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APPROXIMATELY PERIODIC FUNCTIONALS ON C^*-ALGEBRAS AND VON NEUMANN ALGEBRAS

JOHN C. QUIGG

1. Introduction. In the duality for locally compact groups, much use is made of a version of the Hopf algebra technique in the context of von Neumann algebras, culminating in the theory of Kac algebras [**6**], [**14**]. It seems natural to ask whether something like a Hopf algebraic structure can be defined on the pre-dual of a Kac algebra. This leads to the question of whether the multiplication on a von Neumann algebra M, viewed as a linear map m from $M \odot M$ (the algebraic tensor product) to M, can be pre-transposed to give a co-multiplication on the pre-dual M_*, i.e., a linear map m_* from M_* to the completion of $M_* \odot M_*$ with respect to some cross-norm. A related question is whether the multiplication on a C^*-algebra A can be transposed to give a co-multiplication on the dual A^*. Of course, this can be regarded as a special case of the preceding question by taking $M = A^{**}$, where the double dual A^{**} is identified with the enveloping von Neumann algebra of A. Keeping in mind the relationship with Kac algebras, the most desirable choice for the cross-norm on $M_* \odot M_*$ would be the dual of the spatial (least) C^*-norm on $M \odot M$.

To make the problem more precise, note that, since m is continuous with respect to the greatest cross-norm γ on $M \odot M$, we can extend m to the completion $M \otimes_\gamma M$ and then transpose to get a map m^* from M^* to $(M \otimes_\gamma M)^*$. We shall not pause to seek conditions under which the range of m^* is contained in $M^* \otimes_{\gamma^*} M^*$ (see [**13**] for a study of this question in the Banach algebra context). Rather, since we want to work with C^*-tensor products, we should require first of all that m be continuous with respect to the spatial C^*-norm, which turns out to be a surprisingly restrictive condition. Of course, this is not enough for our present purposes, since this only gives

$$m^*: M^* \rightarrow (M \otimes M)^*$$

(here $M \otimes M$ denotes the completion of $M \odot M$ relative to the spatial C^*-norm). We actually need m to extend to a σ-weakly continuous map from $M \bar{\otimes} M$ (the von Neumann algebra tensor product) to M, so that we can pre-transpose to get

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$$m_*: M_* \to M_* \otimes M_*$$

(by which we denote the completion of $M_* \otimes M_*$ relative to the dual of the spatial C^*-norm). To see what this entails, recall that there is an isometric isomorphism

$$T:(M \otimes_\gamma M)^* \to B(M, M^*)$$

(the bounded linear maps from M to M^*) defined by

$$\langle x, T(\phi)y\rangle = \langle x \otimes y, \phi\rangle \quad (x, y \in M, \phi \in (M \otimes M)^*).$$

It is immediate that $T(\phi)$ is finite rank with range in M_* for $\phi \in M_* \odot M_*$, and hence that $T(\phi)$ is compact with range in M_* for $\phi \in M_* \otimes M_*$. We conclude that a necessary condition for m to pre-transpose to give a co-multiplication on M_* is that $T(m^*(\phi))$ be a compact linear map from M to M_*, for each $\phi \in M_*$. Letting M act on M_* via

$$\langle x, y \cdot \phi\rangle = \langle xy, \phi\rangle \quad (x, y \in M, \phi \in M_*),$$

it is straightforward to compute that

$$T(m^*(\phi))y = y \cdot \phi \quad (y \in M, \phi \in M_*).$$

Therefore the above necessary condition can be re-written as: $M_1 \cdot \phi$ is relatively compact in M_* for each $\phi \in M_*$, where M_1 denotes the unit ball of M. For a Banach algebra A, Kitchen [**9**] calls a functional ϕ almost periodic if the linear map $x \to x \cdot \phi$ from A to A^* is compact. If $A = L^1(G)$ for a locally compact group G, then this reduces to the classical notion of almost periodicity for $\phi \in L^\infty(G)$. However, if ϕ is a faithful normal state on a von Neumann algebra, then Connes [**2**] calls ϕ almost periodic if its modular operator Δ_ϕ is diagonalizable. This has nothing to do with the compactness of $x \to x \cdot \phi$. Since we are interested primarily in C^*-algebras and von Neumann algebras, and since we do not wish to introduce terminology conflicting with that of Connes, we will use the term *approximately periodic*.

In Section 2 we characterize those von Neumann algebras M having the property that every normal functional is approximately periodic, and we use this to characterize the set of approximately periodic normal functionals in the pre-dual of any von Neumann algebra.

In Section 3 we specialize to the case where $M = A^{**}$ for a C^*-algebra A, and we characterize those A having the property that every functional is approximately periodic. We use this to define what we call the universal approximately periodic representation, which is a generalization of the construction of the Bohr compactification of a locally compact group.

In Section 4 we characterize those von Neumann algebras whose multiplication is continuous relative to the spatial C^*-norm.

In Section 5 we show that the multiplication on a von Neumann algebra pre-transposes to give a co-multiplication on the pre-dual if and only if every normal functional is approximately periodic and the multiplication is continuous relative to the spatial C^*-norm.

In Section 6 we discuss a certain approximation property satisfied by the approximately periodic functionals on a C^*-algebra.

Finally, Section 7 is devoted to an indication of the way in which the co-multiplication on the pre-dual can be utilized in connection with actions of Kac algebras on von Neumann algebras.

2. Approximately periodic functionals on von Neumann algebras. Let M be a von Neumann algebra, and let

$$AP_*(M) = \{\phi \in M_* | M_1 \cdot \phi \text{ is relatively compact}\}$$

denote the set of approximately periodic normal functionals on M. Thus, ϕ is approximately periodic if and only if the map $x \rightarrow x \cdot \phi$ from M to M_* is compact. We proceed to study the condition $AP_*(M) = M_*$.

LEMMA 2.1. *Let $M = B(H)$, the set of bounded linear operators on the Hilbert space H. Then $AP_*(M) = M_*$ if and only if H is finite dimensional.*

Proof. M_* is identified with the set of trace class operators on H, equipped with the trace norm, via

$$\langle x, \phi \rangle = \mathrm{tr}(x\phi).$$

A trivial computation shows that $x \cdot \phi = x\phi$ $(x \in M, \phi \in M_*)$.

Now, M_* is also identified with the projective tensor product $\bar{H} \otimes_\gamma H$ of the conjugate space to H and H via

$$(\bar{\xi} \otimes \eta)\zeta = (\zeta|\xi)\eta \quad (\xi, \eta, \zeta \in H).$$

Under this identification, if ξ and η have norm one, we have

$$M_1 \cdot (\bar{\xi} \otimes \eta) = \bar{\xi} \otimes H_1,$$

which is relatively compact in $\bar{H} \otimes_\gamma H$ if and only if H_1 is compact in H, which is equivalent to the finite dimensionality of H.

LEMMA 2.2. *If $AP_*(M) = M_*$ and N is a von Neumann subalgebra of M, then $AP_*(N) = N_*$.*

Proof. Identify N_* with the set $M_*|N$ of restrictions to N of elements of M_*. Then, for $\phi \in M_*$,

$$\begin{aligned} N_1 \cdot (\phi|N) &= (N_1 \cdot \phi)\,|N \\ &\subset (M_1 \cdot \phi)\,|N, \end{aligned}$$

which is relatively compact in N_* since restriction to N is a continuous map from M_* to N_*.

LEMMA 2.3. *If M is the direct sum of the von Neumann algebras M_i $(i \in I)$, then $AP_*(M) = M_*$ if and only if*

$$AP_*(M_i) = (M_i)_* \quad \text{for all } i \in I.$$

Proof. As the necessity is immediate from Lemma 2.2, we show only the sufficiency.

M_* is identified with the l^1 direct sum of the pre-duals $(M_i)_*$ via

$$\langle x, \phi \rangle = \sum_{i \in I} \langle x_i, \phi_i \rangle \quad (x \in M, \phi \in M_*).$$

A straightforward computation shows that

$$(x \cdot \phi)_i = x_i \cdot \phi_i \quad (x \in M, \phi \in M_*).$$

In particular, if ϕ has only finitely many non-zero components, then $M_1 \cdot \phi$ is a sum of finitely many relatively compact sets, and is therefore relatively compact. The proof is concluded by observing that such ϕ are dense in M_*.

LEMMA 2.4. *If $M = L^\infty(S, \mu)$, where μ is a Radon measure on the locally compact Hausdorff space S, then $AP_*(M) = M_*$ if and only if μ is purely atomic.*

Proof. By restricting to compact subsets, we may assume without loss of generality that μ is bounded. If M_* is identified with $L^1(S, \mu)$ in the usual way, it is clear that

$$f \cdot \phi = f\phi \quad (f \in M, \phi \in M_*).$$

In particular, if ϕ is the constant function 1, then $M_1 \cdot \phi$ is the image of the unit ball of M under the natural embedding of $L^\infty(S, \mu)$ in $L^1(S, \mu)$, which is relatively compact if and only if μ is purely atomic. This proves the necessity, and the sufficiency follows at once from Lemma 2.3.

THEOREM 2.5. *$AP_*(M) = M_*$ if and only if M is a direct sum of matrix algebras.*

Proof. The sufficiency follows immediately from Lemmas 2.1 and 2.3.

For the necessity, first note that M must be finite, for it would otherwise contain a type I_∞ factor, in violation of Lemmas 2.1 and 2.2.

Similarly, M must be a direct sum of factors, for its center would otherwise be a non-atomic abelian von Neumann algebra, in violation of Lemmas 2.2 and 2.4.

Finally, if M were not type I, then it would contain a type II_1 factor, which would contain the hyperfinite II_1 factor, which would in turn contain a non-atomic abelian von Neumann algebra, again in violation of Lemmas 2.2 and 2.4.

We now proceed to characterize $AP_*(M)$ as a subspace of M_*. Recall that a subset S of M_* is called *invariant* if

$$M \cdot S \cdot M \subset S,$$

where

$$\langle z, x \cdot \phi \cdot y \rangle = \langle yzx, \phi \rangle \quad (x, y, z \in M, \phi \in M_*).$$

LEMMA 2.6. *$AP_*(M)$ is a closed invariant subspace of M_*.*

Proof. This is routine in view of the fact that the maps $x \rightarrow x \cdot \phi \quad (x \in M)$ vary continuously with $\phi \in M_*$.

Recall [**12**] that there is a bijection between the closed invariant subspaces V of M_*, the σ-weakly closed ideals N of M, and the central projections z of M via

$$V^{\circ} = N = M(1 - z),$$

where V° denotes the polar of V. Under this correspondence, $V = z \cdot M_*$, V^* is identified with Mz, and we have the decompositions

$$M = Mz \oplus M(1 - z)$$

$$M_* = (Mz)_* \oplus (M(1 - z))_*$$
$$= z \cdot M_* \oplus (1 - z) \cdot M_*.$$

Also, if V_1 and V_2 are closed invariant subspaces of M_*, then $V_1 \subset V_2$ if and only if $V_1^* \subset V_2^*$.

LEMMA 2.7. *If N is a σ-weakly closed ideal of M, then*

$$AP_*(N) = N_* \cap AP_*(M).$$

Proof. Let $N = Mz$, where z is a central projection of M. Then, for $\phi \in N_* = z \cdot M_*$,

$$N_1 \cdot \phi = (Mz)_1 \cdot \phi = (M_1 z) \cdot \phi = M_1 \cdot (z \cdot \phi) = M_1 \cdot \phi.$$

Therefore, $AP_*(N) = z \cdot AP_*(M)$.

Let M_{ap} be the largest ideal of M which is a direct sum of matrix algebras.

THEOREM 2.8. $(M_{ap})_* = AP_*(M)$.

Proof. By construction and Theorem 2.5,

$$AP_*(M_{ap}) = (M_{ap})_*,$$

so that by Lemma 2.7

$$(M_{ap})_* \subset AP_*(M).$$

On the other hand, if $N = AP_*(M)^*$, then $AP_*(N) = N_*$ by Lemma 2.7, so that N is a direct sum of matrix algebras. But M_{ap} is the largest such ideal, so that we must have $N \subset M_{ap}$ and hence $AP_*(M) \subset (M_{ap})_*$.

3. Approximately periodic functionals on C^*-algebras. Let A be a C^*-algebra and let

$$AP(A) = \{\phi \in A^* | A_1 \cdot \phi \text{ is relatively compact}\}$$

denote the set of approximately periodic functionals on A. Thus, ϕ is approximately periodic if and only if the map $x \rightarrow x \cdot \phi$ from A to A^* is compact.

LEMMA 3.1. $AP_*(A^{**}) = AP(A)$.

Proof. This follows immediately from the observation that the double dual of the map $x \rightarrow x \cdot \phi$ from A to A^* is the map $x \rightarrow x \cdot \phi$ from A^{**} to A^*, so that the compactness of either implies that of the other.

Recall [**8**] that A is called *scattered* if every positive functional on A is a sum of pure positive functionals, or, equivalently, if A^{**} is a direct sum of type I factors. By Theorem 2.5 and Lemma 3.1, we have

THEOREM 3.2. $AP(A) = A^*$ *if and only if A is scattered and its irreducible representations are finite dimensional.*

We now characterize $AP(A)$ by means of a representation which is the analogue of the construction of the Bohr compactification of a locally compact group.

Definition. A representation π of A is *approximately periodic* if $AP_*(\pi(A)'') = \pi(A)''_*$.

Recall that every nondegenerate representation π of A extends uniquely to a normal representation $\tilde{\pi}$ of A^{**}, and that $\tilde{\pi}(A^{**}) = \pi(A)''$. Let z_π be the central projection in A^{**} such that

$$\ker \tilde{\pi} = A^{**}(1 - z_\pi).$$

Then $\pi(A)'' \simeq A^{**}z_\pi$, and we have

PROPOSITION 3.3. *π is approximately periodic if and only if*

$$z_\pi \cdot A^* \subset AP(A).$$

Proof. This follows immediately from Lemma 2.7.

Let π_{ap} be the direct sum of all the finite dimensional cyclic representations obtained from the application of the Gelfand-Naimark-Segal construction to the states of A, and denote the corresponding central projection by z_{ap}.

THEOREM 3.4. $z_{ap} \cdot A^* = AP(A)$.

Proof. It is clear from the construction that $A^{**}z_{ap}$ is the largest ideal of A^{**} which is a direct sum of matrix algebras, i.e.,

$$A^{**}z_{ap} = (A^{**})_{ap}.$$

The result now follows from Theorem 2.8.

COROLLARY 3.5. *A representation of A is approximately periodic if and only if it is quasi-equivalent to a sub-representation of* π_{ap}.

In view of Corollary 3.5, we make the

Definition. π_{ap} is the *universal approximately periodic representation* of A.

Note that if A is the group C^*-algebra of a locally compact group G, then the weak operator closure of $\{\pi_{ap}(s):s \in G\}$ is the Bohr compactification of G.

By analogy with groups, we could call a C^*-algebra *maximally approximately periodic* if its universal approximately periodic representation is faithful. In regard to this, Choi [**1**] has shown that the C^*-algebra of the free group on two generators is maximally approximately periodic.

4. Continuity of the multiplication.

Definition. For a C^*-algebra A, let $m_A:A \odot A \to A$ denote the multiplication, and let $\|m_A\|$ denote the norm of m_A relative to the spatial C^*-norm on $A \odot A$.

In this section we characterize those von Neumann algebras M satisfying $\|m_M\| < \infty$.

LEMMA 4.1. *If* $\|m_A\| < \infty$ *and B is a* C^**-subalgebra of A, then* $\|m_B\| < \infty$.

Proof. The spatial C^*-norm on $B \odot B$ is the same as the restriction of the spatial C^*-norm on $A \odot A$ [**8**]. Since $m_B = m_A|B \odot B$, we conclude that $\|m_B\| \leqq \|m_A\|$.

LEMMA. 4.2. *Let* $\{A_i|i \in I\}$ *be a collection of* C^**-algebras. Then there is a *-monomorphism*

$$\Phi:(\oplus_i A_i) \otimes (\oplus_i A_i) \to \oplus_{i,j} A_i \otimes A_j$$

defined by

$$\Phi(x \otimes y)_{i,j} = x_i \otimes y_j.$$

Proof. If each A_i is faithfully represented on a Hilbert space H_i, then $(\oplus_i A_i) \otimes (\oplus_i A_i)$ and $\oplus_{i,j} A_i \otimes A_j$ are faithfully represented on

$(\bigoplus_i H_i) \bar{\otimes} (\bigoplus_i H_i)$ and $\bigoplus_{i,j} H_i \bar{\otimes} H_j$, respectively (where $\oplus$ here denotes the l^2 direct sum and $\bar{\otimes}$ the Hilbert space tensor product). Define isometries

$$V_i : H_i \to \bigoplus_j H_j \quad \text{and}$$
$$W_{i,j} : H_i \bar{\otimes} H_j \to \bigoplus_{k,l} H_k \bar{\otimes} H_l$$

by

$$(V_i \xi)_j = \begin{cases} \xi, & i = j \\ 0, & \text{else} \end{cases}$$

$$(W_{i,j} \xi)_{k,l} = \begin{cases} \xi, & i = k, j = l \\ 0, & \text{else.} \end{cases}$$

Then $U = \sum_{i,j} W_{i,j} V_i^* \otimes V_j^*$ converges in the strong operator topology and defines a unitary from $(\bigoplus_i H_i) \bar{\otimes} (\bigoplus_i H_i)$ onto $\bigoplus_{i,j} H_i \bar{\otimes} H_j$ which is easily seen to implement the map Φ defined in the statement of the lemma.

Remark. The above map Φ need not be surjective. For example, if $I = \mathbf{N}$ and $A_i = \mathbf{C}$ for all $i \in I$, then

$$\bigoplus_i A_i = l^\infty(\mathbf{N}) \quad \text{and} \quad \bigoplus_{i,j} A_i \otimes A_j = l^\infty(\mathbf{N} \times \mathbf{N})$$

(where $\mathbf{C} \otimes \mathbf{C}$ is identified with $\mathbf{C}$), but

$$l^\infty(\mathbf{N} \times \mathbf{N}) \simeq l^\infty(\mathbf{N}) \bar{\otimes} l^\infty(\mathbf{N}) \neq l^\infty(\mathbf{N}) \otimes l^\infty(\mathbf{N}),$$

where $\bar{\otimes}$ here denotes the von Neumann algebra tensor product.

COROLLARY 4.3. *If* $A = \bigoplus_{i \in I} A_i$, *then*

$$\|m_A\| \leqq \sup_i \|m_{A_i}\|.$$

Proof. Let

$$\sum_k x^{(k)} \otimes y^{(k)} \in A \odot A.$$

Then

$$m_A\left(\sum_k x^{(k)} \otimes y^{(k)}\right)_i = \sum_k x_i^{(k)} y_i^{(k)}$$
$$= m_{A_i}\left(\sum_k x_i^{(k)} \otimes y_i^{(k)}\right),$$

so

$$\left\|m_A\left(\sum_k x^{(k)} \otimes y^{(k)}\right)\right\| = \sup_i \left\|m_A\left(\sum_k x^{(k)} \otimes y^{(k)}\right)_i\right\|$$

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$$= \sup_i \left\| m_{A_i}\left(\sum_k x_i^{(k)} \otimes y_i^{(k)} \right) \right\|$$

$$\leqq \sup_i \|m_{A_i}\| \cdot \sup_i \left\| \sum_k x_i^{(k)} \otimes y_i^{(k)} \right\|.$$

But

$$\sup_i \left\| \sum_k x_i^{(k)} \otimes y_i^{(k)} \right\| \leqq \sup_{i,j} \left\| \sum_k x_i^{(k)} \otimes y_j^{(k)} \right\|$$
$$= \sup_{i,j} \left\| \sum_k \Phi(x^{(k)} \otimes y^{(k)})_{i,j} \right\|$$
$$= \sup_{i,j} \left\| \Phi\left(\sum_k x^{(k)} \otimes y^{(k)} \right)_{i,j} \right\|$$
$$= \left\| \Phi\left(\sum_k x^{(k)} \otimes y^{(k)} \right) \right\|$$
$$= \left\| \sum_k x^{(k)} \otimes y^{(k)} \right\|,$$

by Lemma 4.2.

LEMMA 4.4. $\|m_{B(H)}\| < \infty$ *if and only if H is finite dimensional.*

Proof. Only the necessity is unclear. If H is infinite dimensional, then $B(H)$ contains a non-semidiscrete factor M, so that

$$M \odot M' \subset B(H) \odot B(H).$$

But $\|m_{B(H)}|M \odot M'\| = \infty$ **[4]**.

COROLLARY 4.5. *If A contains a type I_n factor for arbitrarily large n, then* $\|m_A\| = \infty$.

Proof. It suffices to show that

$$\sup_n \|m_{B(\mathbf{C}^n)}\| = \infty.$$

Now, the C^*-algebra $K(H)$ of compact operators on a separable infinite dimensional Hilbert space H contains an increasing sequence of sub-algebras $A_n \simeq B(\mathbf{C}^n)$ whose union is dense. Hence, $U_n A_n \odot A_n = U_n A_n \odot U_n A_n$ is dense in $K(H) \odot K(H)$, so that

$$\|m_{K(H)}\| = \sup_n \|m_{A_n}\| = \sup_n \|m_{B(\mathbf{C}^n)}\|.$$

Suppose that $\|m_{K(H)}\| < \infty$. Then we can extend to get a bounded map

$$m_{K(H)}:K(H) \otimes K(H) \to K(H)$$

whose double dual is a σ-weakly continuous map

$$m_{K(H)}^{**}:B(H) \bar{\otimes} B(H) \to B(H).$$

But, using the separate σ-weak continuity of multiplication, it is easy to see that

$$m_{K(H)}^{**}|B(H) \odot B(H) = m_{B(H)},$$

which is not even norm continuous by Lemma 4.4. This contradiction forces us to conclude that

$$\|m_{K(H)}\| = \|m_{K(H)}^{**}\| = \infty.$$

THEOREM 4.6. *If M is a von Neumann algebra, then $\|m_M\| < \infty$ if and only if M is type I and the factors contained in M have bounded dimension.*

Proof. We start with the necessity. First, M must be finite, since it would otherwise contain a type I_∞ factor, in violation of Lemmas 4.1 and 4.4. Similarly, M must be type I, since a non-type I von Neumann algebra contains type I_n factors for arbitrarily large n, which M cannot contain by Lemma 4.1 and Corollary 4.5. Of course, this last observation shows that the factors contained in M must have bounded dimension.

For the sufficiency, if M is type I and the factors contained in M have bounded dimension, then there is a finite set I and for each $i \in I$ there are an abelian von Neumann algebra N_i and a finite dimensional Hilbert space H_i such that

$$M \simeq \oplus_{i \in I} N_i \bar{\otimes} B(H_i).$$

Hence, by Corollary 4.3, assume without loss of generality that M is of the form $N \bar{\otimes} B(H)$, where N is abelian and H is finite dimensional. Since H is finite dimensional, the von Neumann algebra tensor product reduces to the spatial C^*-tensor product $N \otimes B(H)$. Since N is abelian, there is a hyper-Stonian space S such that $N \simeq C(S)$. Since $C(S)$ is a C^*-subalgebra of $l^\infty(S)$, $C(S) \otimes B(H)$ is a C^*-subalgebra of

$$l^\infty(S) \otimes B(H) = \oplus_{s \in S} B(H).$$

The proof is completed by an appeal to Lemma 4.1 and Corollary 4.3.

5. Co-multiplication on the pre-dual. In the introduction we observed that a necessary condition for the multiplication on M to give rise to a co-multiplication on M_* is that $AP_*(M) = M_*$. We also mentioned that the continuity of the multiplication relative to the spatial C^*-norm is necessary, which we prove in this section. We shall also prove that these two conditions are sufficient.

Let $m = m_M$. Since m is always continuous relative to the greatest cross-norm γ, we can transpose to get

$$m^*:M^* \to (M \bigotimes_\gamma M)^*.$$

Let $\delta = m^*|M_*$. By the definition of the dual cross-norm, $M_* \otimes M_*$ is isometrically embedded in $(M \odot M)^*$, which is (non-isometrically) embedded in $(M \bigotimes_\gamma M)^*$. For m to give rise to a co-multiplication on M_*, we need

$$\delta(M_*) \subset M_* \otimes M_*.$$

THEOREM 5.1. *The following are equivalent*:
(i) $\delta(M_*) \subset M_* \otimes M_*$ *and* $\|\delta\| < \infty$;
(ii) $AP_*(M) = M_*$ *and* $\|m_M\| < \infty$;
(iii) *M is a direct sum of matrix algebras of bounded dimension.*

Proof. (i) $\Rightarrow$ (ii). We have already pointed out the necessity of the condition $AP_*(M) = M_*$. If $\delta(M_*) \subset M_* \otimes M_*$, then

$$\delta^*:M \bar{\bigotimes} M \to M.$$

But by construction $m_M = \delta^*|M \odot M$.

(ii) $\Rightarrow$ (iii). This is immediate from Theorems 2.5 and 4.6.

(iii) $\Rightarrow$ (i). Let M be a direct sum of matrix algebras of bounded dimension. Then $M = A^{**}$, where A is the c_0 direct sum of the matrix algebras. Since $\|m_M\| < \infty$ by Theorem 4.6, we get

$$m_A:A \otimes A \to A.$$

Now, $(A \otimes A)^{**} = A^{**} \bar{\bigotimes} A^{**}$ (something which is false in general), and so we have

$$m_A^{**}:M \bar{\bigotimes} M \to M.$$

But separate σ-weak continuity of multiplication shows that $m_M = m_A^{**}|M \odot M$, and we finally arrive at

$$\delta = m_A^*:M_* \to M_* \otimes M_*.$$

Remarks. (i) The above proof shows that if M is a direct sum of matrix algebras of bounded dimension, then m_M extends to a σ-weakly continuous map on $M_* \bar{\bigotimes} M_*$ (although the multiplication will of course not be jointly continuous in general). This is primarily a consequence of the fact that, under these hypotheses, there is a C^*-algebra A such that

$$M = A^{**} \quad \text{and} \quad M \bar{\bigotimes} M = (A \otimes A)^{**}.$$

In general, $A^{**} \bar{\bigotimes} A^{**}$ can be identified with a σ-weakly closed ideal of $(A \otimes A)^{**}$, and it is not difficult to see that if $\|m_A\| < \infty$, then

$$m_A^{**}|A^{**} \odot A^{**} = m_{A^{**}}$$

if and only if

$$A^{**} \bar{\otimes} A^{**} = (A \otimes A)^{**}.$$

For example, if $A = C[0, 1]$, then $\|m_A\| < \infty$ and $\|m_{A^{**}}\| < \infty$ but

$$m_A^{**}|A^{**} \odot A^{**} \neq m_{A^{**}}.$$

(ii) If $M = \mathrm{VN}(G)$, the von Neumann algebra generated by the left regular representation of a locally compact group G, then M_* is the Fourier algebra $A(G)$ of G [**7**], which is contained in the algebra $C_0(G)$ of continuous functions on G which vanish at infinity. When $\mathrm{VN}(G) \bar{\otimes} \mathrm{VN}(G)$ is identified with $\mathrm{VN}(G \times G)$, the map δ can be computed by

$$\delta(f)(s, t) = f(st),$$

for $f \in A(G)$ and $s, t \in G$. If G is not compact, then $\delta(A(G))$ is not even contained in $C_0(G \times G)$. Theorem 5.1 shows that, even when G is compact, $\delta(A(G))$ will not lie in $A(G \times G)$ unless the irreducible representations of G have bounded dimension. On the other hand, Theorem 5.1 shows that, if the left regular representation of G is a direct sum of irreducibles of bounded dimension, then G is compact. These facts are undoubtedly well-known, although the author could not find a reference.

6. An approximation property. If A is any Banach algebra and X is any two-sided A-module, Kitchen [**9**] says that the *approximation theorem holds* for (X, A) if every closed invariant subspace of $AP(X)$ (obvious definition) is a direct sum of finite dimensional invariant subspaces. Using our characterization of approximately periodic functionals, it is easy to prove:

PROPOSITION 6.1. *If A is a C^*-algebra and M is a von Neumann algebra, then the approximation theorem holds for (A^*, A) and (M_*, M).*

Actually, one of Kitchen's [**9**] results implies that, for any Banach algebra A, if there is a bounded subgroup of the invertible multipliers of A that determines the closed invariant subspaces of $AP(A)$, then the approximation theorem holds for (A^*, A). When A is a C^*-algebra, the group of unitary multipliers will do, and this leads to another proof of Proposition 6.1. It is tempting to conjecture that the same technique would work for any Banach $*$-algebra A which can be isometrically embedded in its multiplier algebra. Of course, this is not trivial, for A could possess unitary multipliers (multipliers u satisfying $uu^* = u^*u = 1$) of norm larger than one. For example, if G is a locally compact group, then, as is well known, the approximation theorem holds for $(L^\infty(G), L^1(G))$, essentially because the multiplier algebra $M(G)$ (the bounded Radon measures on G) of $L^1(G)$ contains a sufficiently large bounded group (namely, the point masses at the elements of G, all of which have norm one). However, $L^1(G)$ has many unitary multipliers of norm greater than one. In fact,

PROPOSITION 6.2. *If A is a Banach *-algebra having the property that*

$$\|x\| = \sup\{\max\{\ \|xy\|, \|yx\|\ \}: y \in A, \|y\| \leqq 1\}, \quad x \in A,$$

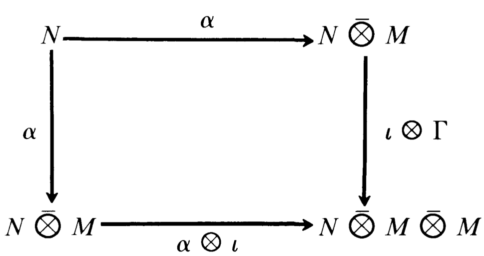
*then the norms of the unitary multipliers of A are equal to one (resp., bounded) if and only if A is (resp. is *-isomorphic to) a C^*-algebra.*

Proof. The hypothesis guarantees that A can be isometrically embedded in its multiplier algebra, so the result follows from Problems 4 and 6 of Section 15.6 in [**3**].

Note that the hypothesis is satisfied when $A = L^1(G)$ or $A(G)$.

We are indebted to J. DeCannière and C. Apostol for discussions concerning an earlier version of the above result (before we found the reference to Dieudonné).

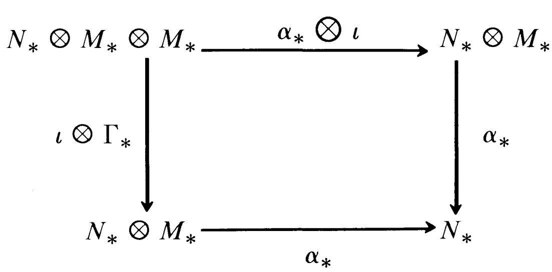
7. An application. If $\mathbf{K} = (M, \Gamma, \kappa, \phi)$ is a Kac algebra and N is a von Neumann algebra, then an action of $\mathbf{K}$ on N is defined as a unital *-monomorphism $\alpha: N \rightarrow N \bar{\otimes} M$ satisfying the co-associativity property expressed in the commutative diagram



(Enock, [**5**]).

Taking pre-duals, we obtain a map $\alpha_*: N_* \otimes M_* \rightarrow N_*$ which makes the following diagram commute:

(7.1)



Keeping in mind that Γ_* is the multiplication on the Banach algebra M_*,

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diagram (7.1) expresses the fact that N_* is a right M_*-module, although we are using the tensor product $\otimes$ rather than the more usual $\otimes_\gamma$. Pursuing this, we see almost immediately how to translate the unicity, injectivity, and involutivity of α into corresponding properties of α_*. However, the homomorphicity of α is not so easily transferred to α_*.

Let us assume that both M and N are direct sums of matrix algebras of bounded dimension. Then so is $N \bar{\otimes} M$, and we can express the homomorphicity of α in the commutative diagram

$$(7.2)\qquad \begin{array}{ccc} N \bar{\otimes} N & \xrightarrow{\alpha \otimes \alpha} & N \bar{\otimes} M \bar{\otimes} N \bar{\otimes} M \\ {\scriptstyle m_N}\big\downarrow & & \big\downarrow{\scriptstyle m_{N\bar{\otimes}M}} \\ N & \xrightarrow[\alpha]{} & N \bar{\otimes} M \end{array}$$

We want to write diagram (7.2) in a more convenient form by removing the cumbersome $m_{N\bar{\otimes}M}$. In order to do so, we define the *tensor product* $\alpha_1 \tilde{\bar{\otimes}} \alpha_2$ of two actions α_1 and α_2 of $\mathbf{K}$ on von Neumann algebras N_1 and N_2 by the commutative diagram

$$(7.3)\qquad \begin{array}{ccc} N_1 \bar{\otimes} N_2 & \xrightarrow{\alpha_1 \tilde{\bar{\otimes}} \alpha_2} & N_1 \bar{\otimes} N_2 \bar{\otimes} M \\ {\scriptstyle \alpha_1 \otimes \alpha_2}\big\downarrow & & \big\uparrow{\scriptstyle \iota \otimes m_M} \\ N_1 \bar{\otimes} M \bar{\otimes} N_2 \bar{\otimes} M & \xrightarrow[\tau]{} & N_1 \bar{\otimes} N_2 \bar{\otimes} M \bar{\otimes} M \end{array}$$

where τ is the "middle two flip":

$$\tau(x \otimes y \otimes z \otimes w) = x \otimes z \otimes y \otimes w.$$

It is tedious but straightforward to check the co-associative law for $\alpha_1 \tilde{\bar{\otimes}} \alpha_2$.

Using this, diagram (7.2) can be re-written as

$$(7.4)\qquad \begin{array}{ccc} N \bar{\otimes} N & \xrightarrow{\alpha \tilde{\bar{\otimes}} \alpha} & N \bar{\otimes} N \bar{\otimes} M \\ {\scriptstyle m_N}\big\downarrow & & \big\downarrow{\scriptstyle m_N \odot \iota} \\ N & \xrightarrow[\alpha]{} & N \bar{\odot} M \end{array}$$

In words, diagram (7.4) expresses the condition that the multiplication on N should "intertwine" the actions $\alpha \widetilde{\otimes} \alpha$ and α.

We can pre-transpose diagram (7.3) to obtain a tensor product of the right M_*-modules N_{1*} and N_{2*}:

$$(7.5)\qquad \begin{array}{ccc} N_{1*} \otimes N_{2*} \otimes M_* & \xrightarrow{\alpha_{1*} \widetilde{\otimes} \alpha_{2*}} & N_{1*} \otimes N_{2*} \\ \Big\downarrow{\scriptstyle \iota \otimes \delta_{M_*}} & & \Big\uparrow{\scriptstyle \alpha_{1*} \otimes \alpha_{2*}} \\ N_{1*} \otimes N_{2*} \otimes M_* \otimes M_* & \xrightarrow[\tau]{} & N_{1*} \otimes M_* \otimes N_{2*} \otimes M_* \end{array}$$

where δ_{M_*} is the co-multiplication on M_*. We can then use this to pre-transpose the homomorphicity of α:

$$(7.6)\qquad \begin{array}{ccc} N_* \otimes M_* & \xrightarrow{\alpha_*} & N_* \\ \Big\downarrow{\scriptstyle \delta_{N_*} \otimes \iota} & & \Big\downarrow{\scriptstyle \delta_{N_*}} \\ N_* \otimes N_* \otimes M_* & \xrightarrow[\alpha_* \widetilde{\otimes} \alpha_*]{} & N_* \otimes N_* \end{array}$$

Diagram (7.6) expresses the requirement that the co-multiplication on N_* be an M_*-module map.

Remarks. (i) The condition that the multiplication on M pre-transpose to give a co-multiplication on M_* is very restrictive. It seems likely that a construction accomplishing a similar purpose can be performed in a more general setting. For example, if G is a locally compact group, then the multiplication on $L^\infty(G)$ gives rise to a map from $L^1(G)$ to $M(L^1(G) \otimes L^1(G))$, the multiplier algebra of $L^1(G) \otimes L^1(G)$. In the general situation where M is the von Neumann algebra of a Kac algebra, this suggests a search for conditions under which the multiplication on M will give rise to a map from M_* to $M(M_* \otimes M_*)$, the multiplier algebra of $M_* \otimes M_*$, which is a Banach algebra since M is a Kac algebra. However, this still seems to require at least the continuity of the multiplication on M relative to the spatial C^*-norm, which is still fairly restrictive. Of course, this is always satisfied for $M = L^\infty(G)$, but is rarely satisfied in the case $M = \mathrm{VN}(G)$, the von Neumann algebra of the regular representation of G.

(ii) Let G be a locally compact group. Then $L^\infty(G)$ is a Kac algebra, and actions of $L^\infty(G)$ correspond to automorphic actions of G. If G is compact with irreducible representations of bounded dimension, our definition of $\alpha_1 \tilde{\otimes} \alpha_2$ via diagram (7.3) generalizes the tensor product of group actions. It would be useful to have a definition of the tensor product of actions of a Kac algebra in the general case. For example, a definition of semi-direct products of Kac algebras can be formulated using tensor products of actions. Nakagami [**11**] has proposed a definition of the tensor product of actions of a Kac algebra; unfortunately, his definition does not entail

$$\alpha_1 \tilde{\otimes} \alpha_2(N_1 \bar{\otimes} N_2) \subset N_1 \bar{\otimes} N_2 \bar{\otimes} M$$

unless K is abelian (the group case), so that he does not actually get an action.

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Arizona State University,
Tempe, Arizona