

Estimation of Signal and Noise Parameters from CMB Polarization Observations

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Abstract. We propose a technique for determination of the spectral parameters of the cosmological signal and pixel noise using observational data on CMB polarization without any additional assumptions. We introduce the notion of so called crossing points in the observational maps and derive the theoretical dependence of the total number of crossing points at different level of the polarization. Finally, we use the statistics of the signal in the vicinities of the singular points, where the polarization of the pure CMB vanishes to correct the final result.

1. Introduction

Recent observational data by BOOMERANG and MAXIMA-1 (De Bernardis et al. 2000, Hanany et al. 2000) opened a new epoch in the investigation of the CMB power spectrum at large multipole numbers l . The angular power spectrum of the CMB (in particular the positions and amplitudes of the first, second and subsequent Doppler peaks) provides a unique chance to construct the most likely cosmological model. This model includes information about the Hubble constant, baryonic and CDM densities, cosmological constant Ω_Λ , ionisation history and so on. In the coming years the measurements of the angular anisotropy of CMB by satellite missions will provide CMB maps with high resolution and sensitivity.

In addition to the anisotropy of the intensity it is possible, though more difficult, to measure polarization of the radiation. Polarization is a secondary effect induced by the scattering of anisotropic radiation on electrons in the cosmic plasma. Importance of the polarization measurements of the relict radiation was pointed out by Rees (1968) and this problem has since been discussed in many papers. Polarization contains an addition to the anisotropy information about the nature of primordial cosmological perturbations and different types of foregrounds. The polarization field is a combination of two randomly distributed Stokes parameters (Q and U) while the anisotropy is just a scalar. In particular, this field is quite sensitive to the presence of tensor perturbations and a deviation

from zero of the so called pseudo scalar or 'magnetic' part of polarization would be an indicator of gravitational waves or vector perturbations.

Analogously to the anisotropy of the CMB, one of the major problems in the future analysis of polarization maps is the separation of the noise from the original cosmological signal. Most of the denoising techniques require significant assumptions about expected signal and noise to be made before the data analyzed. One example of such technique is Wiener filtering (Tegmark and Efstathiou 1996, Bouchet and Gispert 1999). We would like to focus our attention on the following problem: is it possible to find the spectral parameters of at least some kind of noise using the observational data without any additional assumptions. In this case we could use the real spectral parameters instead of the assumed one for the subsequent filtering. We use geometrical and statistical properties of the CMB polarization field for the following purposes:

1. To find the parameters of the signal and pixel noise;
2. To detect noise in the regions of the map where polarization vanishes.

For solving the first problem we suggest investigation of so called up-crossing and down-crossing points of the modulus of polarization as well as Q and U components separately at different levels in the pixelized map. For the pure CMB signal these points are situated along the isopolarization lines and their number is proportional to the total length of these lines. In this case the length of such lines is known analytically. In the presence of pixel noise these 'lines' become wider and are completely destroyed (look like spots) in the vicinities of zero points where the signal is smaller than the noise. The analytical formula (derived by us) for the number of these points in the case of the presence of Gaussian CMB signal and pixel noise gives us a unique possibility to find the spectral parameters of signal and noise with high accuracy.

The second part of our investigation is the natural generalization of the first one. We show that it is useful to study the singular points in polarization where polarization is vanishing. Such singular points of polarization have the common property that in the vicinity of each point the polarization field is formed mainly by a small scale noise (pixel noise and (or) point sources). Noise could manifest itself due to influence on the weak CMB signal in the vicinities of non-polarized points.

2. General properties of the polarization field

Here we will describe some general properties of CMB polarization and present the necessary formalism. We make a simplifying assumption that the relevant angular scales are sufficiently small, so that the corresponding part on the sky is almost flat. In this approximation the polarization field on the sky can be considered as a two dimensional field on the (x, y) -plane. Since Thompson scattering does not produce circular polarization, the resulting field can be completely described in terms of two Stokes parameters Q and U . Without loss of generality we can consider a cosmological model with scalar perturbations only. In this case parameters Q and U are determined by a single scalar field ϕ . These parameters can be written in the following form:

$$\begin{aligned} Q &= \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}, \\ U &= 2 \frac{\partial^2 \phi}{\partial x \partial y}, \end{aligned} \quad (1)$$

where $\phi(x, y)$ is supposed to be in the form of a random Gaussian field. For further investigations we have to introduce the spectral parameters as follows:

$$\begin{aligned} \sigma_0^2 &= \langle Q^2 \rangle = \langle U^2 \rangle, \\ \sigma_1^2 &= \langle Q_x^2 \rangle = \langle U_x^2 \rangle = \langle Q_y^2 \rangle = \langle U_y^2 \rangle. \end{aligned} \quad (2)$$

Using these terms one can write the joint probability distribution functions (PDF) for the fields Q , U , P ($P = \sqrt{Q^2 + U^2}$) and their first derivatives in x -direction (see for details (Naselsky and D. Novikov, 1998)). Since PDF for Q and U are identical, we restrict ourself to theoretical investigation of Q and P fields only:

$$\begin{aligned} F_Q(Q, Q_x) dQ dQ_x &= \frac{1}{2\pi\sigma_0\sigma_1} e^{-\frac{Q^2}{2\sigma_0^2}} e^{-\frac{Q_x^2}{2\sigma_1^2}} dQ dQ_x, \\ F_P(P, P_x) dP dP_x &= \frac{P}{\sqrt{2\pi}\sigma_0^2\sigma_1} e^{-\frac{P^2}{2\sigma_0^2}} e^{-\frac{P_x^2}{2\sigma_1^2}} dP dP_x. \end{aligned} \quad (3)$$

Let us consider the behavior of a smooth continuous two-dimensional random field $f(x, y)$ in the direction x with y kept fixed. We define the crossing point as the point where this field crosses some threshold ν . In the small vicinity of this point $df = f_x dx$ and $f = \nu + f_x \Delta x$. Therefore, the probability Δ to find such a point between x and $x + \Delta x$ is:

$$\begin{aligned} \Delta^+ &= \Delta x \int_0^{+\infty} F(\nu, f_x) f_x df_x && \text{up - crossing,} \\ \Delta^- &= -\Delta x \int_{-\infty}^0 F(\nu, f_x) f_x df_x && \text{down - crossing,} \\ \Delta &= \Delta^+ + \Delta^- = \Delta x \int_{-\infty}^{+\infty} F(\nu, f_x) |f_x| df_x. \end{aligned} \quad (4)$$

Using equations (3,4) we can easily find density of crossing points for Stokes parameters Δ_Q and for the modulus of polarization Δ_P :

$$\begin{aligned} \Delta_q &= \frac{1}{2\pi} \frac{\Delta x}{r_c} e^{-\frac{q^2}{2}}, \\ \Delta_p &= \frac{p}{\sqrt{2\pi}} \frac{\Delta x}{r_c} e^{-\frac{p^2}{2}}. \end{aligned} \quad (5)$$

Here, we use dimensionless values $q = Q/\sigma_0$, $p = P/\sigma_0$ and $r_c = \sigma_0/\sigma_1$ is the correlation radius.

In the two dimensional map these points are along the isolines of q or p respectively and the density of such points is proportional to the total length of these lines in the map (fig. 1). This is one of the so called Minkowski functionals (see Schmalzing and Gorski, 1997).

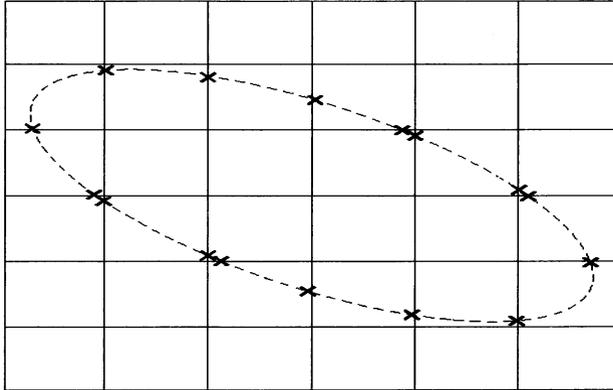


Figure 1. Crossing points in the pixelized map for the smooth field ($h \ll r_c$). Area inside the ellipse corresponds to the region, where $f > \nu$.

3. CMB polarization and noise in the pixelized map.

3.1. Definitions

The real observational datasets have a pixelized form. This means that we should consider the field $f(x,y)$ which is defined at the points x_i, y_j . Without loss of generality one can use rectangular map $N \times N$ pixels with distance h between them: $x_{i+1} - x_i = h$. In this case Δx in the formulae (5) should be replaced by h . The field f crosses some threshold ν between two neighbor pixels (i,j) and $(i+1,j)$ if $f_{i,j} < \nu$ and $f_{i+1,j} > \nu$ (up-crossing) or $f_{i,j} > \nu$ and $f_{i+1,j} < \nu$ (down-crossing). The position of the crossing point x_{cr} can be defined by the linear interpolation of the field between two neighboring pixels:

$$\begin{aligned} x_{cr} &= x_i + h \frac{\nu - f_{i,j}}{f_{i+1,j} - f_{i,j}}, \\ y_{cr} &= y_j. \end{aligned} \tag{6}$$

Analogously we find the positions of crossing points along the y-direction. Finally, we construct the map with the set of crossing points that are placed on the grid lines (fig.1). This set of points obviously form lines of the same level for the field (if this field is smooth enough, namely if $h \ll r_c$ or (the same) $h\sigma_1 \ll \sigma_0$).

The total number of these points in the map is:

$$N_{cr} = 2\Delta N^2, \tag{7}$$

where the 2 occurs in right hand side is because we use two directions for each pixel.

3.2. Statistics of crossing points for signal + noise

We consider uncorrelated Gaussian pixel noise independently occurs in both components of polarization: Q and U with zero mean and variance δ_o . The resulting signal in each pixel can be described as follows:

$$\begin{aligned} Q &= Q_s + Q_n, \\ U &= U_s + U_n, \end{aligned} \tag{8}$$

where indices s and n are for the signal and the noise respectively. Therefore, one part of the signal is strongly correlated from pixel to pixel (CMB) and another part (pixel noise) is completely uncorrelated. The Q and U components of polarization obey the following relations:

$$\begin{aligned} \langle Q^2 \rangle &= \langle U^2 \rangle = \sigma_o^2 + \delta_o^2, \\ \langle (Q_2 - Q_1)^2 \rangle &= \langle (U_2 - U_1)^2 \rangle = h^2 \sigma_1^2 + \delta_o^2, \\ h\sigma_1 &\ll \sigma_o, \end{aligned} \tag{9}$$

where 1,2 denotes the values of Q and U in two neighbor pixels along one of the grid lines. It is useful to introduce parameters: $a = \delta_o / \sqrt{\sigma_o^2 + \delta_o^2}$ and $b = h\sigma_1 / \sqrt{\sigma_o^2 + \delta_o^2}$. We again use the dimensionless values: $q = Q / \sqrt{\sigma_o^2 + \delta_o^2}$, $p = P / \sqrt{\sigma_o^2 + \delta_o^2}$. The probability, that the space between two neighbor pixels contains up-crossing or down-crossing point is equal to the following integrals:

$$\begin{aligned} \Delta_q &= 2 \int_{-\infty}^q dq_1 \int_q^{+\infty} dq_2 F_q(q_1, q_2), \\ \Delta_p &= 2 \int_0^p dp_1 \int_p^{+\infty} dp_2 F_p(p_1, p_2) \end{aligned} \tag{10}$$

for q and p values correspondingly. F_q and F_p are joint probability distribution functions for q and p values in two neighbor pixels (1 and 2). Eq(10) has two obvious asymptotics. If noise is much less than the signal ($a \ll b \ll 1$), then it is useful to make the substitution $\bar{q} = (q_1 + q_2)/2$, $q_x = (q_2 - q_1)/b$ and $\int d\bar{q} \bar{F}_q(\bar{q}, q_x) = b q_x \bar{F}_q(q, q_x)$. Analogous result is, of course, for the p field. Finally, we get the same formulae as in the previous subsection for the pure signal.

On the other hand, if the noise is much bigger than the signal ($a \approx 1$), then random values in neighbor pixels are independent:

$$\begin{aligned} F_q(q_1, q_2) &= \frac{1}{2\pi} e^{-\frac{q_1}{2}} e^{-\frac{q_2}{2}}, \\ F_p(p_1, p_2) &= p_1 p_2 e^{-\frac{p_1}{2}} e^{-\frac{p_2}{2}}, \end{aligned} \tag{11}$$

and we get a very simple result:

$$\begin{aligned} \Delta_q &= 2(1 - \Phi(\frac{q}{\sqrt{2}}))\Phi(\frac{q}{\sqrt{2}}), \\ \Delta_p &= 2(1 - e^{-\frac{p^2}{2}})e^{-\frac{p^2}{2}}. \end{aligned} \tag{12}$$

The result of integration in (10) is quite complicated and can be found in the appendix of the paper (Naselsky et al. 2000). In (fig. 2) we demonstrate the number of crossing points for p values as a function of the level for different values of a ($b=0.07$).

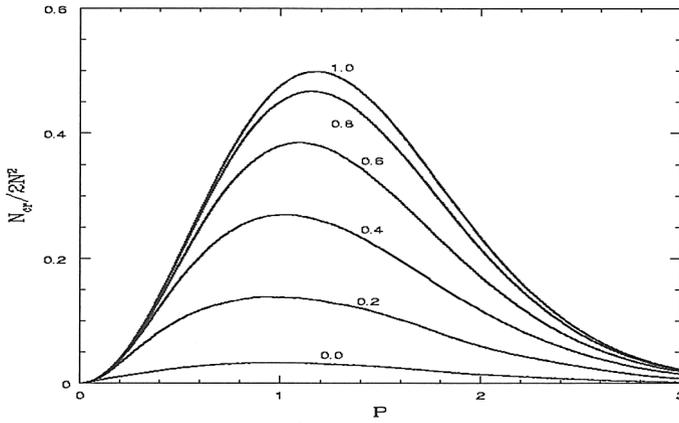


Figure 2. Number of crossing points in the pixelized map divided by the total number of pixels as a function of level. Numbers indicate curves for different values of a .

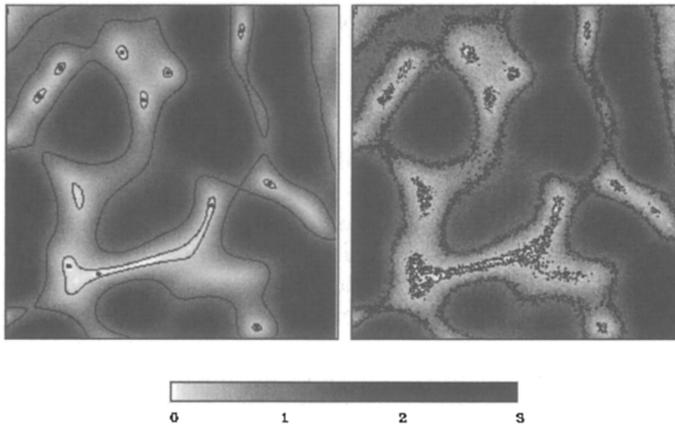


Figure 3. Left panel: pure CMB signal. Right panel: signal+noise. Small points are crossing points at the levels $p = 0.2$ and $p = 1$. Shaded circles show the positions of non-polarized points for the pure CMB signal.

We have simulated $5^\circ \times 5^\circ$, 256×256 pixels map of CMB polarization for the standard CDM cosmological model and the same map with 10% of the noise/signal ratio ($\delta_0/\sigma_0 = 0.1$) (fig. 3). In the second map one can see the 'non-zero width' of the isopolarization lines. Number of crossing points is definitely higher for the map with the noise.

Finally, we suggest using the total number of crossing points in the observational map for different levels in order to construct the best fit line (Eq(10)) with parameters a and b. Therefore, we can find parameters of the signal (σ_0 , σ_1) and noise (δ_0) before the subsequent filtering.

3.3 Non-polarized points in the map

Non-polarized (or singular) points in the map are the points where both components Q and U of a pure CMB signal are equal to zero. These points of polarization are a natural part of the geometrical structure for the CMB signal and their total number N_{np} in the map is $\approx S/r_c^2$, where S is the total area of the map (see for details (Naselsky and Novikov 1998)). Such points can be of three different types: saddles, comets and beaks. Concentrations of singular points of different types in case of a Gaussian signal are $0.5N_{np}$, $0.04N_{np}$ and $0.46N_{np}$ for saddles, beaks and comets correspondingly. Therefore, they can provide the statistical information about the nature of the signal.

In addition to the mentioned properties, these points can be used for the analysis of the noise in their vicinities. At the small distance r ($r \ll r_c$) from such a point signal is sufficiently small $P_s \approx r\sigma_1$. Therefore, in the area, where $r < \delta_0/\sigma_1$ the signal is much smaller, than the noise. Roughly speaking, pixels inside this area indicate the noise only. We suggest using this fact for the estimation of the pixel-pixel correlations in the noise and (if they exist) making an appropriate correction in the final formula (10).

4. Conclusions

In this paper we propose the method of determination of the spectral parameters for the cosmological signal and pixel noise using the observational data of the CMB polarization without any additional assumptions. We also suggest use of singular points of the polarization for the same purpose.

To determine the parameters of the noise we introduced the notion of the crossing points in the observational maps (see section 2). We obtained the formulae for the total number of them at different levels for the modulus of polarization P (N_p) and separately for two components Q and U ($N_{q,u}$) in case of presence of the CMB signal and pixel noise. This formulae includes parameters σ_0 , σ_1 , δ_0 . The theoretical expressions N_p , $N_{q,u}$ that fit the observational data allow us to determine these parameters. On the other hand, the determination of correlations in the observational data in the very vicinities of the points, where the CMB polarization vanishes and where there is the noise only (spot like regions of crossing dots in the map (fig.3)) allows us to correct the final result.

We would like to emphasize, that this approach is applicable for the analysis of the CMB anisotropy as well.

This method allows us to estimate spectral parameters of signal and noise directly from the observational data without any additional assumptions before subsequent filtration.

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