NILPOTENTS IN SEMIGROUPS OF PARTIAL ORDER-PRESERVING TRANSFORMATIONS

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In this paper we extend the results of Garba [1] on $IO_n$, the semigroup of all partial one-one order-preserving maps on $X_n = \{1, \ldots, n\}$, to $PO_n$, the semigroup of all partial order-preserving maps on $X_n$. A description of the subsemigroup of $PO_n$ generated by the set $N$ of all its nilpotent elements is given. The set \{a \in PO_n | \text{im} a \leq r \text{ and } |X_n \setminus \text{dom} a| \geq r\} is shown to be contained in $\langle N \rangle$ if and only if $r \leq \frac{1}{3}n$. The depth of $\langle N \rangle$, which is the unique $k$ for which $\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k$ and $\langle N \rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1}$, is shown to be equal to 3 for all $n \geq 3$. The rank of the subsemigroup \{a \in PO_n | \text{im} a \leq \frac{n-2}{n} \text{ and } a \in \langle N \rangle\} is shown to be equal to $6(n-2)$, and its nilpotent rank to be equal to $7n-15$.


1. Introduction

In 1987, Gomes and Howie [3], and Sullivan [7] independently initiated the study of nilpotent generated subsemigroups of transformation semigroups on the set $X_n = \{1, \ldots, n\}$, by considering $I_n$, the symmetric inverse semigroup and $P_n$, the semigroup of all partial transformations on $X_n$ respectively. In [1] Garba considered $IO_n$, the semigroup of all partial one-one order-preserving maps on $X_n$. We shall now consider the larger semigroup $PO_n$ of all partial order-preserving transformations on $X_n$.

Let $N$ be the set of all nilpotent elements in $PO_n$ and $\langle N \rangle$ the sub-semigroup of $PO_n$ generated by $N$. In Section 2 a description of the elements in $\langle N \rangle$ is given. We show also that the set \{a \in PO_n | \text{im} a \leq r \text{ and } |X_n \setminus \text{dom} a| \geq r\} is contained in $\langle N \rangle$ if and only if $r \leq \frac{1}{3}n$.

Define the depth of $\langle N \rangle$, denoted by $\Delta(\langle N \rangle)$, to be the unique $k$ for which

$$\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$ 

In Section 3 we show that $\Delta(\langle N \rangle) = 3$ for all $n \geq 3$.

By the rank of a semigroup $S$ we shall mean the cardinality of any subset $A$ of minimal order in $S$ such that $\langle A \rangle = S$. If $S$ has a zero and is generated by nilpotents then the cardinality of the smallest subset $A$ consisting of nilpotents for which $\langle A \rangle = S$ is called the nilpotent rank of $S$. In Section 4 we show that the rank of the subsemigroup \{a \in PO_n | \text{im} a \leq n-2 \text{ and } a \in \langle N \rangle\} is equal to $6(n-2)$, and its nilpotent rank is equal to $7n-15$.

2. The nilpotent generated subsemigroup

We will denote an element $a$ in $PO_n$ by
where for each $a_i \in A_i$, $a_i < a_{i+1}$ ($i = 1, \ldots, r$) and $b_1 < b_2 < \cdots < b_r$. Let $x_i = \min \{ x : x \in A_i \}$ and $y_i = \max \{ x : x \in A_i \}$. For $i = 1, \ldots, r$, let $S_i = \{ x : x \in X_n : x_i \leq x \leq y_i \}$, and for $i = 1, \ldots, r - 1$, $T_i = \{ x : x \in X_m : y_i < x < x_{i+1} \}$. Let $T_0 = \{ x : x \in X_m : x < x_1 \}$ and $T_r = \{ x : x \in X_m : x > y_r \}$.

Following [1], we define $j_i(a)$, the length of the $i$th lower jump of $a$, by

$$j_i(a) = b_{i+1} - b_i, \quad (i = 1, \ldots, r - 1), \quad j_0(a) = b_1 - 1.$$

Then let

$$j^*_i(a) = \sum_{i=0}^{r-1} j_i(a).$$

An element $a$ in $PO_n$ is called nilpotent if $a^k = 0$ for some $k \geq 1$. We begin with a generalisation of Lemma 2.1 in [1].

**Lemma 2.1.** An element $a$ in $PO_n$ is nilpotent if and only if for all $x \in \text{dom } a$, $xa \neq x$.

**Proof.** If $a = 0$ (the empty map) the result is trivial. We may therefore suppose that $\text{dom } a \neq 0$. It is clear that if $xa = x$ for some $x \in \text{dom } a$, then $a$ cannot be nilpotent; for we would have

$$x = xa = xa^2 = \cdots.$$

Conversely, suppose that $xa \neq x$ for all $x \in \text{dom } a$. We first show that if $\text{dom } a^k \neq \emptyset$ ($k \geq 2$) then $xa^k \neq x$ for all $x \in \text{dom } a^k$. (Note that if $\text{dom } a^k = \emptyset$ for some $k$ then $a$ is nilpotent.) Let $x \in \text{dom } a^k$. Then $x \in \text{dom } a^t$ for all $t$ such that $1 \leq t \leq k$. In particular $x \in \text{dom } a$, and thus $xa \neq x$. We therefore have $xa < x$ or $xa > x$. By the order-preserving property we have $xa^k < x$ or $xa^k > x$. Thus $xa^k \neq x$.

Let

$$a = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $r = |\text{im } a|$. Now, if $b_r \in \text{dom } a$ then (since $b_r a \neq b_r$) we must have $b_r < x_r$ ($= \min \{ x : x \in A_r \}$), and by the order-preserving property we must have $\text{im } a \cap A_r = \emptyset$. Thus $b_r \in \text{im } a^2$, and so $\text{im } a^2 < a$ (properly). If $b \notin \text{dom } a$ then $|\text{dom } a \cap \text{im } a| < r$, and so $|\text{im } a^2| < r = |\text{im } a|$, which shows that $\text{im } a^2 \subseteq \text{im } a$.

If we now denote by $s$ the cardinality of $\text{im } a^2$, then $a^2$ can be written as

$$a^2 = \begin{pmatrix} A_1' & A_2' & \cdots & A_s' \\ b_1' & b_2' & \cdots & b_s' \end{pmatrix}.$$
Since $x^2 \neq x$ for all $x \in \text{dom } \alpha^2$, repeating the same argument as above with $\alpha^2$ replacing $\alpha$ we obtain $\text{im } \alpha^4 \subseteq \text{im } \alpha^2$. If this process is to continue we will obtain a strict descent

$$\text{im } \alpha \supset \text{im } \alpha^2 \supset \text{im } \alpha^4 \supset \cdots$$

and since $|\text{im } \alpha|$ is finite there exists $m$ such that $\text{im } \alpha^{2m} = \emptyset$, that is such that $\alpha^{2m} = 0$. \hfill \square

This result will be used below without comment.

By analogy with Theorem 2.7 in [1], we have:

**Theorem 2.2.** An element

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

in $PO_n$ is not a product of nilpotents if and only if $\alpha$ satisfies one or both of the following:

(i) $1 \in A_i$, $n \in A_i$ and (for all $i$) $A_i = S_i$ and $|T_i| \leq 1$,

(ii) $b_1 = 1$, $b_r = n$ and all lower jumps of $\alpha$ are of length 1 at most.

**Proof.** Suppose that $\alpha$ satisfies neither (i) and (ii). We distinguish four cases.

**Case 1.** $1 \notin A_1$, $b_1 \neq 1$. Here

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ 1 & 2 & \cdots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

a product of two nilpotents.

**Case 2.** $1 \in A_1$, $b_1 \neq 1$.

(a) if $n \notin A_r$, then

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \cdots & A_{r-1} & A_r \\ n-r+1 & \cdots & n-1 & n \end{pmatrix},$$

$$n_2 = \begin{pmatrix} n-r+1 & \cdots & n-1 & n \\ 1 & \cdots & r-1 & r \end{pmatrix} \text{ and } n_3 = \begin{pmatrix} 1 & \cdots & r-1 & r \\ b_1 & \cdots & b_{r-1} & b_r \end{pmatrix}.$$

(b) $n \in A_r$ and $A_i \neq S_i$ for some $i$. Then there exists $c \in S_i \setminus A_i$ (such that $x_i < c < y_i$) and

$$\alpha = n_1 n_2 n_3,$$
where
\[
  n_1 = \begin{pmatrix}
    A_1 & \cdots & A_{i-2} & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \cdots & A_r \\
    x_2 & \cdots & x_{i-1} & x_i & c & y_i & y_{i+1} & \cdots & y_{r-1}
  \end{pmatrix},
\]
and
\[
  n_2 = \begin{pmatrix}
    x_2 & \cdots & x_i & c & y_i & \cdots & y_{r-1} \\
    1 & \cdots & i-1 & i & i+1 & \cdots & r
  \end{pmatrix}
\]
and
\[
  n_3 = \begin{pmatrix}
    1 & \cdots & i-1 & i & i+1 & \cdots & r \\
    b_1 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_r
  \end{pmatrix}.
\]

(c) \( n \in A_r \) and \( |T_i| \geq 2 \) for some \( i \). Then there exists \( c, d \in T_i \) with \( c < d \), and
\[
  \alpha = n_1 n_2 n_3,
\]
a product of three nilpotents, where
\[
  n_1 = \begin{pmatrix}
    A_1 & \cdots & A_{i-2} & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \cdots & A_r \\
    y_2 & \cdots & y_i & c & d & y_{i+1} & y_{i+2} & \cdots & y_{r-1}
  \end{pmatrix},
\]
and
\[
  n_2 = \begin{pmatrix}
    y_2 & \cdots & y_i & c & d & y_{i+1} & \cdots & y_{r-1} \\
    1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & r
  \end{pmatrix}
\]
and
\[
  n_3 = \begin{pmatrix}
    1 & \cdots & i-1 & i & i+1 & \cdots & r \\
    b_1 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_r
  \end{pmatrix}.
\]

Case 3. \( 1 \notin A_1, b_1 = 1 \).

(a) \( b_r \neq n \). Define \( c_i = b_i + 1 \). Then
\[
  \alpha = \begin{pmatrix}
    A_1 & A_2 & \cdots & A_r \\
    1 & 2 & \cdots & r
  \end{pmatrix} \begin{pmatrix}
    1 & 2 & \cdots & r \\
    c_1 & c_2 & \cdots & c_r
  \end{pmatrix} \begin{pmatrix}
    b_1 & b_2 & \cdots & b_r
  \end{pmatrix},
\]
a product of three nilpotents.

(b) \( b_r = n \). Then \( \alpha \) must have at least one lower jump of length greater than 1. Since \( b_1 = 1 \) we may suppose that the first lower jump of length greater than 1 occurs between \( b_k \) and \( b_{k+1} \). Define
\[
  c_i = \begin{cases} 
    b_i + 1 & \text{if } i \leq k, \\
    b_i - 1 & \text{if } i > k.
  \end{cases}
\]
Note that \( c_{k+1} = b_{k+1} - 1 \geq (b_k + 3) - 1 = b_k + 2 > c_k \). Hence \( c_i < c_{i+1} \) for all \( i \), and
\[ \alpha = \left( \begin{array}{ccc} A_1 & A_2 & \ldots & A_r \\ 1 & 2 & \ldots & r \\ c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{array} \right) \]

a product of three nilpotents.

**Case 4.** \( 1 \in A_1, b_1 = 1. \)

(a) \( n \notin A_r, b_r \neq n. \) Define \( c_i = \max\{y_i, b_i\} + 1 \) for all \( i. \) Then

\[ \alpha = \left( \begin{array}{ccc} A_1 & A_2 & \ldots & A_r \\ c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{array} \right) \]

a product of two nilpotents.

(b) \( n \in A_r, b_r = n. \) Then \( \alpha \) must have at least one lower jump of length greater than 1. We may suppose that the first lower jump of length greater than 1 occurs between \( b_k \) and \( b_{k+1}. \) Define

\[
c_i = \begin{cases} 
  b_i + 1 & \text{if } 1 \leq i \leq k, \\
  b_i - 1 & \text{if } i > k.
\end{cases}
\]

Then

\[ \alpha = \left( \begin{array}{ccc} A_1 & \ldots & A_r \\ n-r+1 & \ldots & n \\ c_1 & \ldots & c_r \\ b_1 & \ldots & b_r \end{array} \right) \]

a product of three nilpotents.

(c) \( n \in A_r, b_r \neq n. \)

(i) \( A_i \neq S_i \) for some \( i. \) Then there exists \( c \) in \( S_i \setminus A_i \) (such that \( x_i < c < y_i), \) and

\[ \alpha = n_1 n_2 n_3, \]

a product of three nilpotents, where

\[ n_1 = \left( \begin{array}{cccc} A_1 & \ldots & A_{i-1} & A_i \\ x_2 & \ldots & x_i & c \\ x_{i+1} & \ldots & y_i & y_{i-1} \end{array} \right), \]

\[ n_2 = \left( \begin{array}{cccc} x_2 & \ldots & x_i & c \\ c_1 & \ldots & c_{i-1} & c_i \\ c_{i+1} & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{array} \right), \]

\[ n_3 = \left( \begin{array}{cccc} c_1 & c_2 & \ldots & c_r \end{array} \right), \]

and

\[
c_j = \begin{cases} 
  \max\{x_{j+1}, b_j\} + 1 & \text{if } 1 \leq j \leq i-1, \\
  \max\{c, b_j\} + 1 & \text{if } j = i, \\
  \max\{y_{j-1}, b_j\} + 1 & \text{if } j > i.
\end{cases}
\]
(ii) \(|T_i| \geq 2\) for some \(i\). Then there exists \(c, d \in T_i\) with \(c < d\) and

\[
\alpha = n_1 n_2 n_3,
\]

a product of three nilpotents, where

\[
n_1 = \begin{pmatrix} A_1 & \cdots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \cdots & A_r \\ x_2 & \cdots & x_{i-1} & c & d & y_{i+1} & \cdots & y_{r-1} \end{pmatrix},
\]

\[
n_2 = \begin{pmatrix} x_2 & \cdots & x_i & c & d & y_{i+1} & \cdots & y_{r-1} \\ c_1 & \cdots & c_{i-1} & c_i & c_{i+1} & c_{i+2} & \cdots & c_r \end{pmatrix},
\]

\[
n_3 = \begin{pmatrix} c_1 & \cdots & c_{i-1} & c_i & c_{i+1} & \cdots & c_r \\ b_1 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_r \end{pmatrix}
\]

and

\[
c^j = \begin{cases} 
\max\{x_{j+1}, b_j\} + 1 & \text{if } 1 \leq j \leq i-1, \\
\max\{c, b_j\} + 1 & \text{if } j = i, \\
\max\{d, b_j\} + 1 & \text{if } j = i+1, \\
\max\{y_{j-1}, b_j\} + 1 & \text{if } j > i+1.
\end{cases}
\]

(d) \(n \in A_r, b_r = n\). Then \(\alpha\) has at least one lower jump of length greater than 1, and either \(A_i \neq S_i\) for some \(i\) or \(|T_i| \geq 2\) for some \(i\). We may assume that the first lower jump of length greater than 1 occurs between \(b_k\) and \(b_{k+1}\). Define

\[
c_j = \begin{cases} 
b_j + 1 & \text{if } 1 \leq j \leq k, \\
b_j - 1 & \text{if } j > k.
\end{cases}
\]

Then

\[
\alpha = n_1 n_2 n_3 n_4,
\]

where

\[
n_1 = \begin{pmatrix} A_1 & \cdots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \cdots & A_r \\ x_2 & \cdots & x_{i-1} & c & d & y_{i+1} & \cdots & y_{r-1} \end{pmatrix},
\]

\[
n_2 = \begin{pmatrix} x_2 & \cdots & x_i & c & d & y_{i+1} & \cdots & y_{r-1} \\ 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & r \end{pmatrix},
\]

\[
n_3 = \begin{pmatrix} 1 & 2 & \cdots & r \\ c_1 & c_2 & \cdots & c_r \end{pmatrix},
\]

\[
n_4 = \begin{pmatrix} c_1 & c_2 & \cdots & c_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.
\]
$c \in S \setminus A_i$ and $d = y_i$ if $A_i \neq S_i$ for some $i$, or $c, d \in T_i$ if $|T_i| \geq 2$ for some $i$ (with $c < d$).

Conversely, suppose that $\alpha$ satisfies condition (i). Without loss of generality we may assume that $\alpha$ is expressible as a product

$$\alpha = n_1 n_2 \ldots n_k$$

of $k$ nilpotents with

$$n_1 = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix}.$$

We must first show by induction that $c_i > y_i$ for all $i$. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leq k$ and that $c_{k+1} < y_{k+1}$. Then since $A_{k+1} = S_{k+1}$ we must have $c_{k+1} < y_{k+1}$. Thus $y_k < c_k < S_{k+1} < y_{k+1}$. But this will mean $|T_{k+1}| \geq 2$, which is a contradiction. So $c_i > y_i$ for all $i$. In particular we have $c_r > y_r = n$, and so $c_r$ does not exist. Hence $\alpha$ is not a product of nilpotents.

Suppose that $\alpha$ satisfies (ii) and $\alpha$ is expressible as a product $\alpha = n_1 n_2 \ldots n_k$ of $k$ nilpotents. We may then assume that

$$n_k = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix},$$

where $\{c_1, \ldots, c_r\} = \text{im} n_{k-1}$. We will begin by showing inductively that $c_i \geq b_i + 1$ for all $i$. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leq k$ and that $c_{k+1} \leq b_{k+1} - 1$. Then since all the lower jumps of $\alpha$ are of length 1 at most, we have $b_{k+1} \leq b_i + 2$. Thus $c_{k+1} \leq b_{k+1} - 1 \leq b_i + 1 \leq c_i$. This is impossible. So $c_i \geq b_i + 1$ for all $i$. In particular we have $c_r \geq b_r + 1 = n + 1$, and so $c_r$ does not exist. Hence $\alpha$ is not a product of nilpotents.

The next result is analogous to Theorem 2.8 in [1].

**Theorem 2.3** The set

$$A = \{ \alpha \in PO_n: |\text{im} \alpha| \leq p \text{ and } |X_n \setminus \text{dom} \alpha| \geq p \}$$

is contained in $\langle N \rangle$ if and only if $p \leq \frac{1}{2} n$.

**Proof.** Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} \in A,$$

and suppose that $p \leq \frac{1}{2} n$. Then by Theorem 2.2, to show that $\alpha \in \langle N \rangle$ we are required to prove the following:
(i) If $1 \in A_i$, $n \in A_r$, then for some $i$ it is the case that $A_i \neq S_i$ or $|T_i| \geq 2$.

(ii) If $b_1 = 1$, $b_r = n$, then $\alpha$ has a lower jump of length greater than 1.

So suppose by way of contradiction that $1 \in A_i$, $n \in A_r$ and that there exists no $i$ for which $A_i \neq S_i$ or $|T_i| \geq 2$. Then $X_n \setminus \text{dom } \alpha = \bigcup_{i=1}^{r-1} T_i$, and

$$r \leq |X_n \setminus \text{dom } \alpha| = \sum_{i=1}^{r-1} |T_i| \leq r - 1 \leq p - 1.$$  

This is a contradiction; thus $\alpha$ satisfies (i).

Now, suppose that $b_1 = 1$, $b_r = n$ and that all lower jumps of $\alpha$ are of length at most 1. Then $j_{\alpha}(\alpha) \leq r - 1 \leq p - 1$. Also $n = b_r = r + j_{\alpha}(\alpha)$ and so

$$j_{\alpha}(\alpha) = n - r \geq n - p \geq p \text{ (since } p \leq \frac{1}{2} n).$$

This is also a contradiction; thus $\alpha$ satisfies (ii).

To complete the proof of the theorem, we now show that if $r > n/2$, then there exists $\alpha \in A$ such that $\alpha \notin \langle N \rangle$.

Consider an element $\alpha$ for which $|\text{im } \alpha| = r \geq n/2 + 1$ and $X_n \setminus \text{im } \alpha = \{2, 4, \ldots, 2s\}$, where $s = n - r$. Then we have

$$2s = 2(n - r) \leq 2n - (n + 2) = n - 2,$$

from which we can conclude that $n \in \text{im } \alpha$, and thus $b_r = n$. It is clear that $b_1 = 1$ and that all lower jumps of $\alpha$ are of length 1. Hence $\alpha$ satisfies condition (ii) in Theorem 3.2. So $\alpha$ is not a product of nilpotents.

3. The depth of the nilpotent-generated subsemigroup

By the proof of Theorem 2.2 we can express $\alpha$ in $\langle N \rangle$ as a product of at most four nilpotents, with elements having $1 \in A_1$, $n \in A_r$, $b_1 = 1$, $b_r = n$ expressible as a product of exactly four nilpotents. As in [1] we now show that even such elements can be expressed as a product of two or three nilpotents.

**Proposition 3.1.** Let $\alpha$ in $\langle N \rangle$ be such that $1 \in A_1$, $n \in A_r$, $b_1 = 1$ and $b_r = n$. Then $\alpha$ is expressible as a product of at most three nilpotents.

**Proof.** By Theorem 2.2 there exists $i$ for which $A_i \neq S_i$ or $|T_i| \geq 2$, and $\alpha$ has a lower jump of length greater than 1. We will assume that the first lower jump of length greater than 1 occurs between $b_k$ and $b_{k+1}$.

Let $c \in S \setminus A_l$ or $c = \min \{x : x \in T_i\}$, and $d \in T_i$ with $d \neq c$. We first show inductively that $c - i + j > y_i$ if $1 \leq j \leq i - 1$ and $c - i + j < x_j$ if $j > i$. The results are true respectively for $j = i - 1$ and $j = i + 1$, since $y_{i-1} < x_i \leq c - 1$ and $c + 1 \leq (y_i$ or $d) < x_{i+1}$. Suppose that they are true (respectively) for $j = s \leq i - 1$ and $j = t > i$; that is, $y_s < c - i + s$ and $x_t > c - i + t$.  

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Then $y_{s-1} \leq y_s - 1 < c - i + s - 1$ and $c - i + t + 1 < x_t + 1 \leq x_t + 1$, as required. Next we show that $b_{k - k + j + 1} > b_j$ if $1 \leq j \leq k$ and $b_{k - k + j + 1} < b_j$ if $j > k$. For $j = k$ and $k + 1$ we have $b_{k + 1} > b_k$ and $b_{k + 2} < b_{k + 1}$. So suppose that the results are true for $j = s \leq k$ and $j = t \geq k + 1$, that is, $b_{k - k + s + 1} > b_s$ and $b_{k - k + t + 1} < b_t$. Then $b_{k - k + s} > b_{s - 1} \geq b_{s - 1}$ and $b_{k - k + t + 2} < b_{t + 1} \leq b_{t + 1}$.

We now distinguish two cases.

**Case 1.** $c - i + k = b_k + 1$. Then $c - i + j = b_k - k + j + 1$ for all $j = 1, \ldots, r$ and

$$\alpha = n_1 n_2,$$

a product of two nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \cdots & A_k & A_{k+1} & \cdots & A_r \\ b_k - k + 2 & \cdots & b_k + 1 & b_k + 2 & \cdots & b_k - k + r + 1 \end{pmatrix},$$

and

$$n_2 = \begin{pmatrix} b_k - k + 2 & \cdots & b_k + 1 & b_k + 2 & \cdots & b_k - k + r + 1 \\ b_1 & \cdots & b_k & b_{k+1} & \cdots & b_r \end{pmatrix}.$$

**Case 2.** $c - i + k \neq b_k + 1$. Then $c - i + j \neq b_k - k + j + 1$ for all $j = 1, \ldots, r$ and

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \cdots & A_k & A_{k+1} & \cdots & A_r \\ c - i + 1 & \cdots & c - i + k & c - i + k + 1 & \cdots & c - i + r \end{pmatrix},$$

$$n_2 = \begin{pmatrix} c - i + 1 & \cdots & c - i + k & c - i + k + 1 & \cdots & c - i + r \\ b_k - k + 2 & \cdots & b_k + 1 & b_k + 2 & \cdots & b_k - k + r + 1 \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} b_k - k + 2 & \cdots & b_k + 1 & b_k + 2 & \cdots & b_k - k + r + 1 \\ b_1 & \cdots & b_k & b_{k+1} & \cdots & b_r \end{pmatrix}.$$

The following Theorem now follows from Proposition 3.1 above and Theorem 3.3 in [1].

**Theorem 3.2.** Let $N$ be the set of all nilpotents in $PO_n$, $\langle N \rangle$ the subsemigroup of $PO_n$ generated by the nilpotent elements, and $\Delta(\langle N \rangle)$ the unique $k$ for which
\[ \langle N \rangle = N \cup N^2 \cup \cdots \cup N^k, \quad \langle N \rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1}. \]

Then \( \Delta(\langle N \rangle) = 3 \) for all \( n \geq 3 \).

4. The nilpotent rank

An element \( \alpha \) in \( PO_n \), and indeed in the larger semigroup \( P_n \) of all partial transformations of \( X_n \), is said to have projection characteristic \((k,r)\) or to belong to the set \([k,r]\) if \(|\text{dom}\, \alpha| = k\) and \(|\text{im}\, \alpha| = r\). We use the standard notation

\[ J_r = \{ \alpha : |\text{im}\, \alpha| = r \} = \bigcup_{r \leq k \leq n} [k,r]. \]

**Lemma 4.1.** Every element \( \alpha \in \langle N \rangle \cap J_r \), where \( r \leq n - 3 \), is expressible as a product of elements in \( \langle N \rangle \cap J_{r+1} \).

**Proof.** Let

\[ \alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \]

be an element in \( \langle N \rangle \) such that \(|\text{im}\, \alpha| = r \leq n - 3\). From Proposition 4.1 in [1], if \( \alpha \in \langle N \rangle \cap [r, r] \) then \( \alpha \) can be expressed as a product of two elements in \( \langle N \rangle \cap [r+1, r+1] \). We will therefore assume that \( \alpha \in \langle N \rangle \cap [k, r] \), \( r + 1 \leq k \leq n - 1 \).

By Theorem 2.2, since \( \alpha \in \langle N \rangle \) then at least one of the following holds:

(i) \( 1 \not\in A_1 \) (that is, \( |T_0| \geq 1 \));

(ii) \( n \not\in A_n \) (that is, \( |T_n| \geq 1 \));

(iii) \( A_i \neq S_i \) for some \( i \) such that \( 1 \leq i \leq r - 1 \);

(iv) \( |T_i| \geq 2 \) for some \( i \) such that \( 1 \leq i \leq r - 1 \).

Suppose that (i) or (ii) or (iv) holds. Then

\[ \alpha = \gamma_1 \gamma_2 \gamma_3, \]

where

\[ \gamma_1 = \begin{pmatrix} A_1 & \cdots & A_{j-1} & x_j & A_j \setminus \{x_j\} & A_{j+1} & \cdots & A_r \\ 1 & \cdots & j-1 & j & j+1 & j+2 & \cdots & r+1 \end{pmatrix}, \]

\[ \gamma_2 = \begin{pmatrix} 1 & \cdots & j-1 & \{j, j+1\} & j+2 & \cdots & r+1 & r+2 \\ 2 & \cdots & j & j+2 & j+3 & \cdots & r+2 & r+3 \end{pmatrix}. \]
\[
\gamma_3 = \begin{pmatrix}
2 & \cdots & j & j+2 & j+3 & \cdots & r+2 \\
b_1 & \cdots & b_{j-1} & b_j & b_{j+1} & \cdots & b_r
\end{pmatrix},
\]
and it is assumed that \( |A_j| \geq 2 \), \( x_j = \min \{ x : x \in A_j \} \). Observe that \( \gamma_3 \in \langle \mathcal{N} \rangle \) by Theorem 2.7 in [1], and that \( \gamma_2 \) is nilpotent by Lemma 2.1. Further, since (i) or (ii) or (iv) holds and \( r+1 \neq n \), it follows from Theorem 2.2 that \( \gamma_1 \in \langle \mathcal{N} \rangle \). Finally, since \( \gamma_3 \in \langle \mathcal{N} \rangle \cap [r, r] \), \( \gamma_3 \) can be expressed as a product of two elements in \( \langle \mathcal{N} \rangle \cap [r+1, r+1] \), by [1, Proposition 4.1]. Thus \( \alpha \) is expressible as a product of (four) elements in \( \langle \mathcal{N} \rangle \cap J_{r+1} \).

Now suppose that (iii) holds: that is, \( A_i \neq S_i \) for some \( i \). Consider first the case where \( k < n - 1 \). Then we may assume that there exists \( x \in \mathcal{X}_n \backslash \text{dom} \alpha \) such that \( y_j < x < y_{j+1} \) for some \( j \), where \( y_j = \max \{ x : x \in A_i \} \). Here we have
\[
\alpha = \beta_1 \beta_2,
\]
where
\[
\beta_1 = \begin{pmatrix}
A_1 & \cdots & A_j & A_{j+1} & \cdots & A_r \\
1 & \cdots & j & j+1 & j+3 & \cdots & r+2
\end{pmatrix},
\]
\[
\beta_2 = \begin{pmatrix}
1 & \cdots & j & j+3 & \cdots & r+2 \\
b_1 & \cdots & b_j & b_{j+1} & \cdots & b_r
\end{pmatrix}.
\]
Observe here too, that \( \beta_2 \) belongs to \( \langle \mathcal{N} \rangle \) and can be expressed as a product of two elements of \( \langle \mathcal{N} \rangle \cap [r+1, r+1] \) by [1, Proposition 4.1]. Also, since \( A_i \neq S_i \) for some \( i \) and \( r+2 \neq n \), we have \( \beta_1 \in \langle \mathcal{N} \rangle \) by Theorem 2.2.

Now consider the case where \( k = n - 1 \). Then it is clear that \( |A_i| \geq 2 \). If \( |A_i| = 2 \) then there exists another block, say \( A_k \), such that \( |A_k| \geq 2 \) (since \( r \leq n - 3 \) by hypothesis), and
\[
\alpha = \delta_1 \delta_2 \delta_3,
\]
where
\[
\delta_1 = \begin{pmatrix}
A_1 & \cdots & A_{k-1} & x_k & A_k \backslash \{ x_k \} & A_{k+1} & \cdots & A_r \\
1 & \cdots & k-1 & k & k+1 & k+2 & \cdots & r+1
\end{pmatrix},
\]
\[
\delta_2 = \begin{pmatrix}
1 & \cdots & k-1 & \{ k, k+1 \} & k+2 & \cdots & r+2 \\
2 & \cdots & k & k+2 & k+3 & \cdots & r+3
\end{pmatrix}
\]
and
\[
\delta_3 = \begin{pmatrix}
2 & \cdots & k & k+2 & k+3 & \cdots & r+2 \\
b_1 & \cdots & b_{k-1} & b_k & b_{k+1} & \cdots & b_r
\end{pmatrix}.
\]
Note that \( \delta_1 \in \langle \mathcal{N} \rangle \) by Theorem 2.2. Also \( \delta_2 \in \langle \mathcal{N} \rangle \) by Lemma 2.1, and \( \delta_3 \) is expressible as the product of two elements in \( \langle \mathcal{N} \rangle \cap [r+1, r+1] \), by [1, Proposition 4.1]. If \( |A_i| > 2 \)
then there exists \( a_i \in A_i \) and \( s_i \in S_i \setminus A_i \) such that either \( x_i < a_i < s_i < y_i \) or \( x_i < s_i < a_i < y_i \). If \( x_i < a_i < s_i < y_i \) then

\[
\alpha = \lambda_1 \lambda_2 \lambda_3,
\]

where

\[
\lambda_1 = \begin{pmatrix}
A_1 & \ldots & A_{i-1} & x_i & A_i \setminus \{x_i\} & A_{i+1} & \ldots & A_r \\
1 & \ldots & i-1 & i & i+1 & i+2 & \ldots & r+1
\end{pmatrix},
\]

\[
\lambda_2 = \begin{pmatrix}
1 & \ldots & i-1 & \{i,i+1\} & i+2 & \ldots & r+2 \\
2 & \ldots & i & i+2 & i+3 & \ldots & r+3
\end{pmatrix}
\]

and

\[
\lambda_3 = \begin{pmatrix}
2 & \ldots & i & i+2 & i+3 & \ldots & r+2 \\
1 & \ldots & b_1 & b_{i-1} & b_i & b_{i+1} & \ldots & b_r
\end{pmatrix}.
\]

If \( x_i < s_i < a_i < y_i \) then

\[
\alpha = \lambda_1 \lambda_2 \lambda_3,
\]

where

\[
\lambda_1 = \begin{pmatrix}
A_1 & \ldots & A_{i-1} & y_i & A_{i+1} & \ldots & A_r \\
1 & \ldots & i-1 & i & i+1 & i+2 & \ldots & r+1
\end{pmatrix},
\]

and where \( \lambda_2 \) and \( \lambda_3 \) are defined as before. Note that by the same argument as in previous cases, \( \lambda_1, \lambda_2, \lambda_3 \in \langle N \rangle \) and \( \lambda_3 \) can be expressed as a product of two elements of \( \langle N \rangle \cap [r+1,r+1] \).

Let \( N_1 \) and \( N_2 \) be the set of all nilpotent elements in \( PO_n \) in \( J_{n-1} \) and in \( J_{n-2} \) respectively. Then, since all the elements in \( N_1 \) are one-one maps, we have by Proposition 4.2 in [1] that \( N_1 \) does not generate \( \langle N \rangle \). However, by Lemma 4.1 above we do have

\[
\langle N_2 \rangle = \langle N \rangle \setminus J_{n-1}.
\]

Our aim here is to determine the rank and the nilpotent rank of \( \langle N_2 \rangle \).

First, notice that from Theorem 2.2 it is easy to verify that \( \langle N \rangle \) is regular. Hence by [6, Proposition II.4.5] two elements of \( \langle N \rangle \) are \( L \)-equivalent in \( \langle N \rangle \) if and only if they have the same image, and are \( S \)-equivalent in \( \langle N \rangle \) if and only if they have the same kernel. This applies also to \( \langle N_2 \rangle = \langle N \rangle \setminus J_{n-1} \), since every element of \( \langle N \rangle \setminus J_{n-1} \) has an inverse in \( \langle N \rangle \setminus J_{n-1} \), and so \( \langle N_2 \rangle \) is again regular.

Now recall from [1, Section 4] that the number of \( S \)-classes and that of \( L \)-classes containing nilpotents, or elements that are expressible as products of nilpotents, in a \( S \)-class, \( J_r \) of \( IO_n \), where \( n/2 < r \leq n-2 \) (notice in passing that \( n/2 < n-2 \) if and only if
n ≥ 5) are both equal to \( \binom{n}{2} - \binom{n-1}{2} \). It therefore follows that the number of \( R \)-classes in \( \langle N_2 \rangle \cap [n-2, n-2] \) is equal to the number of \( \mathcal{L} \)-classes in \( \langle N_2 \rangle \cap J_{n-2} \) and is 
\[
\binom{n}{2} - \binom{n-3}{2} = 3(n-2).
\]

Following [5], we shall refer to an equivalence \( \rho \) on the set \( X_n \) as convex if its classes are convex subsets \( A \) of \( X_n \), where a convex subset of \( X_n \) means a subset \( A \) for which
\[
x, y \in A \text{ and } x \leq z \leq y \Rightarrow z \in A.
\]

By Theorem 2.2 any convex equivalence having \( n-2 \) classes on the subset \( \{1, \ldots, n-1\} \) or \( \{2, \ldots, n\} \) determines an \( R \)-classes in \( \langle N_2 \rangle \cap [n-1, n-2] \). Thus the number of \( R \)-classes in \( \langle N_2 \rangle \cap [n-1, n-2] \) determined by these convex equivalences is \( 2(n-2) \).

On the other hand any convex equivalence having \( n-2 \) classes on a subset containing 1 and \( n \) represents an \( R \)-class in \( \langle N_2 \rangle \cap [n-1, n-2] \) if and only if \( i \) and \( i+2 \) belong to the same equivalence class for some \( i \) in \( \{1, \ldots, n-2\} \). This follows from Theorem 2.2, because \( |T| \geq 2 \) is not possible for an element of \( [n-1, n-2] \) and so the only possibility for such an \( \alpha \) to be in \( \langle N \rangle \) is for some \( A_i \) to be distinct from \( S_i \). Thus the number of such convex equivalences is \( n-2 \). Hence the number of \( R \)-classes in \( \langle N_2 \rangle \cap [n-1, n-2] \) is \( 3(n-2) \). We therefore have \( 6(n-2) \) as the number of \( R \)-classes in \( \langle N_2 \rangle \cap J_{n-2} \).

We now show that every element \( \alpha \in \langle N_2 \rangle \cap [n-1, n-2] \) is expressible in terms of a fixed element in its own \( R \)-class and an element in \( \langle N_2 \rangle \cap [n-2, n-2] \). More generally we shall show:

**Lemma 4.2.** Every element \( \alpha \in \langle N_2 \rangle \cap [k, r] \), \( r < k \leq n-1 \) is expressible as a product of a nilpotent in \( \langle N_2 \rangle \cap [k, r] \) and an element in \( \langle N_2 \rangle \cap [r, r] \).

**Proof.** Let \( \alpha \in \langle N_2 \rangle \cap [k, r] \) and suppose that
\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.
\]

We shall distinguish four cases.

**Case 1.** \( 1 \notin A_1 \). Then
\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ 1 & 2 & \cdots & r \end{pmatrix} \begin{pmatrix} 1 & 2 \cdots & r \\ b_1 & b_2 \cdots & b_r \end{pmatrix}.
\]

**Case 2.** \( n \notin A_r \). Then
\[
\alpha = \beta \gamma
\]
where
\[
\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_{r-1} & A_r \\ n-r+1 & n-r+2 & \cdots & n-1 & n \end{pmatrix}.
\]
That $\gamma \in \langle N \rangle$ follows from [1, Theorem 2.6].

**Case 3.** $1 \in A_1$, $n \in A_r$ and $A_i \not= S_i$ for some $i$. Let $c$ be a fixed element in $S_i \setminus A_i$. Then

$$\gamma = \begin{pmatrix} n-r+1 & n-r+2 & \cdots & n-1 & n \\ b_1 & b_2 & \cdots & b_{r-1} & b_r \end{pmatrix}.$$ 

where

$$\lambda = \begin{pmatrix} A_1 & \cdots & A_{i-1} & A_i & A_{i+1} & \cdots & A_r \\ x_2 & \cdots & x_i & c & y_i & \cdots & y_{r-1} \end{pmatrix},$$

$$\mu = \begin{pmatrix} x_2 & \cdots & x_i & c & y_i & \cdots & y_{r-1} \\ b_1 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_r \end{pmatrix}.$$ 

The latter element is in $\langle N \rangle$ by [1, Theorem 2.6].

**Case 4.** $1 \in A_1$, $n \in A_r$, $A_i = S_i$ for all $i$ and $|T_i| \geq 2$ for some $i$. Let $c$, $d$ be two fixed elements in $S_i$ with $c < d$. Then

$$\alpha = \zeta \xi$$

where

$$\zeta = \begin{pmatrix} A_1 & \cdots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \cdots & A_r \\ y_2 & \cdots & y_i & c & d & y_{i+1} & \cdots & y_{r-1} \end{pmatrix},$$

$$\xi = \begin{pmatrix} y_2 & \cdots & y_i & c & d & y_{i+1} & \cdots & y_{r-1} \\ b_1 & \cdots & b_{i-1} & b_i & b_{i+1} & b_{i+2} & \cdots & b_r \end{pmatrix}.$$ 

\[\square\]

**Theorem 4.3.** Let $n \geq 5$. Then $\text{rank } (\langle N_2 \rangle) = 6(n-2)$.

**Proof.** Since $\langle N_2 \rangle \cap J_{n-2}$ has $6(n-2)$ $\mathcal{R}$-classes we have

$$\text{rank } (\langle N_2 \rangle) \geq 6(n-2).$$

By Proposition 2.4 in [2], $[n-2, n-2] \cap \langle N_2 \rangle$ is generated by a set of $3(n-2)$ elements. If we now choose a set of $3(n-2)$ elements to cover the $\mathcal{R}$-classes in $[n-1, n-2]$ as in Lemma 4.2, we obtain a generating set of $\langle N_2 \rangle$ consisting of $6(n-2)$ elements. The result follows. \[\square\]

**Lemma 4.4.** Every $\mathcal{L}$-class in $J_{n-2}$ whose elements have image
for \( i = 2, \ldots, n - 2 \) contains a single nilpotent. Thus there are at least \( n - 3 \) \( L \)-classes in \( J_{n-2} \) containing only one nilpotent.

**Proof.** Let \( \alpha \) be an element whose \( L \)-class is represented by \( \{1, \ldots, i-1, i+2, \ldots, n\} \). Then the only domain for which \( \alpha \) is nilpotent is that represented by the set \( \{2, \ldots, n-1\} \).

**Theorem 4.5.** \( \text{nilrank}(\langle N_2 \rangle) = 7n - 15. \)

**Proof.** Since any generating set of \( \langle N_2 \rangle \) must cover the \( L \)-classes in \( \langle N_2 \rangle \cap J_{n-2} \), the \( n - 3 \) nilpotents whose image set is \( \{1, \ldots, i-1, i+2, \ldots, n\} \) for \( i = 2, \ldots, n-2 \) must be contained in a generating set consisting of only nilpotent elements (see Lemma 4.4). By the same Lemma 4.4, (proof) all the \( n - 3 \) nilpotents belong to the same \( R \)-class, determined by the set \( \{2, \ldots, n-1\} \). For the generating set to cover all the \( R \)-classes we must now choose \( 6(n-2) - 1 \) nilpotents from the remaining \( R \)-classes, making a total of \( 7n - 16 \) nilpotents. However the \( 7n - 16 \) nilpotents cannot generate \( \langle N_2 \rangle \). For if \( \alpha \) is an element in the same \( R \)-class as the \( n - 3 \) nilpotents (that is the \( R \)-class represented by the set \( \{2, \ldots, n-1\} \)) and if we suppose that

\[
\alpha = n_1 n_2 \cdots n_k
\]

is the decomposition of \( \alpha \) in terms of nilpotents from the chosen \( 7n - 16 \) nilpotents, then we must have

\[
n_1 = \begin{pmatrix} 2 & 3 & \ldots & i & i+1 & \ldots & n-1 \\ 1 & 2 & \ldots & i-1 & i+2 & \ldots & n \end{pmatrix},
\]

\[
n_2 = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i+2 & \ldots & n \\ 2 & 3 & \ldots & i & i+1 & \ldots & n-1 \end{pmatrix}
\]

and

\[
n_3 = \begin{pmatrix} 2 & 3 & \ldots & j & j+1 & \ldots & n-1 \\ 1 & 2 & \ldots & j-1 & j+2 & \ldots & n \end{pmatrix}
\]

for some \( i, j = 2, \ldots, n - 2 \). But then \( n_1 n_2 \) is a left identity for \( n_3 \), and so

\[
\alpha = n_3 n_4 \cdots n_k.
\]

By the same reasoning we must also have

\[
n_4 = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j+2 & \ldots & n \\ 2 & 3 & \ldots & j & j+1 & \ldots & n-1 \end{pmatrix}
\]
and

\[
n_5 = \begin{pmatrix}
2 & 3 & \ldots & l & l+1 & \ldots & n-1 \\
1 & 2 & \ldots & l-1 & l+2 & \ldots & n
\end{pmatrix},
\]

But again \(n_3n_4\) is then a left identity for \(n_5\), and

Continuing this way we obtain

\[
\alpha = n_5 \cdots n_k.
\]

Thus if \(\alpha\) is not any of the \(n-3\) nilpotents in its \(R\)-class, and is not the left identity in the \(R\)-class, then \(\alpha\) cannot be expressed as a product of nilpotents from the chosen \(7n-16\) nilpotents. We therefore have

\[
nilrank(\langle N_2 \rangle) \geq 7n-15.
\]

We now show that we can choose \(7n-15\) nilpotents in \(N_2\) that can generate \(\langle N_2 \rangle\). Denote by \(A_{i,j}\) the subset \(X_n \setminus \{i, j\}\) of cardinality \(n-2\), and by \(\alpha_{i,j}^k\) the element whose domain is \(A_{i,j}\) and image \(A_{i,j}\). Then arrange the \(3(n-2)\) subsets of \(X_n\) of cardinality \(n-2\), representing the \(2\)- and the \(R\)-classes in \(\langle N_2 \rangle \cap [n-2, n-2]\) as follows:

\[
A_2, n, A_1, 3, A_3, n, \ldots, A_{1, i}, A_{i, n-1}, A_{n-1, i}, A_{1, n}, A_{n, 2}, A_{2, 3}, A_3, 4, \ldots, A_{n-2, n-1}, A_{1, 2}.
\]

By [2, Proposition 2.4], \(\langle N_2 \rangle \cap [n-2, n-2]\) is generated by the set

\[
B = \{\alpha_{2, 3}^n, \alpha_{3, 4}^n, \alpha_{1, i}^n, \alpha_{i, i+1}^n, \alpha_{i+1, i}^n \cdots, \alpha_{n-1, n}^n, \alpha_{1, n}^n, \alpha_{n, 1}^n, \alpha_{2, n}^n, \alpha_{n-2, n-1}^n, \alpha_{n-1, 2}^n, \alpha_{n-2, 2}^n \}
\]

It is easy to see that \(\alpha_{i, i}^n, \alpha_{i+1, i}^n\) (for \(i = 3, \ldots, n-1\)), \(\alpha_{2, 3}^n, \alpha_{2, n}^n\) and \(\alpha_{2, 2}^n\) are all nilpotents. It is also not difficult to see that

\[
\alpha_{2, 3}^n, \ldots, \alpha_{n-3, n-2}^n, \alpha_{n-2, 2}^n, \alpha_{1, 2}^{n-1, n}
\]

are all non-nilpotent. In fact \(n\) is fixed by all of these elements. Let us denote by \(B'\) the set of all nilpotent elements in \(B\). Let \(T\) be the set of \(4(n-2)-1\) elements given by

\[
T = B' \cup \{\alpha_{3, 4}^n, \ldots, \alpha_{n-2, n-1}^n, \alpha_{1, 2}^n, \alpha_{2, 1}^n, \ldots, \alpha_{n-2, n-1}^{n-1}\}.
\]

It is easy here too, to see that all the elements in \(T\) are nilpotents. Next we observe that
the non-nilpotent elements in \( B \), given by (4.6) are expressible as products of elements in \( T \). In fact we have

\[
\alpha_{i+1,i+2}^{i+1} = \alpha_{i+1,i+2}^{i} \alpha_{i+1,i+2}^{i+1} \quad \text{for } i = 2, \ldots, n-3
\]

and

\[
\alpha_{1,2}^{n-2,n-1} = \alpha_{1,2}^{n-2,n-1} \alpha_{1,2}^{n-2,n-1}.
\]

Thus

\[
\langle B \rangle = \langle T \rangle.
\]

If we now choose a set \( H \) of \( 3(n-2) \) nilpotents to cover the \( \mathcal{R} \)-classes in \( \langle N_2 \rangle \cap [n-1,n-2] \) as in Lemma 4.2 we obtain a generating set \( H \cup T \) of \( \langle N_2 \rangle \) consisting of nilpotent elements. Since \( |H \cup T| = 7n - 15 \) the proof is complete. 

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REFERENCES


