or $*19)(211^4$ or $*211^3)$	
$2^{14}113 \cdot 29^2 421 \cdot 211^2$		excess 113's
$2^{14}113 \cdot 29^2 421 \cdot 211$	$3 \cdot 5 \cdot 19 \cdot (53^4$ or $*53^3$ or $*53^2)$	
$2^{14}113 \cdot 29^2 421 \cdot 211 \cdot 53$	$3^4 5 \cdot (19^4$ or $*19^3$ or $*19^2)$	
$2^{14}113 \cdot 29^2 421 \cdot 211 \cdot 53 \cdot 19$	$3^4 5^3$ $[3^4 5^2$ known$]$	(4)
$2^{14}113 \cdot 29^2 421 \cdot 211 \cdot 53 \cdot 19 \cdot 5^2$	$(13^2$ or $*13)(3^6$ or $*3^5$ or $*3^4)$	
$2^{14}113 \cdot 29$	$3^3 5^3$ $[3^2 5^2$ known$]$	(4)
$2^{14}113 \cdot 29 \cdot 3^2 5^3$		excess 3's
$2^{14}113 \cdot 29 \cdot 3^2$	$5^4 19$	(5)
$2^{14}113 \cdot 29 \cdot 5^2 3^2$		excess 5's
$2^{14}113 \cdot 29 \cdot 5^2 19$		excess 5's
$2^{14}113 \cdot 29 \cdot 5^2$	$3^3 13 \cdot 19^2$	(5)
2^{15}	$3^2 11 \cdot (331^6$ or $*331^5$ or $*331^4)$	
$2^{15}331^3$	$3^2 11 \cdot 19 \cdot 83 \cdot (5749^2$ or $*5749)$	
$2^{15}331^2$	$3^2 11 \cdot (29^2$ or $*29)(1889^3$ or $*1889^2)$	
$2^{15}331^2 1889$	$3^5 5 \cdot 7 \cdot 11 \cdot (29^5$ or $*29^4$ or $*29^3$ or $*29^2)$	
$2^{15}331^2 1889 \cdot 29$	$3^6 5^3$ $[3^6 5^2$ known$]$	(4)
$2^{15}331^2 1889 \cdot 29 \cdot 5^2$	$3^6 7 \cdot 11 \cdot (13^3$ or $*13^2/5)$	
$2^{15}331^2 1889 \cdot 29 \cdot 5^2 13 \cdot 7^2$		excess 5's
$2^{15}331^2 1889 \cdot 29 \cdot 5^2 13$	$3^6 7^3 11$	
$2^{15}331$	$3^2(11^2$ or $*11)(83^8$ or $*83^e$ for $4 \leq e \leq 7)$	
$2^{15}331 \cdot 83^3$	$3^4 7 \cdot (11^2$ or $*11)(2269^2$ or $*2269)$	
$2^{15}331 \cdot 83^2$	$3^2(5^2$ or $*5)(11^2$ or $*11)(13^2$ or $*13)$ $(53^4$ or $*53^3)$	
$2^{15}331 \cdot 83^2 53^2$	$3^2 5^2 11 \cdot 13 \cdot (281^2$ or $*281)$	
$2^{15}331 \cdot 83^2 53$	$3^5 5^3$ $[3^5 5$ known$]$	(4)
$2^{15}331 \cdot 83^2 53 \cdot 5^2$	$3^5 11 \cdot 13^2$	(5)

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Table 1 contd.

Unitary Divisors	Other Divisors	Elimination if not (3)
$2^{15}331 \cdot 83^2 53 \cdot 5$	$3^6 11 \cdot 13$	(5)
$2^{15}331 \cdot 83$	$3^3 5^3$ (or N prime to 5) [3^3 known]	(4)
$2^{15}331 \cdot 83 \cdot 5^2$	$3^3 7 \cdot 11 \cdot 13$	(5)
$2^{15}331 \cdot 83 \cdot 5$	$3^4 7^3 11$	(5)
$2^{15}331 \cdot 83 \cdot 5 \cdot 7^2$		excess 5's
$2^{15}331 \cdot 83 \;\; 5 \cdot 7 \cdot 13$		excess 7's
$2^{15}331 \cdot 83 \cdot 5 \cdot 7$	$3^4 11 \cdot 13^2$ (or N prime to 13)	(5)
2^{16}	65537^3	(6)
$2^{16}65537^2$	2147549185	
$2^{16}65537$	$3^2 11 \cdot 331$	see (12) in next section
2^{17}	$3 \cdot 43691^3$	(6)
$2^{17}43691^2$	$3 \cdot 954451741$	
$2^{17}43691$	$3^2 11 \cdot 331$	see (12) in next section
2^{18}	$5 \cdot 13 \cdot 37 \cdot 109^5$	(6)
$2^{18}109^4$	$5 \cdot 13 \cdot 37 \cdot 70579081$	
$2^{18}109^3$	$5^2 11 \cdot (13^2$ or $*13)(61^2$ or $*61)(193^2$ or $*193)$	
$2^{18}109^2$	$5 \cdot (13^3$ or $*13^2)(37^2$ or $*37)(457^3$ or $*457^2)$	
$2^{18}109^2 457$	$5 \cdot (13^3$ or $*13^2)(37^2$ or $*37)(229^2$ or $*229)$	
$2^{18}109$	$5^2 11 \cdot 13 \cdot (37^8$ or $*37^7$ or $*37^6$ or $*37^5$ or $*37^4)$	
$2^{18}109 \cdot 37^3$	$5^2(13^2$ or $*13)(19^2$ or $*19) \cdot 31$ $\cdot (43^2 11$ or $43 \cdot 11^2$ or $43 \cdot 11^2 61)$	
$2^{18}109 \cdot 37^2$	$5^3 11 \cdot (13^2$ or $*13)(137^4$ or $*137^3$ or $*137^2)$	
$2^{18}109 \cdot 37^2 137$	$3 \cdot 5(11^2$ or $*11)(23^5$ or $*23^4$ or $*23^3)$	
$2^{18}109 \cdot 37^2 137 \cdot 23^2$	$3 \cdot 5^4(11^2$ or $*11)(53^2$ or $*53)$	
$2^{18}109 \cdot 37^2 137 \cdot 23$	$3^3 5^3$ [$3^2 5^3$ known]	(4)
$2^{18}109 \cdot 37^2 137 \cdot 23 \cdot 3^2 11$		excess 3's
$2^{18}109 \cdot 37^2 137 \cdot 23 \cdot 3^2$	$5^4 11^2 13$	see (14) in next section
$2^{18}109 \cdot 37 \cdot 5^2 13^2$		excess 5's
$2^{18}109 \cdot 37 \cdot 5^2 19^{odd}$		excess 5's
$2^{18}109 \cdot 37 \cdot 5^2$	$11 \cdot 13 \cdot (19^8$ or $*19^6$ or $*19^4)$	
$2^{18}109 \cdot 37 \cdot 5^2 19^2$	$11 \cdot 181 \cdot (13^7$ or $*13^5$ or $*13^4)$	
$2^{18}109 \cdot 37 \cdot 5^2 19^2 13^6$		excess 5's
$2^{18}109 \cdot 37 \cdot 5^2 19^2 13^3$	$7 \cdot 11 \cdot 181 \cdot (157^2$ or $*157)$	
$2^{18}109 \cdot 37$	$5^3 3^3$ (or N prime to 3) [5^3 known]	(4)
$2^{18}109 \cdot 37$	$3 \cdot 11 \cdot 13 \cdot 19 \cdot (5^{15}$ or $*5^e$ for $8 \le e \le 14)$	
$2^{18}109 \cdot 37 \cdot 5^7$	$3 \cdot 11 \cdot 13 \cdot 19 \cdot 29 \cdot 449^2$	
$2^{18}109 \cdot 37 \cdot 5^7 449$	3^3 [$3^3 5^7$ known]	(4)
$2^{18}109 \cdot 37 \cdot 5^6$	$3 \cdot 11 \cdot 13^2 19 \cdot (601^2$ or $*601)$	
$2^{18}109 \cdot 37 \cdot 5^5$	$3 \cdot 11 \cdot 13 \cdot 19 \cdot (521^3$ or $*521^2)$	
$2^{18}109 \cdot 37 \cdot 5^5 521$	3^3 [$3^3 5^5$ known]	(4)
$2^{18}109 \cdot 37 \cdot 5^3$	3^3 [$3^2 5^3$ known]	(4)
$2^{18}109 \cdot 37 \cdot 5^3 3^2 19$		excess 5's
$2^{18}109 \cdot 37 \cdot 5^3 3^2 11$		excess 3's
$2^{18}109 \cdot 37 \cdot 5^3 3^2$	$7 \cdot 11^2 13 \cdot 19^2$	(5)
$2^{18}109 \cdot 37 \cdot 5^4$	$3 \cdot 11 \cdot 13 \cdot 19 \cdot (313^4$ or $*313^3$ or $*313^2)$	
$2^{18}109 \cdot 37 \cdot 5^4 313$	$3 \cdot 11 \cdot 13 \cdot 19 \cdot (157^3$ or $*157^2)$	
$2^{18}109 \cdot 37 \cdot 5^4 313 \cdot 157$	$3 \cdot 11 \cdot 13 \cdot 19 \cdot 79^2$	
$2^{18}109 \cdot 37 \cdot 5^4 313 \cdot 157 \cdot 79$	$3 \cdot 11 \cdot 13 \cdot 19$	see (15) in next section
2^{19}	$3 \cdot (174763^4$ or $*174763^3$ or $*174763^2)$	
$2^{19}174763$	$3 \cdot 43691$	see (13) in next section

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Table 1. contd.

Unitary Divisors	Other Divisors	Elimination if not (3)
2^{20}	$17 \cdot (61681^4$ or $*61681^3$ or $*61681^2)$	
$2^{20}61681$	$17 \cdot (30841^3$ or $*30841^2)$	
$2^{20}61681 \cdot 30841$	$7 \cdot 17 \cdot (2203^2$ or $*2203)$	
2^{21}	$3^2(43^2$ or $*43)(5419^4$ or $*5419^3$ or $*5419^2)$	
$2^{21}5419$	$3^25 \cdot (43^2$ or $*43)(271^4$ or $*271^3$ or $*271^2)$	
$2^{21}5419 \cdot 271$	$3^25 \cdot 17 \cdot (43^5$ or $*43^4$ or $*43^3)$	
$2^{21}5419 \cdot 271 \cdot 43^2$	$3^25^3(17^2$ or $*17)(37^2$ or $*37)$	
$2^{21}5419 \cdot 271 \cdot 43$	$3^25 \cdot (11^2$ or $*11)(17^5$ or $*17^4$ or $*17^3)$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17^2$	$3^25^2(11^2$ or $*11) \cdot 29^2$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17^229$	3^35^3 [known]	(4)
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17$	$3^45 \cdot (11^5$ or $*11^4$ or $*11^3)$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17 \cdot 11^2$	$3^45 \cdot (61^2$ or $*61)$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17 \cdot 11$	$5 \cdot (3^{13}$ or $*3^e$ for $9 \leq e \leq 12$ or $*3^8/17$ or $*3^7)$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17 \cdot 11 \cdot 3^6$	$5^2(73^2$ or $*73)$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17 \cdot 11 \cdot 3^5$	$5 \cdot (61^3$ or $*61^2)$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17 \cdot 11 \cdot 3^561$	$5 \cdot 31^2$	
$2^{21}5419 \cdot 271 \cdot 43 \cdot 17 \cdot 11 \cdot 3^561 \cdot 31$	5	(5)
2^{22}	$5 \cdot (397^2$ or $*397)(2113^4$ or $*2113^3$ or $*2113^2)$	
$2^{22}2113$	$5 \cdot 7 \cdot (151^2$ or $*151)(397^4$ or $*397^3$ or $*397^2)$	
$2^{22}2113 \cdot 397$	$5 \cdot 7 \cdot (151^2$ or $*151)(199^3$ or $*199^2)$	
$2^{22}2113 \cdot 397 \cdot 199$	$5^37 \cdot (151^3$ or $*151^2)$	
$2^{22}2113 \cdot 397 \cdot 199 \cdot 151$	$5^37 \cdot (19^3$ or $*19^2)$	
$2^{22}2113 \cdot 397 \cdot 199 \cdot 151 \cdot 19$	$7 \cdot (5^6$ or $*5^5$ or $*5^4)$	
2^{23}	$3 \cdot 2796203^2$	
$2^{23}2796203$	$3^243 \cdot 5419^2$	
$2^{23}2796203 \cdot 5419$	$3^25 \cdot 43 \cdot (271^2$ or $*271)$	
2^{24}	$97 \cdot (257^2$ or $*257)(673^3$ or $*673^2)$	
$2^{24}673$	$(97^2$ or $*97)(257^2$ or $*257) \cdot 337^2$	
$2^{24}673 \cdot 337$	$(97^2$ or $*97)(257^2$ or $*257)(13^3$ or $*13^2)$	
2^{25}	$3 \cdot 11 \cdot 251 \cdot (4051^3$ or $*4051^2)$	
$2^{25}4051$	$3 \cdot 11 \cdot 251 \cdot (1013^3$ or $*1013^2)$	
$2^{25}4051 \cdot 1013$	$3^211 \cdot (13^3$ or $*13^2)(251^2$ or $*251)$	
2^{26}	$5 \cdot 53 \cdot 157 \cdot (1613^3$ or $*1613^2)$	
$2^{26}1613$	$5 \cdot (53^2$ or $*53)(157^2$ or $*157) \cdot 269^2$	
$2^{26}1613 \cdot 269$	$3^45^2(53^2$ or $*53)(157^2$ or $*157)$	
2^{27}	$3^419 \cdot (87211^3$ or $*87211^2)$	
$2^{27}87211$	$3^419 \cdot (21803^2$ or $*21803)$	
2^{30}	$5^213 \cdot 41 \cdot 61 \cdot (1321^3$ or $*1321^2)$	
$2^{30}1321$	$5^213 \cdot 41 \cdot 61 \cdot (661^2$ or $*661)$	

(11) If $2^{13}2731 \cdot 683 \| N$, we have a contradiction as in (8), since $f(2^{13}2731 \cdot 683) = f(2^9)$.

(12) Note that $f(2^{16}65537) = f(2^{17}43691) = f(2^{15})$. If $2^{16}65537m \leq W$ or $2^{17}43691n \leq W$ is unitary perfect with $(m, 2 \cdot 65537) = (n, 2 \cdot 43691) = 1$, then $2^{15}m$ or $2^{15}n$ is unitary perfect and smaller than W. But $A = 15$ has already been eliminated.

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(13) Since $f(2^{19}174763) = f(2^{17})$, $A \neq 17$ implies $A \neq 19$ by reasoning similar to that in (12).

(14) If $N = 2^{18}109 \cdot 37^2 137 \cdot 23 \cdot 3^2 5^4 11^2 13 \cdot m \leq W$ with $(m, 6) = 1$, then $m \leq 133$. Either $11^2 \| N$ and hence $61 \mid m$, or $11 \mid m$; either $13 \| N$ and hence $7 \mid m$, or $13 \mid m$. The only possible value for m is 77, but this requires $f(N) < 2$.

(15) We write $N = 2^{18}109 \cdot 37 \cdot 5^4 313 \cdot 157 \cdot 79 \cdot 3 \cdot 11 \cdot 13 \cdot 19 \cdot m \leq W$. Then $m \leq 7$. Since $13 \nmid m$, $13 \| N$ and hence $7 \mid m$. Thus only $m = 7$ is possible, and hence $N = W$.

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University of South Carolina