ON \(k\)-QUASIHYPONORMAL OPERATORS II

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An operator \(T\) on a Hilbert space is in the class of \(k\)-quasi-
hyponormal operators \(Q(k)\), if \(T^* (T^*T - TT^*)T^k \geq 0\). It is
shown that if \(T\) is in \(Q(k)\) and \(S\) is normal such that
\(TX = XS\), where \(X\) is one to one with dense range, then \(T\) is
normal; and is unitarily equivalent to \(S\). It is proved that \(S\) can be replaced by a cohyponormal operator, if \(T\) in \(Q(1)\)
is one to one. It is also shown that two quasisimilar operators
in \(Q(k)\) have equal spectra, and every reductive operator quasi-
similar to a normal operator is normal.

A bounded linear operator \(T\) on a Hilbert space \(H\) is called
\(k\)-quasihyponormal if \(T^* (T^*T - TT^*)T^k \geq 0\), or equivalently,
\[\|T^k x\| \leq \|T^k+1 x\| \] for every \(x \in H\), where \(k\) is a positive integer.
Clearly, the class \(Q(k)\) of all \(k\)-quasihyponormal operators on \(H\)
contains all hyponormal operators and forms a strictly increasing sequence
in \(k\). The class \(Q(1)\) is the class of quasihyponormal operators [10].
For an operator \(T \in Q(k)\) the following representation was obtained in
[4].

THEOREM A. An operator \(T\) is in \(Q(k)\) if and only if \(T\) has
matrix representation

\[
T = \begin{pmatrix}
T_1 & T_2 \\
0 & T_3
\end{pmatrix}
\]

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with respect to a pair of complementary orthogonal subspaces of the Hilbert space $H$, where

\[(a) \quad T^* T_1 - T_1 T^*_1 \geq T^*_1 T_1, \quad \text{and} \]

\[(b) \quad T^k_3 = 0.\]

The representation (*) is not unique; however, we can always take

$T_1 = T[R(T^k)], \quad T$ restricted to $R(T^k).$ 

Using the representation (*) it was shown that eigenspaces corresponding to non-zero eigenvalues are reducing and several structure theorems for operators in $Q(k)$ were proved. Further, it was shown that there is a non-hyponormal operator in $Q(1)$ with reducing kernel; and since restriction of an operator $T \in Q(1)$ to an invariant subspace is again in $Q(1)$, this also gives a one to one non-hyponormal operator in $Q(1)$. In this paper, we continue the study of operators in $Q(k)$.

We denote the kernel, the range, the spectrum and the closure of the numerical range of an operator $T$ by $N(T)$, $R(T)$, $\sigma(T)$ and $\overline{W(T)}$, respectively. The norm closure of a subspace $M$ of $H$ is denoted by $\overline{M}$ and the Banach algebra of all operators on a Hilbert space $H$ by $B(H)$.

It is shown in this paper that if $T \in Q(k)$ and $S$ is normal such that $TX = XS$ where $N(X) = N(X^*) = \{0\}$ then $T$ is normal, and is unitarily equivalent to $S$. If in addition, $T$ is in $Q(1)$ with $N(T) = \{0\}$ then the normal operator $S$ can be replaced by a cohyponormal operator without affecting the conclusion. In case $T$ is an arbitrary hyponormal operator these results are due to Stampfli and Wadhwa [11] and Radjabali pour [8].

It is known that two quasimilar hyponormal operators have equal spectra [5], and every reductive operator similar to a normal operator is normal [6]. We show that two quasisimilar operators in $Q(k)$ have equal spectra, and every reductive operator which is quasisimilar to a normal operator is normal.

For our purpose, we mention the following which is an easy modification of Theorem 1 in [11].

**Theorem B.** Let $T \in B(H)$ be hyponormal and let $S \in B(K)$ be
normal. If $TX = XS$ where $X : K \rightarrow H$ is a one to one bounded linear operator with dense range then $T$ is normal and is unitarily equivalent to $S$.

**THEOREM 1.** Let $T \in Q(k)$, $S$ a normal operator and let $TX = XS$ where $X$ is a one to one operator with dense range. Then $T$ is a normal operator unitarily equivalent to $S$.

**Proof.** Let $T_1 = T|R(T^k)$ and $S_1 = S|R(S^k)$. Then by Theorem A, we have

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$$

and

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $S_1$ is normal, $T_3^* = 0$ and $T_1^* T_1 - T_1 T_1^* \geq T_2 T_2^*$. Since $T^k X = XS^k$ and $X$ has dense range, $X(R(S^k)) = R(T^k)$. If we denote the restriction of $X$ to $R(S^k)$ by $X_1$ then $X_1 : R(S^k) \rightarrow R(T^k)$ is one to one and has dense range and for every $x \in R(S^k)$, $X_1 S_1 x = XSx = TXx = T_1 X_1 x$ so that $X_1 S_1 = T_1 X_1$. Now since $T_1$ is hyponormal it follows from Theorem B that $T_1$ is a normal operator unitarily equivalent to $S_1$. But then

$$T_2 T_2^* = 0,$$

which implies that $T_2 = 0$ and therefore $R(T^k)$ reduces $T$.

Since $X^*(R(T^k)) \subset N(S^k)$, for each $x \in N(T^*)$, we have $X^* T_3 x = X^* T_3 x = S x x x = 0$. But $X$ has dense range and so $X^*$ is one to one. Therefore $T_3^* x = 0$ for every $x \in N(T^k)$. Thus $T_3 = 0$. Hence $T = T_1 \oplus 0$. This completes the proof.

As an application, we get the following version of [8, Corollary 1] for operators in $Q(1)$.

**THEOREM 2.** Let $T \in Q(1)$ be one to one, $S$ a cohyponormal operator and let $X$ be a one to one operator with dense range such that $TX = XS$. Then $T$ and $S$ are unitarily equivalent normal operators.

**Proof.** Suppose $S$ is not normal. Then by Theorem 1 of [7] there exists a non-zero vector $x \in H$ and a bounded function $f : \mathbb{C} \rightarrow H$ such
that \((S-\lambda I)f(\lambda) \equiv x\). Then it follows that \(Xf : \mathbb{C} \to H\) is a bounded function such that \((T-\lambda I)Xf(\lambda) = Xx\). Let \(Xf(\lambda) = f_1(\lambda) \oplus f_2(\lambda)\) and \(Xx = x_1 \oplus x_2\) be the decompositions of \(Xf(\lambda)\) and \(Xx\) relative to the decomposition \(H = \overline{R(T)} \oplus N(T^4)\). Then Theorem A gives
\[
\left( T_1 - \lambda I \right) f_1(\lambda) + T_2 f_2(\lambda) = x_1
\]
and
\[
-\lambda f_2(\lambda) = x_2
\]
for all \(\lambda \in \mathbb{C}\).

In particular, if \(\lambda = 0\) then \(x_2 = 0\). Therefore \(f_2(\lambda) = 0\) if \(\lambda \neq 0\) and \(x_1\) is a non-zero vector. So \(\left( T_1 - \lambda I \right) f_1(\lambda) = x_1\) for all \(\lambda \neq 0\). Now Theorem 1 and Proposition 2 of [8] imply that
\[
\chi^{T_1}_1(0) = \{x \in H : \text{there exists an analytic function } f_x : \mathbb{C} \setminus \{0\} \to H \text{ such that } (T_1 - \lambda I) f_x(\lambda) = x \}
\]
is a closed invariant subspace of \(T_1\) containing the non-zero vector \(x_1\) and \(\sigma(T_1 | \chi^{T_1}_1(0)) = \{0\}\). Since \(T_1\) is hyponormal, \(T_1 | \chi^{T_1}_1(0) = 0\). So \(T x_1 = T_1 x_1 = 0\). But \(T\) is one to one and therefore \(x_1 = 0\), a contradiction. Hence \(S\) must be normal and the result follows from Theorem 1. //

If \(T\) and \(T^4\) both are hyponormal then \(T\) is normal. For \(T \in Q(k)\), we have the following:

**THEOREM 3.** If \(T \in Q(k)\) is cohyponormal then \(T\) is normal.

**Proof.** Since \(T\) is cohyponormal, \(N(T^4) = N(T^4)^{\perp}\) reduces \(T\). Therefore \(T = T_1 \oplus T_3\), where \(T_1 = T | \overline{R(T^k)}\) and \(T_3\) both are hyponormal. Also \(T_3^4\) is nilpotent and hyponormal. Hence \(T_3 = 0\) and \(T\) is normal. //

Let \(0\) be the class of all operators \(T\) on \(H\) for which \(T^4 T\) and \(T + T^4\) commute. In [2] Campbell proved the following:

**THEOREM C.** If \(T \in 0\) and \(T^4\) is hyponormal then \(T\) is normal.
THEOREM D. If \( T \in \Theta \) is hyponormal then \( T \) is subnormal.

Theorem C remains valid even if \( T^* \in Q(1) \).

THEOREM 4. If \( T^* \in Q(1) \) and \( T \in \Theta \) then \( T \) is normal.

Proof. Let \( R = T^* | R(T^*) \). Then \( R \) is hyponormal. Also since \( T \in \Theta \), \( N(T) \) is reducing [3] and therefore Theorem A implies that \( T^* = R \oplus 0 \) is hyponormal. Hence by Theorem C, \( T \) is normal. //

Question. If \( T \in \Theta \cap Q(1) \), must it be subnormal?

THEOREM 5. If \( A, B \in Q(k) \) are quasisimilar then they have equal spectra.

Proof. Suppose \( X \) and \( Y \) are one to one operators on \( H \) with dense range such that \( XA = BX \) and \( YB = AY \). Let \( A_1 = A \upharpoonright R(A_1^k) \) and \( B_1 = B \upharpoonright R(B_1^k) \). Then

\[
A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}
\]

where \( A_1, B_1 \) are hyponormal and \( A_3, B_3 \) are nilpotents. Therefore \( \sigma(A) = \sigma(A_1) \cup \{0\} \) and \( \sigma(B) = \sigma(B_1) \cup \{0\} \). In view of the fact that quasisimilar hyponormal operators have equal spectra [5], it suffices to show that \( A_1 \) and \( B_1 \) are quasisimilar.

Since \( XA_1^k = B_1^kX \) and \( YB_1^k = A_1^kY \), it follows that the restrictions \( X : R(A_1^k) \to R(B_1^k) \) and \( Y : R(B_1^k) \to R(A_1^k) \) are one to one and have dense range. Now for any \( x \in R(A_1^k) \), \( XA_1x = XAx = BXx = B_1x \) and similarly for any \( y \in R(B_1^k) \), \( YB_1y = A_1y \). Thus \( A_1 \) and \( B_1 \) are quasisimilar.

Hence the result. //

In [9], Sheth proved that if \( T \) is hyponormal and \( S^{-1}TS = T^* \) where \( 0 \notin \overline{W(S)} \) then \( T \) is self-adjoint. We prove the following:

THEOREM 6. If \( T \in Q(k) \) is such that \( S^{-1}TS = T^* \) where \( 0 \notin \overline{W(S)} \) then \( T \) is direct sum of a self-adjoint operator and a nilpotent operator.
Proof. Write

\[ T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \]

as usual. Then \( \sigma(T) = \sigma(T_1) \cup \{0\} \). Since \( S^{-1}TS = T^* \) and \( 0 \notin \overline{W(S)} \), by Theorem 1 of [12], \( \sigma(T) \) and hence \( \sigma(T_1) \) is real. Thus \( T_1 \) is self-adjoint. But then \( T_2 = 0 \), and we are done. \(/\)

Recall that an operator \( T \) is reductive if every invariant subspace of \( T \) is reducing. Every reductive operator similar to a normal operator is normal [6, Lemma 2.4]. The following shows that in this result similarity condition can be weakened to quasisimilarity.

**THEOREM 7.** If \( T \) is reductive and quasisimilar to a normal operator then \( T \) is normal.

Proof. Since \( T \) is reductive and quasisimilar to a normal operator by a result of Apostol [1] there exists a basic system \( \{X_n\} \) of reducing subspaces such that each \( S_n = T|X_n \) is reductive and similar to a normal operator and therefore \( S_n \) itself is normal for each \( n \). Since

\[ \bigvee_{n=1}^\infty X_n = H \], for each \( x \in H \), we have \( x = \lim_{m \to \infty} \left( \sum_{n=1}^\infty x_{mn} \right) \), where \( x_{mn} \in X_n \)

and for each \( m \), \( x_{mn} = 0 \) for all but finitely many \( n \)'s. Therefore for each \( x \in H \),

\[ TT^*x = \lim_{m \to \infty} \left( \sum_{n=1}^\infty TT^*x_{mn} \right) \]

\[ = \lim_{m \to \infty} \left( \sum_{n=1}^\infty S_n S^*_n x_{mn} \right) \]

\[ = \lim_{m \to \infty} \left( \sum_{n=1}^\infty T^*T x_{mn} \right) \]

\[ = T^*Tx. \]

Thus \( T \) is normal. \(/\)
k-quasihipo-normal operators

References


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