CONVEX SUM OF UNIVALENT FUNCTIONS

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1. Introduction

Let $f(z) = z + \cdots$ be regular in the unit disc $|z| < 1$ (hereafter called E). In a recent paper Trimble [7] has proved that if $f(z)$ be convex in E, then $F(z) = (1-\lambda)z + \lambda f(z)$ is starlike with respect to the origin in E for $(2/3) \leqq \lambda \leqq 1$. The purpose of this note is to show that if certain additional restrictions be imposed on $f(z)$, then $F(z)$ becomes starlike for all λ, $0 \leqq \lambda \leqq 1$. Also we consider some related problems.

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THEOREM 1. *If $f(z)$ be convex in E, then*

$$F(z) = \frac{2\lambda}{z}\int_0^z f(z)\,dz + (1-\lambda)z \tag{1}$$

is starlike w.r.t. the origin in E for all λ, $0 \leqq \lambda \leqq 1$.

PROOF. Let

$$g(z) = \frac{2}{z}\int_0^z f(z)\,dz. \tag{2}$$

Then it is known that $g(z)$ is also convex in E [2]. From (1), we have

$$\frac{zF'(z)}{F(z)} = \frac{\mu z + 2f(z) - g(z)}{\mu z + g(z)} \tag{3}$$

where $\mu = (1-\lambda)/\lambda$. Now

$$\frac{zF'(z)}{F(z)} - 1 = \frac{2(f(z) - g(z))}{\mu z + g(z)}$$

and

$$\frac{zF'(z)}{F(z)} + 1 = \frac{2(\mu z + f(z))}{\mu z + g(z)}.$$

For $F(z)$ to be starlike w.r.t. the origin in E, it is both necessary and sufficient that $\operatorname{Re}(zF'(z)/F(z)) > 0$ for all z in E. This condition is satisfied if

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < \left|\frac{zF'(z)}{F(z)} + 1\right|,$$

for all z in E. The above condition in our case is equivalent to

$$(4) \qquad \left|1 - \frac{g(z)}{f(z)}\right| < \left|1 + \frac{\mu z}{f(z)}\right|.$$

From (2), we have

$$\frac{f(z)}{g(z)} = \frac{1}{2}\left[\frac{zg'(z)}{g(z)} + 1\right].$$

Since $g(z)$ is convex (and in particular, starlike w.r.t. the origin in E), therefore $\operatorname{Re}(zg'(z)/g(z)) > 0$. Consequently

$$\operatorname{Re}(f(z)/g(z)) > \tfrac{1}{2},$$

for all z in E. This is equivalent to

$$(5) \qquad \left|\frac{g(z)}{f(z)} - 1\right| < 1,$$

for all z in E. Also $f(z)$ being convex, $\operatorname{Re}(f(z)/z) > \frac{1}{2} > 0$ [6] for all z in E. Therefore

$$(6) \qquad \left|1 + \frac{\mu z}{f(z)}\right| \geqq \operatorname{Re}\left(1 + \frac{\mu z}{f(z)}\right) \geqq 1,$$

for z in E. From (5) and (6), it follows that (4) is satisfied for all z in E.

REMARK. To prove the above theorem, we have in fact made use of much weaker assumptions, viz., (i) $f(z)$ is starlike, and (ii) $\operatorname{Re}(f(z)/z) > 0$. For if $f(z)$ be starlike in E then $g(z)$ is also starlike in E [2].

The following simple theorem leads to interesting results.

THEOREM 2. *If $f(z)$ be starlike and $\operatorname{Re} f'(z) > 0$ for z in E, then*

$$(7) \qquad F(z) = (1-\lambda)z + \lambda f(z)$$

is starlike and $\operatorname{Re} F'(z) > 0$ for z in E.

PROOF. From (7), we have

$$(8) \qquad \frac{zF'(z)}{F(z)} = \frac{\mu z + zf'(z)}{\mu z + f(z)}$$

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$$= \frac{\mu}{\mu + \frac{f(z)}{z}} + \frac{1}{\frac{\mu}{f'(z)} + \frac{f(z)}{zf'(z)}},$$

where $\mu = (1-\lambda)/\lambda$. Since $\mathrm{Re} f'(z) > 0$ for z in E, therefore $\mathrm{Re}(f(z)/z) > 0$ for z in E [5]. Making use of this and the given facts, it is now easy to see that $\mathrm{Re}(zF'(z)/F(z)) > 0$ for all z in E.

COROLLARY 1. *If $f(z)$ be convex in E, then*

$$F(z) = (1-\lambda)z + \lambda \int_0^z \frac{f(z)}{z} dz$$

is starlike in E for all λ, $0 \leqq \lambda \leqq 1$. Also $F(z)$ is univalent in E for all λ, $0 \leqq \lambda \leqq 2$.

The corollary follows from Theorem (2) on writing

$$g(z) = \int_0^z \frac{f(z)}{z} dz$$

and noting that $\mathrm{Re}\, g'(z) = \mathrm{Re}(f(z)/z) > \frac{1}{2} > 0$ [6] for z in E. The last statement in the above corollary follows from a result of Noshiro [3], on noting that $\mathrm{Re}\, F'(z) > 0$ in E for all λ, $0 \leqq \lambda \leqq 2$.

From Trimble's result [7] it follows that $\{\lambda/n\}$ is a convexity preserving sequence (For the definition of c.p. (convexity preserving) sequence, see, for example [1]) for $\lambda \geqq 2/3$, whereas from corollary 1, it follows that $\{\lambda/n^2\}$ is a c.p. sequence for all λ, $0 \leqq \lambda \leqq 1$. Combining this result with the well-known fact that $\{l/n\}$ is c.p. sequence, it follws that $\{\lambda/n^p\}$ is c.p. sequence for all λ, $0 \leqq \lambda \leqq 1$, and for all $p \geqq 2$.

COROLLARY 2. *If $f(z)$ be an odd convex function in E, then*

$$F(z) = (1-\lambda)z + \lambda f(z)$$

is starlike in E for all λ, $0 \leqq \lambda \leqq 1$. Also $F(z)$ is univalent in E for all λ, $0 \leqq \lambda \leqq 2$.

LEMMA. *If $f(z)$ be an odd convex function in E, then* $\mathrm{Re} f'(z) > \frac{1}{2}$ *for z in E.*

PROOF. Let $h(z) = z + \cdots$ be regular in E and $g(z) = (h(z^2))^{\frac{1}{2}}$. Then $g(z)$ is an odd starlike function in E if and only if $h(z)$ is starlike in E. For a starlike function $h(z)$, we have $\mathrm{Re}(h(z)/z)^{\frac{1}{2}} > \frac{1}{2}$ [6] for all z in E. Therefore $\mathrm{Re}(g(z)/z) = \mathrm{Re}(h(z^2)/z^2)^{\frac{1}{2}} > \frac{1}{2}$ for z in E. Now the lemma follows on noting that $f(z)$ is an odd convex function in E if and only if $zf'(z)$ is an odd starlike function in E.

The corollary 2 follows from the above lemma and Theorem 2. The last statement in the corollary follows from a result of Noshiro [3] on noting that $\mathrm{Re}\, F'(z) > 0$ for z in E for all λ, $0 \leqq \lambda \leqq 2$.

COROLLARY 3. *If $f(z)$ be starlike in E, then*

$$F(z) = (1-\lambda)z + \lambda \int_0^z (f(z)/z)^\alpha dz$$

is starlike for all λ, $0 \leqq \lambda \leqq 1$, *and for all* α, $0 \leqq \alpha \leqq \frac{1}{2}$.

PROOF. Let $g(z) = \int_0^z (f(z)/z)^\alpha dz$. Then it is easy to see that $g(z)$ is convex in E. Also $f(z)$ being starlike, we have $\operatorname{Re}(f(z)/z)^{\frac{1}{2}} > \frac{1}{2}$, from which it follows that $|\arg(f(z)/z)| \leqq \pi\alpha \leqq \pi/2$ for $0 \leqq \alpha \leqq \frac{1}{2}$. Thus we see that $\operatorname{Re} g'(z) > 0$ for z in E. Now the corollary follows from Theorem 2.

Let $\bar{S}$ denote the class of starlike functions $f(z)$ which are regular and starlike in E and satisfy the condition $|(zf'(z)/f(z)) - 1| < 1$ for z in E. This class has been studied by one of the authors [4].

THEOREM 3. *If $f(z)$ belongs to $\bar{S}$, then*

$$F(z) = (1-\lambda)z + \lambda f(z) \tag{9}$$

belongs to $\bar{S}$ for all λ, $0 \leqq \lambda \leqq 1$.

PROOF. The function $f(z)$ belongs to $\bar{S}$ if and only if $f(z)$ has the representation

$$\log(f(z)/z) = \int_0^z \phi(t)\,dt, \tag{10}$$

where $\phi(z)$ is regular and $|\phi(z)| \leqq 1$ for z in E [4]. From (10), we have

$$\begin{aligned} |\arg(f(z)/z)| &= |\operatorname{Im}\log(f(z)/z)| \\ &\leqq \left|\int_0^z \phi(t)\,dt\right| \\ &\leqq r, \end{aligned}$$

whence $\operatorname{Re}(f(z)/z) > 0$ for z in A.

Now from (9), we have

$$\left|\frac{zF'(z)}{F(z)} - 1\right| \leqq \frac{\left|\dfrac{zf'(z)}{f(z)} - 1\right|}{\left|\dfrac{\mu z}{f(z)} + \right|} < 1,$$

Since $|(zf'(z)/f(z)) - 1| < 1$ and $\operatorname{Re}(f(z)/z) > 0$ for z in E.

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