ON GROUPS WITH SMALL ORDERS OF ELEMENTS

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To Bernhard Neumann on his 90th birthday

For a periodic group $G$, denote by $\omega(G)$ the set of orders of elements in $G$. We prove that if $\omega(G)$ is a proper subset of the set $\{1, 2, 3, 4, 5\}$ then either $G$ is locally finite or $G$ contains a nilpotent normal subgroup $N$ such that $G/N$ is a 5-group.

Let $G$ be a periodic group. Denote by $\omega(G)$ the set of orders of elements in $G$. It is obvious that a group with $\omega(G) = \{1, 2\}$ is elementary Abelian. Levi and van der Waerden [5] proved that if $\omega(G) = \{1, 3\}$ then $G$ is nilpotent of class at most 3. B.H. Neumann [6] described the groups with $\omega(G) = \{1, 2, 3\}$. Sanov [8] and M. Hall [1] stated that a group $G$ with $\omega(G) \subseteq \{1, 2, 3, 4\}$, respectively with $\omega(G) \subseteq \{1, 2, 3, 6\}$, is locally finite. Nothing is known about local finiteness of groups of exponent 5, but it follows from [11] that every group $G$ with $\omega(G) = \{1, 2, 3, 5\}$ is isomorphic to the alternating group $A_5$.

In this direction, we prove the following results.

**THEOREM 1.** Let $G$ be a group with $\omega(G) = \{1, 3, 5\}$. Then one of the following holds:

(i) $G = FT$ where $F$ is a normal 5-subgroup which is nilpotent of class at most 2 and $|T| = 3$;

(ii) $G$ contains a normal 3-subgroup $T$ which is nilpotent of class at most 3 such that $G/T$ is a 5-group.

**THEOREM 2.** If $\omega(G) = \{1, 2, 5\}$ then $G$ contains either an elementary Abelian 5-subgroup of index 2, or an elementary abelian normal Sylow 2-subgroup.

**THEOREM 3.** If $\omega(G) = \{1, 2, 4, 5\}$ then one of the following holds:

(i) $G = TD$ where $T$ is a non-trivial elementary Abelian 2-group and $D$ is a non-Abelian group of order 10;

(ii) $G = FT$ where $F$ is an elementary Abelian normal 5-subgroup and $T$ is isomorphic to a subgroup of a quaternion group of order 8.

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(iii) $G$ contains a normal 2-subgroup $T$ which is nilpotent of class at most 6 such that \( G/T \) is a 5-group.

In a forthcoming paper, we prove that every group $G$ with $\omega(G) = \{1,2,3,4,5\}$ is locally finite. In this connection, we propose a conjecture that every group in the conclusion of Theorems 1-3 is also locally finite. This is equivalent to the following

**CONJECTURE 1.** Let $A$ be an automorphism group of an elementary Abelian $\{2,3\}$-group $G$ such that every non-trivial element of $A$ fixes in $G$ only the trivial element. If $A$ is of exponent 5 then $A$ is cyclic.

**NOTATION AND PRELIMINARY RESULTS**

If $H$ is a subgroup of a group $G$, $x,y \in G$, $X,Y$ are subsets of $G$ then $x^y = y^{-1}xy$, $X^y = \{y^{-1}xy \mid x \in X\}$, $[x,y] = x^{-1}y^{-1}xy$, $X^Y = \{x^y \mid x \in X, y \in Y\}$, $N_H(X) = \{g \in H \mid X^g = X\}$, $\langle X \rangle$ is the subgroup generated by $X$, $[X,Y] = \langle \{[x,y] \mid x \in X, y \in Y\}\rangle$, $C_H(X) = \{h \in H \mid (\forall x \in X)[h,x] = 1\}$, $Z(G) = C_G(G)$. For a prime $p$, $O_p(G)$ is the product of all normal $p$-subgroups of $G$, $A_m$ and $S_m$ denote, respectively, the alternating and symmetric group on $m$ letters.

An automorphism group of a group is said to be regular if every non-trivial element of it is fixed-point-free.

**LEMMA 1.** If $R = \langle r \rangle$ is a regular automorphism group of order 3 of a finite group $H$ then, for every Abelian subgroup $A$ of $H$, $\langle A^R \rangle = \langle A, A^r \rangle$ and $\langle A, A^r \rangle$ is Abelian.

**PROOF:** Let $HR$ be the natural semi-direct product of $H$ and $R$. Then $HR$ is a Frobenius group and hence $(hr^{-1})^3 = 1$ for every element $h \in H$. Since $(hr^{-1})^3 = hh^{-1}h^{-1}$. Therefore $A^{r^2} = \langle A, A^r \rangle$. Let $a,b \in A$. Then $1 = ab(ab)^r(ab)^{r^2} = ab(a^r b^r) a^{r^2} b^{r^2} = a(b b^r)(a^r a^{r^2}) b^{r^2} = a(b^{-1})^r a^{-1} b^{r^2} = [a^{-1}, b^{r^2}]$. Therefore $[a, b^{r^2}] = 1$ and $[a^r, b] = 1$. This means that $A^r$ centralises $A$. The lemma is proved.

An element of order 2 in a group is called an involution. The proof of the following well-known lemma is straightforward.

**LEMMA 2.** Let $i, j$ be involutions. Then the following hold:

1. For every $x \in \langle ij \rangle$, $x^i = x^j = x^{-1}$.
2. If the order of $ij$ is finite and odd then $i, j$ are conjugate by an element in $\langle ij \rangle$ and by an involution in $\langle i(j) \rangle$.
3. If the order of $ij$ is finite and even, then $\langle ij \rangle$ contains an involution which commutes with $i$ and $j$.

**LEMMA 3.** Suppose that $t$ is an involution in the automorphism group $A$ of a periodic group $G$. If $t$ is fixed-point-free then, for every $g \in G$, $g^t = g^{-1}$, $G$ is Abelian and $t$ lies in the centre of $A$. 

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PROOF: Let \( H = G(t) \) be the natural semi-direct product and \( g \in G \). Since \( C_H(t) = \{ t \} \) and \( t^g = g^{-1} t g = g^{-1} g' \in G \), the order of \( t^g \) is odd, by part 3 of Lemma 2. By part 2 of Lemma 2 there exists an involution \( i \in tG \) such that \( t^i = t \) and hence \( gi = 1 \) or \( gi = t \). If \( gi = 1 \) then \( g = i \in tG \) which is not the case. Thus \( gi = t \), \( g = ti \) and by part 1 of Lemma 2, \( g^i = g^{-1} \in G \). If \( a \in A \) then \( g^a = (g^{-1})^a = (g^a)^{-1} = g^{at} \) and hence \( ta = at \). The lemma is proved.

**Lemma 4.** Let \( A \) be a proper subgroup of a group \( H \). If \( A \) contains no elements of order 3 and every element in \( H \setminus A \) is a 3-element then \( A \) is normal in \( H \) and \( A \) is nilpotent of class at most 2.

**Proof:** It is obvious that \( A \) is normal in \( H \). Let \( x \) be an element of order 3 in \( H \). Then, for every \( a \in A \), \( ax^{-1} \notin A \), so \((ax^{-1})^3 = 1 \) and hence \( aa^2a^2 = 1 \). It was proved by B.H. Neumann [7] that, in this situation, every element of \( A \) commutes with each of its conjugates. By a result of Levi [4], \( A \) is nilpotent of class at most 3 and the third term \( T \) of the lower central series of \( A \) is of exponent 3. By assumption, \( T = 1 \). The lemma is proved.

**Lemma 5.** Let \( G \) be a finite group with \( \omega(G) = \{ 1, 3, 5 \} \). Then \( G \) contains a normal Sylow subgroup of prime index with non-cyclic centre.

**Proof:** Since the order of \( G \) is divisible only by two distinct prime numbers, \( G \) is soluble and \( O_p(G) \neq 1 \) for \( p = 3 \) or \( p = 5 \). If \( F/O_p(G) \) is a minimal normal subgroup of \( G/O_p(G) \) then \( F \) is a Frobenius group and \( Z(O_p(G)) \) is non-cyclic. By [3, Theorem V.8.15], \( F/O_p(G) \) is cyclic and hence \( F = G \).

**Lemma 6.** Let \( \omega(G) = \{ 1, 2, 4, 5 \} \). If \( V = O_2(G) \neq 1 \) and \( G = VD \) where \( D \) is a dihedral group of order 10 generated by an involution \( t \) and an element \( r \) of order 5 then \( V \) is elementary Abelian, \( [V, t] = C_V(t) \) and \( |V : C_V(t)| > 2 \).

**Proof:** Suppose that \( V \) is elementary Abelian. If \( v \in C_V(t) \) then \( W = \langle v^r \rangle \) is a \( D \)-invariant subgroup of order 16, \( v \in [W, t] \) and \( 2 < |W : C_W(t)| \leq |V : C_V(t)| \). Thus, it suffices to prove that \( V \) is elementary Abelian. Since \( G \) is locally finite, we can assume that \( G \) is finite and proceed by induction on \( |G| \). Suppose that \( V \) is not elementary Abelian. Let \( Z \) be a minimal normal subgroup of \( G \). Then \( Z \leq Z(V) \), \( |Z| = 16 \) and \( V/Z \) is elementary Abelian. If \( C/Z = C_{V/Z}(t) \) and \( C \) contains an element \( u \) of order 4 then \( U = \langle u^r \rangle Z \) is a \( D \)-invariant subgroup and all elements in \( U \setminus Z \) are of order 4. But then \( G \) contains an element of order 8. Thus \( C \) is elementary Abelian. Let \( v \in V \) be an element of order 4. Then, by induction, \( V = \langle v^r \rangle Z \), \( |V| = 2^{12} \) and \( |C| = 2^8 \). If \( c \in C \setminus Z \) then \( U_c = \langle c^r \rangle Z \) is an elementary Abelian group of order \( 2^8 \) and \( U_{c_1} \cap U_{c_2} = Z \) if \( c_1 \neq c_2 \). Thus \( V = U_{c_1} U_{c_2} \) for some \( c_1, c_2 \) and there exists a uniquely defined \( D \)-homomorphism \( \phi \) of the tensor product \( X \otimes Y \) of a \( D \)-module \( X = U_{c_1}/D \) and \( D \)-module \( Y = U_{c_2}/Z \) into a \( D \)-module \( Z \) which extends the map \( xZ \otimes yZ \to [x, y] \). Let \( F \) be a splitting field of \( D \) over a
field $F_2$ of order 2. \( \overline{X} = X \otimes F_2, \overline{Y} = Y \otimes F_2, \overline{Z} = Z \otimes F_2 \). Then there exists a uniquely defined homomorphism \( \overline{\phi} \) of \( \overline{X} \otimes \overline{Y} \) into \( \overline{Z} \) which extends \( \phi \). Let \( 1 \neq \lambda \in F, \lambda^5 = 1 \). We can choose bases \( \{ x_i \mid i = 1, \ldots, 4 \}, \{ y_i \mid i = 1, \ldots, 4 \} \) and \( \{ z_i \mid i = 1, \ldots, 4 \} \) of \( \overline{X}, \overline{Y}, \overline{Z} \), respectively, such that \( x_i' = \lambda^i x_i, y_i' = \lambda^i y_i, z_i' = \lambda^i z_i, i = 1, \ldots, 4 \) and \( x_1' = x_4, x_2' = x_3, y_1' = y_4, y_2' = y_3, z_1' = z_4, z_2' = z_3 \). Denote \( (x, y) = \overline{\phi}(x \otimes y) \). Since \( (x_i, y_j)' = (x_i', y_j') = (\lambda^i x_i, \lambda^j y_j) = \lambda^{i+j}(x_i, y_j) \), we see that

\[
(1) \quad (x_i, y_j)' = 0 \text{ for } (i, j) \in \{(1, 4), (2, 3), (3, 2), (4, 1)\}, \text{ and, for other pairs } (i, j), (x_i, x_j) = \alpha_{ij} z_k \text{ where } \alpha_{ij} \in F \text{ and } k \text{ is defined by } \lambda^{i+j} = \lambda^k, 1 \leq k \leq 4.
\]

Since \( C \) is Abelian,

\[
(2) \quad (x_1 + x_4, y_1 + y_4) = (x_1 + x_4, y_2 + y_3) = (x_2 + x_3, y_1 + y_4) = 0.
\]

By (1), (2) gives \( 0 = (x_1 + x_4, y_1 + y_4) = (x_1, y_1) + (x_4, y_4) = \alpha_{11} z_2 + \alpha_{44} z_2 \) and hence

\[
(x_1, y_1) = (x_4, y_4) = 0.
\]

Similarly,

\[
(x_1, y_2) = (x_4, y_2) = (x_1, y_3) = (x_4, y_3) = 0,
(x_2, y_1) = (x_3, y_1) = (x_2, y_4) = (x_3, y_4) = 0,
(x_2, y_2) = (x_3, y_2) = 0.
\]

Thus \( \overline{\phi} \) is the zero-homomorphism, \( [x, y] = 1 \) for \( x \in U_{c_1}, y \in U_{c_2} \) and \( V \) is elementary Abelian. The lemma is proved. \( \square \)

**Lemma 7.** Let \( \omega(G) \subseteq \{1, 2, 4, 5\} \). If \( G \) is locally finite then either \( G \) has a normal Sylow subgroup or \( G = VD \) where \( V = O_2(G) \) is a non-trivial elementary Abelian group and \( D \) is a dihedral group of order 10.

**Proof:** Suppose first that \( G \) is finite and proceed by induction on \( G \). If \( V = O_2(G) \neq 1 \) then, by induction, \( G/V \) contains a normal Sylow 5-subgroup \( P/V \). If \( P = G \) then the conclusion is true. If \( P \neq G \), then \( P \neq V \) and, by [3, Theorem V.8.15], \( |P : V| = 5 \), \( G/V \) is a Frobenius group of order 10 or 20. In particular, there exist involutions \( x, y \in G \setminus P \) such that \( xy \) is not a 2-element and hence \( D = \langle x, y \rangle \) is a dihedral group of order 10. Let \( H = O_2(G)D \). By Lemma 6, \( V = O_2(H) \) is elementary Abelian. If \( G/V \) contains an element of order 4 then \( G \) contains an element of order 8. Thus \( H = G \) and the conclusion is true. If \( O_2(G) = 1 \) then \( P = O_3(G) \neq 1 \) and \( C_G(P) \leq P \). By Lemma 3, \( Z(G/P) \) contains an element of order 2 and hence \( G/P \) is a 2-group.

Suppose \( G \) is infinite. If the product of every two 2-elements (every two 5-elements) in \( G \) is a 2-element (respectively, 5-element) then a Sylow 2-subgroup (respectively, a Sylow 5-subgroup) of \( G \) is normal in \( G \). Suppose that there exist elements \( x, y, z, t \) such that \( x, y \) are 2-elements, \( z, t \) are 5-elements, \( xy \) is not a 2-element and \( zt \) is not
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a 5-element. Then \( H = \langle x, y, z, t \rangle \) is a finite group without non-trivial normal Sylow subgroups. Therefore, \( V = O_2(\overline{H}) \) is elementary Abelian and \( H = VD \) where \( D \) is a dihedral group of order 10. Let \( C = C_G(V), N = CD \). By Lemma 6, every element in \( C \) is of order 2. If \( u \in G \) then \( \langle u, H \rangle \) is finite and hence \( H \leq CD = N \). Thus \( N = G \) and the lemma is proved. 

The following four lemmas can be verified by the coset enumeration algorithm (see, for instance \([9]\)):

**Lemma 8.** Let \( A = \langle a, b \mid R \rangle \). Table 1 gives the order of \( A \) for some defining relations \( R \).

| \( R \)         | \( |A| \)         | \( R \)         | \( |A| \)         |
|-----------------|------------------|-----------------|------------------|
| \( a^3, b^3, (ab)^3, (ab^{-1})^3 \) | 27               | \( [a, b]^5, (ab^{-1}ab)^3 \) | 5               |
| \( a^3, b^3, (ab)^3, (ab^{-1})^5 \) | 75               | \( [a, b]^3, (ab^{-1}ab)^3 \) | \( 3^9 \cdot 5 \) |
| \( a^3, b^5, (ab)^5, (ab^{-1})^5, (aba^{-1}b)^3 \) | 1                | \( [a, b]^3, (ab^{-1}ab)^5 \) | 5               |
| \( a^3, b^5, (ab)^5, (ab^{-1})^5, (aba^{-1}b)^5 \) | 62400            |                 |                 |

**Lemma 9.** Let \( A = \langle a, b \mid a^3, b^5, (ab)^5, (ab^{-1})^5, (ab^{-1})^5, (ab^2)^5, (ab^{-2})^5, R \rangle \). Table 2 gives the order of \( A \) for various values of \( R \).

**Lemma 10.** Let \( A = \langle a, b, c \mid a^3, b^3, c^3, aba^{-1}b^{-1}, (ac)^5, (ac^{-1})^3, (bc)^3, R \rangle \). Table 3 gives the order of \( A \) for various values of \( R \).

**Lemma 11.** Let \( A = \langle a, b \mid a^2, b^5, R \rangle \). Table 4 gives the order of \( A \) for various values of \( R \).

### Proofs of Main Results

Let \( G \) be a group with \( \omega(G) = \{1, 3, 5\} \). Note, that if \( x^3 = y^3 = 1 \) for \( x, y \in G \) then \( (xy)^3 = 1 \) or \( (x^{-1}y)^3 = 1 \). Indeed, \( X = \langle x, y \rangle \) is finite by Lemma 8 and if \( X \neq 1 \) then, by Lemma 5, \( X \) contains a normal 5-subgroup \( Y \) of index 3, hence one of the elements \( xy, x^{-1}y \) is not contained in \( Y \). Since every element in \( X \setminus Y \) is of order 3, the assertion follows.

If \( G \) contains a normal Sylow 5-subgroup \( P \) then, by Lemma 4, \( P \) is nilpotent of class at most 2 and hence \( G \) is locally finite. By Lemma 5, \( |G/P| = 3 \). Suppose that there exist elements \( a, b \in G \) of order 5 such that \( ab \) is not a 5-element. Then the order of \( c = ab \) is equal to 3 and \( \langle a, b \rangle = \langle c, a \rangle \). If \( (ca^i)^3 = 1 \) for \( i = 1, 2, 3 \) or 4 then, for \( d = ca^i \), \( \langle a, b \rangle = \langle c, a \rangle = \langle c, d \rangle \) is finite by Lemma 8. Suppose that, for \( i = 1, 2, 3, 4 \), \( (ca^i)^5 = 1 \). Since \( (c^{-1}ca)^3 = 1 \) or \( (cc^2)^3 = 1 \), by Lemma 9, \( \langle a, b \rangle = \langle c, a \rangle \) is finite. By Lemma 5, \( \langle a, b \rangle \) contains a subgroup \( T \) of order 9. Let \( H \) be a maximal 3-subgroup of \( G \) which contains \( T \). Then \( H \) is nilpotent of class at most 3 by \([5]\). Suppose that \( H \) is not a normal

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subgroup of $G$. Then there exists an element $u \notin H$ of order 3 and an element $v \in H$ such that $vu$ is not a 3-element. Then the order of $vu$ is equal to 5. Since $H$ is non-cyclic nilpotent, there exists an element $t \in H$ such that $\langle v, t \rangle$ is an elementary Abelian group of order 9. As above, $\langle t, u \rangle$ is finite and one of the elements $tu, t^{-1}u$ is of order 3. We can assume that $(tu)^3 = 1$. By Lemma 10, $\langle u, v, t \rangle$ is finite which is impossible by Lemma 5. Theorem 1 is proved.

\[ \begin{array}{c|c} \text{$R$} & |A| \\ \hline (bc^2)^3, (abc)^3, (abc^2)^3, (ab^2 c)^3, (ba^2 c)^3 & 9 \\ (bc^2)^3, (abc)^3, (abc)^3, (ab^2 c)^3, (ba^2 c)^5 & 3 \\ (bc^2)^3, (abc)^3, (abc)^2, (ab^2 c)^3, (ba^2 c)^3 & 3 \\ (bc^2)^3, (abc)^3, (abc)^5, (ab^2 c)^3, (ba^2 c)^3 & 3 \\ (bc^2)^3, (abc)^3, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 1 \\ (bc^2)^3, (abc)^5, (abc)^3, (ab^2 c)^3, (ba^2 c)^3 & 1 \\ (bc^2)^3, (abc)^5, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 1 \\ (bc^2)^3, (abc)^3, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 1 \\ (bc^2)^3, (abc)^5, (abc)^3, (ab^2 c)^3, (ba^2 c)^5 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^3, (ba^2 c)^3 & 75 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 3 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^3, (ba^2 c)^5 & 75 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^5, (ba^2 c)^3 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^3, (ba)^5 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^3, (ba)^5 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^3, (ba)^5 & 1 \\ (bc^2)^5, (abc)^3, (abc)^3, (ab^2 c)^3, (ba)^5 & 1 \end{array} \]

\[ \begin{array}{c|c} \text{$R$} & |A| \\ \hline (ab)^5, [a, b]^5, (bab)^5 & 5 \\ (ab)^5, [a, b]^4, (bab)^5 & 2^5 \cdot 5 \\ (ab)^5, [a, b]^5, (bab)^4 & 360 \\ (ab)^4, [a, b]^4, (bab)^4 & 1 \\ (ab)^4, [a, b]^5, (bab)^4 & 160 \\ (ab)^4, [a, b]^4, (bab)^4 & 2 \end{array} \]

**PROOF OF THEOREM 2:** Let $G$ be a group with $\omega(G) = \{1, 2, 5\}$. By Lemma 11, every subgroup $H$ of $G$ generated by an element of order 5 and an element of order 2 is finite. If $H$ contains a normal Sylow 5-subgroup $P$ then, by Lemma 3, $P$ is elementary Abelian and hence $H$ is a dihedral group of order 10. Suppose first that all subgroups of $G$ generated by an element of order 5 and an element of order 2 are of this type. Then the product of every two 5-elements is a 5-element and hence $O_5(G) \neq 1$. By Lemma 3, $G/O_5(G)$ contains at most one involution and hence is a 2-group. Thus there exists an $H$ containing a non-trivial Sylow 2-subgroup $T$, and hence $G$ contains an elementary Abelian subgroup $V$ of order 4. Let $F$ be the subgroup of $G$ generated by all involutions in $G$. If $F$ is a 2-group then the conclusion of the theorem is true.

Suppose that $F$ is not a 2-group. Then there exists an element $x \in G$ of order 5 such
that } x = t_1t_2 \cdots t_s \text{ where every } t_i, \ i = 1, 2, \ldots, s \text{ is an involution. Choose } x \text{ such that } s \text{ is minimal. Then } t_1 \cdots t_{s-1} \text{ is an involution and } X = \langle x, t_s \rangle \text{ is a dihedral group of order 10. Let } t = t_s. \text{ If } C_G(t) \text{ contains only one involution then, by Lemma 2, every involution in } G \text{ is a conjugate of } t, \ t \text{ is contained in a subgroup which is a conjugate of } V \text{ and hence } C_G(t) \text{ contains an involution } u \neq t. \text{ If } (ux)^2 = 1 \text{ then the involution } ut \text{ centralises } x \text{ which is impossible by assumption. Thus } \langle u, x \rangle \text{ is a finite subgroup which has a non-trivial normal Sylow 2-subgroup. This subgroup is } t\text{-invariant and hence } X = \langle u, x, t \rangle \text{ is a finite group which has no non-trivial normal Sylow subgroups. It is easy to see that } H \text{ must contain an element of order 4 contrary to the assumption. Thus, } G \text{ contains a non-trivial normal Sylow } p\text{-subgroup } P. \text{ If } p = 5 \text{ then, by Lemma 3, } P \text{ is Abelian, } G/P \text{ contains only one involution and (i) holds. If } p = 2 \text{ then } P \text{ is elementary Abelian. Theorem 2 is proved.}$

**Proof of Theorem 3:** Let } G \text{ be a group with } \omega(G) = \{1, 2, 4, 5\}.

**Lemma 12.** Suppose that every finite non-trivial subgroup of } G \text{ contains a non-trivial normal Sylow subgroup. Then } G \text{ contains a non-trivial normal Sylow subgroup and (ii) or (iii) in the conclusion of Theorem 3 holds.}

**Proof:** Let } F \text{ be the subgroup of } G \text{ generated by all involutions in } G. \text{ If } F \text{ is a 2-group then } G/F \text{ does not contain an element of order 4 or 10. By Theorem 3, the conclusion of the theorem is true. Hence } F \text{ is not a 2-group. It follows that there exists an element } x \in G \text{ of order 5 such that } x = t_1t_2 \cdots t_s \text{ where every } t_i, \ i = 1, 2, \ldots, s \text{ is an involution. Choose } x \text{ such that } s \text{ is minimal. Then } t_1 \cdots t_{s-1} \text{ is a non-trivial 2-element and, by Lemma 11, } X = \langle x, t_s \rangle \text{ is a finite group which cannot contain a normal Sylow 2-subgroup. Thus } X \text{ is a dihedral group of order 10. Let } t = t_s. \text{ If } C_G(t) \text{ contains only one involution then } C_G(t) \text{ is a finite 2-group, by } [10], \ G \text{ is locally finite and hence contains, by Lemma 7, a non-trivial normal Sylow 5-subgroup. Suppose that } C_G(t) \text{ contains an involution } u \neq t. \text{ If } (ux)^2 = 1 \text{ then the involution } ut \text{ centralises } x \text{ which is impossible by assumption. Thus } \langle u, x \rangle \text{ has a normal Sylow 2-subgroup and hence } (ux)^5 = 1. \text{ Thus } \langle u, x \rangle \text{ is a finite subgroup which has a non-trivial normal Sylow 2-subgroup. This subgroup is } t\text{-invariant and hence } \langle u, x, t \rangle \text{ is a finite group which has no non-trivial normal Sylow subgroups. This contradicts the assumption. Therefore, } G \text{ contains a non-trivial normal Sylow } p\text{-subgroup } P. \text{ If } p = 5 \text{ then, by Lemma 3, } P \text{ is Abelian, } G/P \text{ contains only one involution and, by } [3, \text{ Theorem V.8.15}], \ (ii) \text{ holds. If } p = 2 \text{ then } P \text{ is locally finite. Let } x_1, \ldots, x_7 \in P, \ y \in G \setminus P. \text{ Then the order of } y \text{ is 5 and } Y = \langle x_1, \ldots, x_7, y \rangle \text{ is a finite group with a normal Sylow 2-subgroup } Z = P \cap Y. \text{ By assumption, } (y) \text{ acts regularly on } Z \text{ and } Z \text{ is nilpotent of class at most 6 by } [2]. \text{ In particular, } [[\ldots [[x_1, x_2], x_3], \ldots], x_7] = 1. \text{ This means that } P \text{ is nilpotent of class at most 6 and (iii) holds. The lemma is proved.}$

**Suppose that } G \text{ does not contain a non-trivial normal Sylow subgroup.**

**Lemma 13.** There exists a non-trivial elementary Abelian subgroup } V \text{ in } G \text{ such
that $N = N_G(V) = VD$ where $D$ is generated by an element $r$ of order 5 and an involution $t$ with $r^4 = t^{-1}$. Furthermore, if $V_0$ is a non-trivial normal subgroup of $N$ which is contained in $V$ then $N_G(V_0) = N$.

**Proof:** By Lemmas 7 and 12, there exists a finite subgroup $H$ of $G$ such that $U = O_2(H)$ is a non-trivial elementary Abelian group and $H = UD$ where $D$ is generated by an element $r$ of order 5 and an involution $t$ with $r^4 = t^{-1}$. Let $V = C_G(U)$. Then $V$ is a locally finite 2-group and $N = VD$ is also locally finite. By Lemma 7, $V$ is elementary Abelian and, since $U \leq V$, $C_G(V) = V$. Therefore $O_2(N_G(V)) = O_2(N_G(V))D$ is also elementary Abelian and hence $O_2(N_G(V)) = V$. If $N_G(V)/V$ contains an invariant Sylow 5-subgroup then, by Lemma 12, $N_G(V)$ is locally finite. By Lemma 7, $N_G(V)/V \simeq D$.

If $N_G(V)/V$ does not contain an invariant Sylow 5-subgroup then, by Lemma 7, $N_G(V)/V$ contains a subgroup $S$ such that $O_2(S) \neq 1$ and $S/O_2(S) \simeq D$. The full preimage $U$ of $O_2(S)$ in $G$ is again elementary Abelian and hence $U = V$ contrary to the choice of $S$. Thus $N_G(V) = N$.

Suppose that $1 \neq V_0 \leq V$ and $V_0$ is normal in $N$. Then $V \leq C_G(V_0)$, $C_G(V_0)$ is a $N$-invariant 2-group and $C_G(V_0)D$ is locally finite. Again, by Lemma 7, $C_G(V_0)$ is elementary Abelian and $C_G(V_0) \leq C_G(V) = V$. Thus $C_G(V) = V$ and $N_G(V_0) \leq N_G(C_G(V_0)) = N_G(V) = N$. The lemma is proved.

Throughout the rest of the proof, $N, V, D, r, t$ are the subgroups and elements of $G$ defined in Lemma 13.

**Lemma 14.** If $v$ is an involution in $V$ then $C_G(v) \leq N$.

**Proof:** Suppose that there exists $x \in C_G(v) \setminus N$. Then $V_0 = \langle v^D, x \rangle$ is a finite 2-subgroup in $C_G(v)$ and $V_0 \not\leq N$. Let $V_1 = V \cap V_0$. Then $V \leq C_G(V_1) \leq C_G(\langle v^D \rangle) \leq V$. If $V_1$ is normal in $V_0$ then $V_0 \leq N_G(V_1) \leq N_G(C_G(V_1)) = N_G(V) = V$ contrary to the choice of $V_0$. Thus $V_2 = N_{V_0}(V_1) \neq V_2 = N_{V_0}(V_2)$. Let $y \in V_2 \setminus V_2$. Then $V_1^y \neq V_1, V_1^y \notin V, V_1^y \leq V_2 \leq N$ and $|V_1 : V_1^y : V_1| = 2$. But then $|V_1 : C_{V_1}(V_1^y)| = 2$, contradicting Lemma 6.

**Lemma 15.** If $v$ is an involution in $N \setminus V$ then $C_G(v) \leq N$.

**Proof:** Since $\langle v, v^t \rangle/V \simeq D$, $vv^t$ is of order 5 and hence $v \in N_N(R)$ for some Sylow 5-subgroup $R$ of $N$. Therefore $v$ is a conjugate of $t$ in $N$ and we can assume that $v = t$. Let $V_0$ be a subgroup of order 4 in $C_v(t)$. Then $V_1 = \langle V_0, t \rangle$ is an elementary Abelian group of order 8 in $C_G(t)$. Let $x$ be an element in $C_G(t) \setminus N$. Then $V_2 = \langle V_1, x \rangle$ is a finite subgroup in $C_G(t)$. Let $V_3 = V_2 \cap N$. Then $V_3 = V_4 \times \langle t \rangle$ where $V_4 \leq V$ and there exists $y \in N_{V_4}(V_3) \setminus N$. Since $|V_3 : V_4| = 2 < |V_4|$, $C_{V_4}(y) \neq 1$. This contradicts Lemma 14.

**Lemma 16.** $N = G$.

**Proof:** There exists an involution $v \in N$ such that $C_N(v)$ is non-Abelian. Since, by Lemma 15, $C_G(t)$ is Abelian, not all involutions of $N$ are conjugate in $G$.
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involution of $G$ are contained in $N$ then $N$ is normal in $G$ and hence $G \leq N_G(V) = N$. Suppose that $N \neq G$ and let $x$ be an involution in $G \setminus N$. Then there exists an involution $y \in N$ which is not a conjugate of $x$ in $G$. By Lemma 2, there exists an involution $z \in Z((x,y))$. By Lemmas 14 and 15, $z \in C_G(y) \leq N$ and $x \in C_G(z) \leq N$ contrary to the choice of $x$. The lemma and Theorem 3 are proved.

REFERENCES


