DETERMINATION OF BRAUER CHARACTERS

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The purpose of this note is to show that the values of an irreducible (Brauer) character are the characteristic values of a matrix with non-negative rational integers. The construction of these integral matrices is done by a description of a representation of the Grothendieck ring of the category of modules over the group algebra. In particular a result of Solomon on characters and a result of Burnside on vanishing of a non-linear character on some conjugate class are generalized.

Let \( G \) be a finite group of order \( g \), \( A \) a splitting field of characteristic \( p \) (which may be 0) for \( G \), \( R = AG \) the group algebra of \( G \) over \( A \), \( n \) the number of distinct \( p \)-regular classes of \( G \) and \( \mathcal{C} \) the category of all finite dimensional (right) \( R \)-modules. Then the isomorphism of \( R \)-modules is an equivalence relation in \( \mathcal{C} \). The equivalence class determined by an \( R \)-module \( M \) in \( \mathcal{C} \) will be denoted by \( \overline{M} \). If \( T = \{ \overline{M} | M \in \mathcal{C} \} \), then the Grothendieck group \( K(R) \) is defined to be the additive abelian group generated by \( T \) subject to the defining relations \( \overline{M} = \overline{N} + \overline{P} \) whenever

\[
0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0
\]

is an exact sequence of \( R \)-modules. The Grothendieck group \( K(R) \) is a free abelian group of rank \( n \) [6]. In fact if \( \{ \overline{M}_1, \overline{M}_2, \ldots, \overline{M}_n \} \) is a full set of pairwise non-isomorphic irreducible \( R \)-modules, then \( \{ \overline{M}_1, \overline{M}_2, \ldots, \overline{M}_n \} \) is a basis of \( K(R) \) over the ring \( \mathbb{Z} \) of integers.

If \( M \) and \( N \) are any modules in \( \mathcal{C} \), then we denote their inner tensor product over \( A \) by \( M \otimes N \). Obviously \( M \otimes N \) is a member of \( \mathcal{C} \). If we define \( \overline{M} \overline{N} = \overline{M} \otimes \overline{N} \), then \( K(R) \) becomes a commutative associative ring with unity \( \overline{I} \), where \( I \) is the \( R \)-module which affords the principal irreducible representation; that is the one-dimensional representation of \( G \) in which every element of \( G \) is mapped on the identity. The ring \( K(R) \) is called the Grothendieck ring of the category \( \mathcal{C} \) of \( R \)-modules. The integer \( n \) will be called the rank of the ring \( K(R) \).

**Theorem 1.** Let \( K(R) \) be the Grothendieck ring of the category of \( R \)-modules, \( n \) the rank of \( K(R) \) and \( \mathbb{Z}_n \) the ring of \( n \) by \( n \) matrices over the ring \( \mathbb{Z} \) of integers. Then there is a monomorphism of \( K(R) \) into \( \mathbb{Z}_n \).

**Proof.** Let \( \{ \overline{M}_1, \overline{M}_2, \ldots, \overline{M}_n \} \) be a fixed basis of \( K(R) \) over \( \mathbb{Z} \). Then every element of \( K(R) \) can be written uniquely in the form \( \sum_{t=1}^{n} a_t \overline{M}_t \) where \( a_t \).
belongs to $Z$. Each element $\bar{M}$ of $K(R)$ determines a unique matrix $(a_{ij})$ of $Z_n$ by

$$\bar{M}_i\bar{M} = \sum_{j=1}^{n} a_{ij} M_j$$

for $i = 1, 2, \ldots, n$. Then we show that the mapping $f$ from $K(R)$ to $Z_n$ defined by $\bar{M}f = (a_{ij})$ is a ring monomorphism.

Suppose that $\bar{M}$ and $\bar{N}$ are arbitrary elements of $K(R)$. Then $\bar{M}_i\bar{M} = \sum_{j=1}^{n} a_{ij} M_j$ and $\bar{M}_i\bar{N} = \sum_{j=1}^{n} a_{ij} N_j$ for all $i$. Therefore we have

$$\bar{M}_i(\bar{M} + \bar{N}) = \bar{M}_i\bar{M} + \bar{M}_i\bar{N} = \sum_{j=1}^{n} a_{ij} M_j + \sum_{j=1}^{n} a_{ij} N_j$$

so that $(\bar{M} + \bar{N})f = (a_{ij} + a_{ij}) = (a_{ij}) + (a_{ij}) = \bar{M}f + \bar{N}f$. Hence $f$ is a group homomorphism. Also we have

$$\bar{M}_i(\bar{M}\bar{N}) = \bar{M}_i(M \bar{N}) = \bar{M}_i(M \bar{N}) = \bar{M}_i(M) \bar{N}$$

from which it follows that $\bar{M}\bar{N}f = \bar{M}f\bar{N}f$. Thus we have proved that $f$ is a ring homomorphism.

It remains to show that the homomorphism $f$ is injective. If $\bar{M}$ is any element in the kernel of $f$, then $\bar{M}f = 0$. There are unique integers $a_i$ in $Z$ such that

$$\bar{M} = \sum_{i=1}^{n} a_i \bar{M}_i.$$ 

Let $\bar{I}$ be the identity of $K(R)$. Since $\bar{M}_i\bar{M} = 0$ for all $i$, it follows that $\bar{I}\bar{M} = 0$; that is

$$\sum_{i=1}^{n} a_i \bar{I} \bar{M} = \sum_{i=1}^{n} a_i \bar{M}_i = 0.$$ 

Since $\{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\}$ is a basis of $K(R)$, it follows that $\bar{M} = 0$. Hence $f$ is injective. This completes the proof.

This proposition is a generalization for representations over an arbitrary field of a similar result proved earlier by Robinson [4; 5] for ordinary representations.

With each element $\bar{M}$ of $K(R)$, we associate an $n$-tuple

$$\chi^M = (\chi_1^M, \chi_2^M, \ldots, \chi_n^M)$$
of complex numbers, where \( \chi_1^M, \chi_2^M, \ldots, \chi_n^M \) are the characteristic values of the matrix \( (a_{ij}^M) \). Clearly \( \chi^M \) is independent of a basis of \( K(R) \) over \( Z \) and it is uniquely determined by \( \bar{M} \).

The irreducible Brauer characters of \( G \) will be denoted by \( \varphi^1, \varphi^2, \ldots, \varphi^n \) (in some fixed order) and the \( p \)-regular classes of \( G \) will be denoted by \( C_1 = \{1\}, C_2, \ldots, C_n \) (in some fixed order). If \( p = 0 \) or prime to the order \( g \) of \( G \), then every class of \( G \) is \( p \)-regular. The Brauer character \( \varphi^i \) can also be considered as an \( n \)-tuple \( \varphi^i = (\varphi_1^i, \varphi_2^i, \ldots, \varphi_n^i) \) where \( \varphi_j^i = \varphi^i(x) \) with \( x \) in \( C_j \). The \( n \times n \) matrix \( \Phi = (\varphi_{ji}) \) whose rows are \( \varphi^i, i = 1, 2, \ldots, n \) is called the Brauer character table. Since the rows of \( \Phi \) are linearly independent \([2]\), the matrix is a non-singular (complex) matrix. Without loss of generality, we may assume that \( \varphi^1 \) is the principal character of \( G \).

**Theorem 2.** Let \( M \) be an \( R \)-module, \( n \) the rank of \( K(R) \) and \( \chi^M \) the \( n \)-tuple associated with \( \bar{M} \). Then there is a permutation \( \sigma \) of \{1, 2, \ldots, \( n \)\} such that \( (\chi_{\sigma 1}^M, \chi_{\sigma 2}^M, \ldots, \chi_{\sigma n}^M) \) is a Brauer character of \( G \).

**Proof.** Let \( \{M_1, M_2, \ldots, M_n\} \) be a set of pairwise non-isomorphic irreducible \( R \)-modules such that \( \{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_n\} \) is a basis of \( K(R) \). Then

\[
\bar{M} = \sum_{j=1}^{n} d_j \bar{M}_j
\]

for some non-negative integers \( d_j \). Let

\[
\varphi^M = \sum_{j=1}^{n} d_j \varphi^j.
\]

Then we shall show that the components of \( \varphi^M \) will coincide with those of \( \chi^M \) (except possibly for a permutation).

Let \( M = N_1 \supset \ldots \supset N_t \supset N_{t+1} = 0 \) be a composition series of \( M \). Then

\[
\bar{M} = \bigoplus_{i=1}^{t} N_i/N_{i+1} = \sum_{i=1}^{t} \frac{N_i}{N_{i+1}} = \sum_{j=1}^{n} d_j \bar{M}_j.
\]

Therefore \( \varphi^M \) depends only on \( \bar{M} \) and \( \varphi^M \) is the Brauer character associated with the \( R \)-module \( M \). By [1, p. 577], it follows that \( \varphi^i \varphi^M \) depends only on \( M_i \otimes M = \bar{M}_i \bar{M} \) and \( \varphi^i \varphi^M \) is the Brauer character associated with \( M_i \otimes M \). Therefore the equations

\[
\bar{M}_i \bar{M} = \sum_{j=1}^{n} a_{ij}^M \bar{M}_j, \quad i = 1, 2, \ldots, n
\]

are equivalent to the equations

\[
\varphi_i \varphi^M = \sum_{j=1}^{n} a_{ij}^M \varphi^j
\]

for \( i, t = 1, 2, \ldots, n \). If \( D^M \) is the diagonal matrix whose diagonal entries are \( \varphi_1^M, \varphi_2^M, \ldots, \varphi_n^M \), then the above equations can be written in the matrix

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form $\Phi D^M = (a_{ij}^M)\Phi$ where $\Phi$ is the Brauer character table. Hence $\Phi^{-1}(a_{ij}^M)\Phi = D^M$, so that there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that

$$\varphi^M = (\chi_{1\sigma}^M, \chi_{2\sigma}^M, \ldots, \chi_{n\sigma}^M)$$

where $i\sigma$ is the image of $i$ under $\sigma$. This completes the proof.

**Corollary.** For any Brauer character $\varphi^i$, the quantity $\sum_{j=1}^{n} |\varphi_j|^2$ is a positive integer.

**Proposition 3.** The $t$-th column $c_t$ of the Brauer character table $\Phi$ is a characteristic vector of $(a_{ij}^k)$ corresponding to the characteristic value $\varphi_t^k$ for $k = 1, 2, \ldots, n$ where $(a_{ij}^k)$ is the matrix corresponding to the element $\bar{M}_k$ of $K(R)$ determined by an irreducible $R$-module $M_k$.

**Proof.** This follows by rewriting the equations

$$\varphi_i^t \varphi_i^k = \sum_{k=1}^{n} a_{ij}^k \varphi_i^j$$

for $i, k = 1, 2, \ldots, n$ in the matrix form $(a_{ij}^k)c_i = \varphi_i^kc_i$ for $t = 1, 2, \ldots, n$. (Cf. [5].)

**Lemma 4.** Let $M_k$ be an irreducible $R$-module of dimension $f$, $(a_{ij}^k)$ the matrix corresponding to $\bar{M}_k$ of $K(R)$ and $\omega_1, \omega_2, \ldots, \omega_r$ the non-zero characteristic values of $(a_{ij}^k)$. Then $\omega_1 + \omega_2 + \ldots + \omega_r$ is a non-negative rational integer and $\omega_1\omega_2 \ldots \omega_r/f$ is a rational integer.

**Proof.** Let $\det(a_{ij}^k - \lambda\delta_{ij})$ be the characteristic polynomial of the matrix $(a_{ij}^k)$. Since $a_{ij}^k$ are integers, the coefficients of $\det(a_{ij}^k - \lambda\delta_{ij})$ are rational integers. Then $-(\omega_1 + \omega_2 + \ldots + \omega_r)$ is the coefficient of $\lambda^{n-1}$ and $\pm \omega_1\omega_2, \ldots, \omega_r$ is the coefficient of $\lambda^{-1}$. Hence they are rational integers. But $\omega_1 + \omega_2 + \ldots + \omega_r$ is the sum of the diagonal entries of the matrix $(a_{ij}^k)$ and hence it is a non-negative integer.

**Corollary 1.** Let $\varphi^M$ be any Brauer character of $G$ and $\varphi_1^M, \varphi_2^M, \ldots, \varphi_r^M$ the non-zero values of $\varphi^M$. Then $\sum_{i=1}^{r} \varphi_i^M$ is a non-negative integer and $\varphi_2^M \ldots \varphi_r^M$ is a non-zero integer.

**Corollary (Solomon [7]).** If $A$ is the complex field, then $\sum_{i=1}^{n} \varphi_i^M$ is a rational integer.

**Theorem 5.** Let the characteristic $p$ of $A$ be either 0 or prime to $g = |G|$, $M_k$ an irreducible $R$-module and $(a_{ij}^k)$ the corresponding matrix of $\bar{M}_k$ of $K(R)$. Then $(a_{ij}^k)$ is non-singular if and only if the dimension of $M_k$ is 1.

**Proof.** If the dimension of $M_k$ is 1, then $\det(a_{ij}^k) = \omega_1\omega_2 \ldots \omega_n = 1 \neq 0$, where $\omega_1, \omega_2, \ldots, \omega_n$ are the characteristic values of $(a_{ij}^k)$. Hence $(a_{ij}^k)$ is non-singular.

Conversely assume that $(a_{ij}^k)$ is non-singular. Let $\xi^k$ be the Brauer character of $G$ corresponding to $M_k$. Then by Theorem 2, we may assume that
\[ \zeta^k = (\zeta_1^k, \zeta_2^k, \ldots, \zeta_n^k) \]

where \( \zeta_i^k = \dim M_k \), \( \zeta_2^k, \ldots, \zeta_n^k \) are the characteristic values of the matrix \( (a_{ij}^k) \). Since the arithmetic mean of \( n \) positive real numbers is greater than or equal to the geometric mean, we obtain from the orthogonality relations \([2]\) for characters, that

\[
g = \sum_{i=1}^{n} g_i |\zeta_i^k|^2 = (\zeta_1^k)^2 + \sum_{i=2}^{n} g_i |\zeta_i^k|^2 \\
\geq (\zeta_1^k)^2 + (g - 1)(|\zeta_2^k|^2 + |\zeta_3^k|^2 + \cdots + |\zeta_n^k|^2)^{1/p-1}
\]

where \( g_i \) is the number of elements in the \( i \)-th conjugate class of \( G \). By Lemma 4, the above inequality implies \( g \geq (\zeta_1^k)^2 + g - 1 \) so that \( \zeta_1^k = 1 \). Hence the dimension of \( M_k \) is 1.

**Corollary (Burnside)** 1. If \( p = 0 \) and \( \zeta^k \) is an irreducible character of degree \( > 1 \), then \( \zeta_1^k = 0 \) for some conjugate class \( C_i \).

**Corollary 2.** Let \( p = 0 \) or \( (p, g) = 1 \). If \( M_1, M_2, \ldots, M_n \) are pairwise non-isomorphic irreducible \( R \)-modules, then the number of units in the basis \( \{\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_n\} \) of \( K(R) \) is equal to the order of \( G/G' \) where \( G' \) is the commutator subgroup of \( G \).

**Corollary 3.** Let \( p = 0 \) or \( (p, g) = 1 \), \( M_k \) an irreducible \( R \)-module and \( (a_{ij}^k) \) the corresponding matrix of \( \overline{M}_k \) of \( K(R) \). Then \( (a_{ij}^k) \) is invertible if and only if \( (a_{ij}^k) \) is a permutation matrix.

The Corollary (Burnside) 1 is not true if \( p \) is a factor of \( g = |G| \). By this we mean that the value of an irreducible Brauer character \( \varphi^k \) may not be 0 on any \( p \)-regular conjugate class. As an illustration, we consider an example.

**Example.** Let \( G = L_2(5) \) and \( p = 5 \). The group \( G \) has three 5-regular conjugate classes and hence \( K(R) \) has rank 3. If \( I, J \) and \( K \) are irreducible \( R \)-modules of dimensions 1, 3 and 5 over \( A \), then \( \{\overline{I}, \overline{J}, \overline{K}\} \) is a basis of \( K(R) \) over \( Z \). Then we have

\[
\begin{align*}
II &= I &IJ &= J &IK &= K \\
JI &= J &JJ &= I + J + K &JK &= I + 3J + K \\
KI &= K &KJ &= I + 3J + K &KK &= 3I + 4J + 2K.
\end{align*}
\]

Therefore the corresponding matrices \( (a_{ij}^k) \) are

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
1 & 3 & 1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & 0 & 1 \\
1 & 3 & 1 \\
3 & 4 & 2
\end{bmatrix}.
\]

These matrices determine the irreducible Brauer characters \( \varphi^1 = (1, 1, 1) \), \( \varphi^2 = (3, -1, 0) \) and \( \varphi^3 = (5, 1, -1) \) respectively. Contrary to the situation in Corollary 1, the irreducible Brauer character \( \varphi^3 \) does not vanish on any 5-regular conjugate class of \( G \).
If the field $A$ is a splitting algebraic number field for $G$, then the irreducible Brauer characters of $G$ are called the irreducible ordinary characters of $G$. Let $\text{Int}(A)$ be the ring of all algebraic integers of $A$, $\mathfrak{p}$ a fixed prime ideal of $\text{Int}(A)$ containing a given prime $\mathfrak{p}$ and $D = [\alpha/\beta] | \alpha, \beta \in \text{Int}(A)$ and $\beta \not\in \mathfrak{p}$. Then $D$ is an integral domain with a unique maximal ideal $P$ such that $\mathfrak{p} \subseteq P$ and $A$ is the quotient field of $D$. The field $\overline{A} = D/P$ has a finite number of elements and it is a splitting field for $G$ [2]. Let $R = AG$ and $\overline{R} = \overline{AG}$ be the group rings over $A$ and $\overline{A}$ respectively. The Brauer characters depend on the prime $p$. The Brauer characters determined by $K(\overline{R})$ will be denoted by $\varphi^1, \varphi^2, \ldots, \varphi^n$ and the ordinary irreducible characters of $G$ will be denoted by $\xi^1, \xi^2, \ldots, \xi^m$.

Again assume that $C_1, C_2, \ldots, C_n$ are the $p$-regular classes and $C_{n+1}, C_{n+2}, \ldots, C_m$ are the remaining conjugate classes of $G$. The value of $\xi^i$ at $x$ of $C_j$ will be denoted by $\xi^i_j$ and the number of elements in $C_j$ will be denoted by $g_j$.

**Theorem 6.** Let $\xi^k$ be an irreducible ordinary character of $G$ and $\partial^k$ the $n \times n$ diagonal matrix whose diagonal entries are $\xi^k_1, \xi^k_2, \ldots, \xi^k_n$. If $\Phi$ is the Brauer character table of $G$ corresponding to the prime $p$, then the coefficients of $\Phi \partial^k \Phi^{-1}$ are non-negative rational integers.

**Proof.** There are non-negative rational integers $d_{kj}$ such that

$$
\xi^k_i = \sum_{j=1}^n d_{kj} \varphi^j_i \quad \text{for } i = 1, 2, \ldots, n \ [2].
$$

Hence we have $\partial^k = \sum_{j=1}^n d_{kj} D^j$ where $D^j$ is the diagonal matrix whose entries are $\varphi^1_j, \varphi^2_j, \ldots, \varphi^n_j$. If $(a_{ij})$ are the $n$ matrices determined by a basis of $K(\overline{R})$, then by Theorem 2, we have $\Phi^{-1}(a_{ij}) \Phi = D^j$. Therefore

$$
\partial^k = \sum_{j=1}^n d_{kj} \Phi^{-1}(a_{ij}) \Phi
$$

so that

$$
\Phi \partial^k \Phi^{-1} = \sum_{j=1}^n d_{kj}(a_{ij}).
$$

Since the coefficients of the matrix on the right hand side of this equation are non-negative integers, it follows that the coefficients of $\Phi \partial^k \Phi^{-1}$ are non-negative rational integers.

**Corollary.** Let $\xi^k$ be any irreducible ordinary character of $G$, $\varphi^i$ an irreducible Brauer character of $G$ and $\eta^i$ a projective indecomposable Brauer character of $G$. Then

$$
\frac{1}{g} \sum_{j=1}^n g_j \varphi^j_i \xi^k_j \eta^j_j
$$

is a non-negative integer.
Proof. For the definition of a projective indecomposable Brauer character, see [2]. According to [2], the irreducible Brauer character \( \varphi^i \) and the indecomposable Brauer character \( \eta^l \) of \( G \) are connected by the relations

\[
\sum_{i=1}^{n} \varphi^i \left( \frac{g \eta^l}{g} \right) = \delta_{i l}
\]

where \( \delta_{i l} = 0 \) if \( i \neq l \) and \( 1 \) if \( i = l \). Therefore it follows that

\[
\Phi^{-1} = \left( \frac{g \eta^l}{g} \right)
\]

where \( t \) ranges over the rows and \( l \) ranges over the columns. The \((i, l)\)th coefficient of the matrix \( \Phi \eta^k \Phi^{-1} \) is

\[
\frac{1}{g} \sum_{i=1}^{n} g \varphi^i \left( \frac{g \eta^l}{g} \right) \eta^l
\]

which is a non-negative rational integer.

There are groups for which the matrices \( (a_{i l}^k) \) can be calculated without an elaborate use of representation theory. For example, there is a method for computing the matrices \( (a_{i l}^k) \) in the case of a symmetric group [3]. It appears that the result of this paper relating to ordinary representation theory may have some appeal to physicists [4; 5], although the same may not be true of those relating to modular representation theory.

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References

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