Y. Abe
Nagoya Math. J.
Vol. 116 (1989), 17-24

HOMOGENEOUS LINE BUNDLES OVER A TOROIDAL GROUP

YUKITAKA ABE

§ 0. Introduction

A connected complex Lie group without non-constant holomorphic functions is called a toroidal group ([5]) or an (H, C)-group ([9]). Let X be an n-dimensional toroidal group. Since a toroidal group is commutative ([5], [9] and [10]), X is isomorphic to the quotient group C^n/Γ by a lattice of C^n. A complex torus is a compact toroidal group. Cousin first studied a non-compact toroidal group ([2]).

Let L be a holomorphic line bundle over X. L is said to be homogeneous if T_x^*L is isomorphic to L for all $x \in X$, where T_x is the translation defined by $x \in X$. It is well-known that if X is a complex torus, then the following assertions are equivalent:

(1) L is topologically trivial.

(2) L is given by a representation of Γ.

(3) L is homogeneous.

But this is not always true for a toroidal group. Vogt showed in [11] that every topologically trivial holomorphic line bundle over X is homogeneous if and only if $\dim H^1(X, \mathcal{O}) < \infty$ ([6]). The cohomology groups $H^p(X, \mathcal{O})$ were classified by Kazama [3] and Kazama-Umeno [4].

In this paper we shall show the equivalence of conditions (2) and (3). In the case that X is a complex torus, a similar equivalence was proved for a vector bundle ([7] and [8]). We state our theorem.

THEOREM. *Let $X = C^n/\Gamma$ be a toroidal group. Then every homogeneous line bundle over X is given by a 1-dimensional representation of Γ.*

The converse of the above theorem is easily seen by the definitions ([11, Proposition 6]). We shall prove the theorem by virtue of the following proposition.

PROPOSITION. *Every homogeneous line bundle over a toroidal group is topologically trivial.*

Received July 27, 1987.

https://doi.org/10.1017/S0027763000001665 Published online by Cambridge University Press

§1. Preliminaries

We state some results concerning toroidal groups and fix the notations used in this paper.

If $X = C^n/\Gamma$ is a toroidal group, then the rank of Γ is $n + m$, $0 < m \leqq n$, Let $p^1 = (p_{11}, \cdots, p_{n,1}), \cdots, p^{n+m} = (p_{1,n+m}, \cdots, p_{n,n+m}) \in C^n$ be generators of Γ. The $n \times (n + m)$ matrix

$$P = ({}^t p^1, \cdots, {}^t p^{n+m})$$

is called a period matrix of Γ. We may assume by Proposition 2 in [11] that Γ has a period matrix P as follows

$$P = \begin{pmatrix} 0 & T \\ I_{n-m} & R \end{pmatrix}, \tag{1.1}$$

where I_{n-m} is the $(n - m) \times (n - m)$ unit matrix, T is a period matrix of an m-dimensional complex torus and R is a real $(n - m) \times 2m$ matrix with

$$\sigma R \not\equiv 0 \mod Z^{2m} \qquad \text{for all } \sigma \in Z^{n-m}\backslash\{0\}. \tag{1.2}$$

Let R_Γ^{n+m} be the real-linear subspace of C^n spanned by Γ. We denote by C_Γ^m the maximal complex-linear subspace contained in R_Γ^{n+m}. When a period matrix P of Γ has the form as (1.1), C_Γ^m is the space of the first m variables. Then we take the coordinates of $C^n = C_\Gamma^m \times C^{n-m}$ as (z, w) with $z \in C_\Gamma^m$, $w \in C^{n-m}$.

We refer the reader to [11] for the definitions of factors of automorphy and summands of automorphy.

Lemma 1 ([11, Proposition 8]). *Let $X = C^n/\Gamma$ be a toroidal group. Then every summand of automorphy b: $\Gamma \times C^n \to C$ is equivalent to a summand of automorphy a: $\Gamma \times C^n \to C$ with the following properties*:

a) $a(\gamma; z, w) = a(\gamma, w)$ *for all* $\gamma \in \Gamma$.

b) $a(\gamma; z, w) = 0$ *for* $\gamma \in (0 \quad Z^{n-m})$.

c) *For all* $\gamma \in \Gamma$ *the holomorphic function* $a_\gamma(w) := a(\gamma, w)$ *is* Z^{n-m}-*periodic.*

A homomorphism α: $\Gamma \to C^*$ is called a (1-dimensional) representation of Γ. Two representations α and β of Γ are equivalent if there exists a holomorphic function g: $C^n \to C^*$ such that

$$g(x + \gamma)\alpha(\gamma)g(x)^{-1} = \beta(\gamma)$$

for all $\gamma \in \Gamma$ and $x = (z, w) \in \boldsymbol{C}^n$.

LEMMA 2. *Let $X = \boldsymbol{C}^n/\Gamma$ be a toroidal group and let $\alpha\colon \Gamma \to \boldsymbol{C}_1^\times = \{\zeta \in \boldsymbol{C}; |\zeta| = 1\}$ be a homomorphism. If α is equivalent to the constant map 1, then there exists a $\boldsymbol{C}$-linear form L on $\boldsymbol{C}^n$ depending only on w such that*

$$\alpha(\gamma) = \boldsymbol{e}(L(\gamma)) \qquad \textit{for all } \gamma \in \Gamma\,,$$

where $\boldsymbol{e}(x) = \exp(2\pi\sqrt{-1}\,x)$.

Proof. By the assumption, there exists a holomorphic function g: $\boldsymbol{C}^n \to \boldsymbol{C}^*$ such that

$$g(x + \gamma)\alpha(\gamma)g(x)^{-1} = 1 \qquad \text{for all } \gamma \in \Gamma \quad \text{and} \quad x \in \boldsymbol{C}^n\,. \tag{1.3}$$

We have a holomorphic function h: $\boldsymbol{C}^n \to \boldsymbol{C}$ with $g(x) = \boldsymbol{e}(h(x))$. All first order derivatives of h are Γ-periodic by (1.3). Then we can write $h(x) = -\mathscr{L}(x) + c$, where $\mathscr{L}(x)$ is a $\boldsymbol{C}$-linear form on $\boldsymbol{C}^n$ and c is a complex number. Using (1.3) again, we have $\alpha(\gamma) = \boldsymbol{e}(\mathscr{L}(\gamma))$. Since $|\alpha(\gamma)| = 1$ for all $\gamma \in \Gamma$, L is real-valued on $\boldsymbol{R}_\Gamma^{n+m}$. Then L is constant on $\boldsymbol{C}_\Gamma^m$.

A factor of automorphy $\alpha(\gamma; z, w)$ is called a theta factor if it is expressed by a linear polynomial $\ell_\gamma(z, w)$ on (z, w) as $\alpha(\gamma; z, w) = \boldsymbol{e}(\ell_\gamma(z, w))$.

LEMMA 3 ([5]). *Let $\rho(\gamma; z, w)$ be a theta factor for Γ on $\boldsymbol{C}^n$. Then there exist a hermitian form $\mathscr{H}$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ with $\mathscr{A} := \operatorname{Im}\mathscr{H}$ $\boldsymbol{Z}$-valued on $\Gamma \times \Gamma$, a $\boldsymbol{C}$-bilinear symmetric form $\mathscr{Q}$, a $\boldsymbol{C}$-linear form $\mathscr{L}$ and a semi-character ψ of Γ associated with $\mathscr{A}|_{\Gamma\times\Gamma}$ such that*

$$\rho(\gamma; z, w) = \psi(\gamma)\boldsymbol{e}\left[\frac{1}{2\sqrt{-1}}(\mathscr{H} + \mathscr{Q})(\gamma; z, w) + \frac{1}{4\sqrt{-1}}(\mathscr{H} + \mathscr{Q})(\gamma, \gamma) + \mathscr{L}(\gamma)\right]$$

for all $\gamma \in \Gamma$ and $(z, w) \in \boldsymbol{C}^n$. We say that ρ is of type $(\mathscr{H}, \psi, \mathscr{Q}, \mathscr{L})$ when it has an expression as the above.

Remark. If rank $\Gamma = 2n$, then ρ is of the unique type. But in general, a type of ρ is not uniquely decided. Let $\boldsymbol{R}_\Gamma^{n+m} = \boldsymbol{C}_\Gamma^m \oplus V$, where V is a real-linear subspace of $\boldsymbol{R}_\Gamma^{n+m}$. Then $\boldsymbol{C}^n = \boldsymbol{C}_\Gamma^m \oplus V \oplus \sqrt{-1}\,V$. A hermitian form $\mathscr{H}$ changes according to the choice of $\mathscr{A}|_{V\times\sqrt{-1}V}$. We may assume that $\mathscr{A}|_{V\times\sqrt{-1}V} = 0$.

§ 2. Proof of the proposition

Let L be a homogeneous line bundle over a toroidal group $X = \boldsymbol{C}^n/\Gamma$. We may assume by a result of Vogt ([12], see also [1]) that $L = L_\alpha \otimes L_\rho$,

https://doi.org/10.1017/S0027763000001665 Published online by Cambridge University Press

where L_α is a topologically trivial holomorphic line bundle given by a factor of automorphy α and L_ρ is a theta bundle given by a theta factor ρ. Furthermore we may assume that ρ is reduced, i.e. ρ is of type $(\mathscr{H}, \psi) = (\mathscr{H}, \psi, 0, 0)$, and α has the properties in Lemma 1.

Let $\pi: \boldsymbol{C}^n \to X$ be the projection. Take any $x^* = (x_1^*, x_2^*) \in \boldsymbol{C}_\Gamma^m \times \boldsymbol{Z}^{n-m}$, and set $x = \pi(x^*)$. The pull-back $T_x^* L$ of L by a translation T_x is given by a factor of automorphy $\alpha(\gamma, w - x_2^*)\rho(\gamma; z - x_1^*, w - x_2^*)$. Since $\alpha(\gamma, w)$ is $\boldsymbol{Z}^{n-m}$-periodic, we have $T_x^* L_\alpha = L_\alpha$. Then $T_x^* L_\rho \cong L_\rho$. We set $a := -x^*$ and $\rho_1(\gamma; z, w) := \rho(\gamma; z - x_1^*, w - x_2^*)$. Then ρ_1 is of type $(\mathscr{H}, \psi_1, 0, \mathscr{L}_1)$, where

$$\psi_1(\gamma) := \psi(\gamma)\boldsymbol{e}(-\mathscr{A}(a, \gamma)),$$

$$\mathscr{L}_1(z, w) := \frac{1}{2\sqrt{-1}}\mathscr{H}(a; z, w).$$

We define a homomorphism $\beta: \Gamma \to \boldsymbol{C}_1^\times$ by

$$\beta(\gamma) := \psi(\gamma)\psi_1(\gamma)^{-1} = \boldsymbol{e}(\mathscr{A}(a, \gamma)).$$

Since $\rho \cdot \rho_1^{-1}$ is equivalent to 1, β is also equivalent to 1. By Lemma 2 there exists a $\boldsymbol{C}$-linear form $\mathscr{L}$ on $\boldsymbol{C}^n$ depending only on w such that

$$\beta(\gamma) = \boldsymbol{e}(\mathscr{L}(\gamma)) \qquad \text{for all } \gamma \in \Gamma.$$

It follows immediately from the above equality that

$$\mathscr{A}(a, x) = \mathscr{L}(x) \qquad \text{for all } x \in \boldsymbol{R}_\Gamma^{n+m}.$$

Since $a \in \boldsymbol{C}_\Gamma^m \times \boldsymbol{Z}^{n-m}$ is arbitrary, have

$$\mathscr{A}(x, y) = 0 \qquad \text{for all } x \in \boldsymbol{R}_\Gamma^{n+m} \quad \text{and} \quad y \in \boldsymbol{C}_\Gamma^m.$$

By Remark below Lemma 3 we may assume that $\mathscr{A}|_{V \times \sqrt{-1}V} = 0$. Then we have

$$\mathscr{A}|_{\boldsymbol{C}_\Gamma^m \times \boldsymbol{C}^n} = 0 \quad \text{and} \quad \mathscr{A}|_{\boldsymbol{C}^n \times \boldsymbol{C}_\Gamma^m} = 0, \tag{2.1}$$

because $\mathscr{A}$ is the imaginary part of a hermitian form $\mathscr{H}$. By (2.1) a hermitian form $\mathscr{H}$ is regarded as a hermitian form on $\boldsymbol{C}^{n-m} \times \boldsymbol{C}^{n-m}$.

We set $(I_{n-m} \ R) = ({}^t e_1, \cdots, {}^t e_{n-m}, {}^t r_1, \cdots, {}^t r_{2m})$ in the period matrix (1.1). Every r_k is represented as

$$r_k = \sum_{j=1}^{n-m} r_{j,k} e_j, \qquad r_{j,k} \in \boldsymbol{R}.$$

For any i and k we have

https://doi.org/10.1017/S0027763000001665 Published online by Cambridge University Press

$$\mathscr{A}(e_i, r_k) = \sum_{j=1}^{n-m} r_{j,k}\mathscr{A}(e_i, e_j) \in \boldsymbol{Z}\,.$$

Since $X = \boldsymbol{C}^n/\Gamma$ is a toroidal group, we obtain by (1.2) that

$$\mathscr{A}(e_i, e_j) = 0 \qquad \text{for all } i, j = 1, \cdots, n-m\,. \tag{2.2}$$

By (2.1) and (2.2) we conclude

$$\mathscr{A} = 0 \qquad \text{on } \boldsymbol{C}^n \times \boldsymbol{C}^n\,, \tag{2.3}$$

hence $\mathscr{H} = 0$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$. This means that L_ρ is given by a representation of Γ, therefore L_ρ is topologically trivial.

§ 3. Proof of the theorem

Let L be a homogeneous line bundle over a toroidal group $X = \boldsymbol{C}^n/\Gamma$. By Proposition L is topologically trivial. Then L is given by a factor of automorphy $\alpha(\gamma, w) = \exp(a(\gamma, w))$, where a summand of automorphy $a(\gamma, w)$ has the properties in Lemma 1. Since L is homogeneous, $a(\gamma, w + x)$ and $a(\gamma, w)$ are equivalent for all $x \in \boldsymbol{C}^{n-m}$. That is, there exist a holomorphic function $g_x\colon \boldsymbol{C}^n \to \boldsymbol{C}$ and a homomorphism $n_x\colon \Gamma \to \boldsymbol{Z}$ for any x such that

$$g_x(z + \gamma_1, w + \gamma_2) - g_x(z, w) = a(\gamma, w + x) - a(\gamma, w) + 2\pi\sqrt{-1}\,n_x(\gamma) \tag{3.1}$$

for all $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ and $(z, w) \in \boldsymbol{C}^n$. We see by (3.1) that all first order derivatives of g_x with respect to z are Γ-periodic. Then g_x is expressed as

$$g_x(z, w) = \ell_x(z) + h_x(w)\,,$$

where $\ell_x(z)$ is a $\boldsymbol{C}$-linear form on $\boldsymbol{C}^m_\Gamma$ and $h_x(w)$ is a holomorphic function on $\boldsymbol{C}^{n-m}$. By (3.1) it holds that

$$\begin{aligned} h_x(w + \gamma_2) - h_x(w) &= a(\gamma, w + x) - a(\gamma, w) + 2\pi\sqrt{-1}\,n_x(\gamma) - \ell_x(\gamma_1) \\ &= a(\gamma, w + x) - a(\gamma, w) + c_x(\gamma) \end{aligned} \tag{3.2}$$

for all $\gamma \in \Gamma$ and $w \in \boldsymbol{C}^{n-m}$, where we set $c_x(\gamma) = 2\pi\sqrt{-1}\,n_x(\gamma) - \ell_x(\gamma_1)$.

Let $p^j = (p_1^j, p_2^j) \in \boldsymbol{C}^m_\Gamma \times \boldsymbol{C}^{n-m}$. We define a $\boldsymbol{C}$-linear form $\mathscr{L}_x(w)$ on $\boldsymbol{C}^{n-m}$ by

$$\mathscr{L}_x(w) := \sum_{j-1}^{n-m} c_x(p^j) w_j\,.$$

Putting $\tilde{g}_x(w) := h_x(w) - \mathscr{L}_x(w)$, we have by (3.2) that

$$\tilde{g}_x(w + \gamma_2) - \tilde{g}_x(w) = a(\gamma, w + x) - a(\gamma, w) + c_x(\gamma) - \mathscr{L}_x(\gamma_2)$$

for all $\gamma \in \Gamma$ and $w \in C^{n-m}$. We set newly $g_x(w) = \tilde{g}_x(w)$ and $c_x(\gamma) = c_x(\gamma) - \mathscr{L}_x(\gamma_2)$. Then we have

$$(3.1') \qquad g_x(w + \gamma_2) - g_x(w) = a(\gamma, w + x) - a(\gamma, w) + c_x(\gamma)$$

for all $\gamma \in \Gamma$ and $w \in C^{n-m}$, where $c_x(\gamma) = 0$ for $\gamma \in (0 \quad Z^{n-m})$ and $g_x(w)$ is a Z^{n-m}-periodic holomorphic function on C^{n-m}.

We set $(I_{n-m} \quad R) = ({}^t s_1, \cdots, {}^t s_{n+m})$, i.e. $s_j = p_2^j$ and define

$$b_x^j(w) := a(p^j, w + x) - a(p^j, w) + c_x(p^j)\,.$$

Then $b_x^j(w)$ is a Z^{n-m}-periodic holomorphic function on C^{n-m}. We obtain by (3.1′) that

$$(3.3) \qquad g_x(w + s_j) - g_x(w) = b_x^j(w)\,, \qquad j = 1, \cdots, n + m\,.$$

We put

$$a(p^j, w) = \sum_{\sigma \in Z^{n-m}} a_{j,\sigma} e(\sigma^t w)\,,$$

$$b_x^j(w) = \sum_{\sigma \in Z^{n-m}} b_{x,\sigma}^j e(\sigma^t w)$$

and

$$g_x(w) = \sum_{\sigma \in Z^{n-m}} g_{x,\sigma} e(\sigma^t w)\,.$$

Since $g_x(w)$ is a solution of the system of difference equations (3.3), we have

$$b_{x,0}^j = c_x(p^j) = 0$$

and

$$g_{x,\sigma} = \frac{b_{x,\sigma}^j}{e(\sigma^t s_j) - 1}\,, \qquad \sigma \neq 0$$

for j with $\sigma^t s_j \notin Z$ ([11, Lemma 2]). The system of difference equations (3.3) is independent of $g_{x,0}$. So we may assume that $g_{x,0} = 0$. It follows from the definition of b_x^j that

$$(3.4) \qquad g_{x,\sigma} = a_{j,\sigma} \frac{e(\sigma^t x) - 1}{e(\sigma^t s_j) - 1}\,, \qquad \sigma \neq 0\,.$$

For any $\gamma \in \Gamma$ we have

$$e(\sigma^t(x + \gamma_2)) - 1 = e(\sigma^t \gamma_2)(e(\sigma^t x) - 1) + e(\sigma^t \gamma_2) - 1\,.$$

Using (3.1′), (3.4) and the above equality, we get

$$(3.5) \qquad g_{x+\gamma_2}(w) - g_x(w) = a(\gamma, w + x) - a(\gamma, w) + g_{\gamma_2}(w)$$

for all $\gamma \in \Gamma$ and $w \in \boldsymbol{C}^{n-m}$.

The series $\sum_{\sigma \in \boldsymbol{Z}^{n-m}} g_{x,\sigma}$ is absolutely convergent at each point $x \in \boldsymbol{C}^{n-m}$. We shall show that this series is uniformly absolutely convergent in the wider sense on $\boldsymbol{C}^{n-m}$. Let

$$A_\sigma := \begin{cases} \dfrac{a_{j,\sigma}}{e(\sigma^t s_j) - 1} & \text{if } \sigma \neq 0 \\ 0 & \text{if } \sigma = 0 \,. \end{cases}$$

Then

$$g_{x,\sigma} = A_\sigma(e(\sigma^t x) - 1) \qquad \text{for } \sigma \neq 0 \,.$$

It suffices to show that $\sum_{\sigma \in \boldsymbol{Z}^{n-m}} A_\sigma X^\sigma$ is uniformly absolutely convergent in the wider sense of $\boldsymbol{C}^{n-m}$. Now we set

$$r_\sigma(x) := \exp\,(-\,2\pi\sigma\,{}^t\mathrm{Im}\, x)\,.$$

Then we have

$$|g_{x,\sigma}| \geqq |A_\sigma||r_\sigma(x) - 1|\,.$$

We can write $r_\sigma(x) = r_1(x_1)^{\sigma_1 \cdots r_{n-m}}(x_{n-m})^{\sigma_{n-m}}$, where $r_i(x_i) := \exp\,(-\,2\pi\,\mathrm{Im}\, x_i)$, $i = 1, \cdots, n-m$. There exists a positive number C such that for sufficiently large $r_1(x_1), \cdots, r_{n-m}(x_{n-m})$

$$|r_\sigma(x) - 1| \geqq C r_1(x_1)^{\sigma_1 \cdots r_{n-m}}(x_{n-m})^{\sigma_{n-m}}$$

for all $\sigma_1 > 0, \cdots, \sigma_{n-m} > 0$. Thus we have

$$\text{(3.6)} \qquad \sum_{\sigma_1 \geqq 0, \cdots, \sigma_{n-m} \geqq 0} |A_\sigma||r_\sigma(x) - 1| \geqq C \sum_{\sigma_1 > 0, \cdots, \sigma_{n-m} \geqq 0} |A_\sigma| r_1(x_1)^{\sigma_1 \cdots r_{n-m}}(x_{n-m})^{\sigma_{n-m}}\,.$$

This implies that the series $\sum_{\sigma_1 \geqq 0, \cdots, \sigma_{n-m} \geqq 0} A_\sigma X^\sigma$ is absolutely convergent in the wider sense on $\boldsymbol{C}^{n-m}$. Also we have

$$\text{(3.7)} \qquad \sum_{\sigma \in \boldsymbol{Z}^{n-m}} A_\sigma X^\sigma = \sum_{\sigma_1 \geqq 0, \cdots, \sigma_{n-m} \geqq 0} A_\sigma X^\sigma + \sum_{\sigma_1 < 0, \sigma_2 \geqq 0, \cdots, \sigma_{n-m} \geqq 0} A_\sigma X^\sigma + \cdots + \sum_{\sigma_1 < 0, \cdots, \sigma_{n-m} < 0} A_\sigma X^\sigma\,.$$

Since we can write $r_i(x_i)^{\sigma_i} = r_i(-\,x_i)^{-\sigma_i}$ when $\sigma_i < 0$, we obtain similar inequalities as (3.6) and each term in the right side of (3.7) is uniformly absolutely convergent in the wider sense on $\boldsymbol{C}^{n-m}$. Hence $\sum_{\sigma \in \boldsymbol{Z}^{n-m}} g_{x,\sigma}$ is uniformly absolutely convergent in the wider sense on C^{n-m}. Let $G(x) := g_x(0)$. Since each $g_{x,\sigma}$ is holomorphic, $G(X)$ is a holomorphic function on C^{n-m}. It follows from (3.5) that

$$G(x+\gamma_2) - G(x) = a(\gamma, x) - a(\gamma, 0) + G(\gamma_2) \tag{3.8}$$

for all $\gamma \in \Gamma$. This implies that a factor of automorphy $\alpha(\gamma, x) = \exp(a(\gamma, x))$ is equivalent to a representation $\exp(\phi(\gamma))$ of Γ, where $\phi(\gamma) := a(\gamma, 0) - G(\gamma_2)$.

References

[1] Abe, Y., (H, C)-groups with positive line bundles, Nagoya Math. J., **107** (1987), 1–11. (in press)
[2] Cousin, P., Sur les fonctions triplement périodiques de deux variables, Acta Math., 1–11.
[3] Kazama, H., $\bar{\partial}$ Cohomology of (H, C)-groups, Publ. R.I.M.S. Kyoto Univ., **20** (1984), 297–317.
[4] Kazama, H. and Umeno, T., Complex abelian Lie groups with finite-dimensional cohomology groups, J. Math. Soc. Japan, **36** (1984), 91–106.
[5] Kopfermann, K., Maximale Untergruppen Abelscher komplexer Liescher Gruppen, Schr. Math. Inst. Univ. Münster, **29** (1964).
[6] Malgrange, B., La cohomologie d'une variété analytique complexe à bord pseudo-convexe n'est pas nécessairement séparée, C.R. Acad. Sci. Paris Sér. A, **280** (1975), 93–95.
[7] Matsushima, Y., Fibrés holomorphes sur une tore complexe, Nagoya Math. J., **14** (1959), 1–14.
[8] Morimoto, A., Sur la classification des espaces fibrés vectories holomorphes sur une tore complexe admettant des connexions holomorphes, Nagoya Math. J., **15** (1959), 83–154.
[9] ——, Non-compact complex Lie groups without non-constant holomorphic functions, Proc. Conf. on Complex Analysis (Minneapolis 1964), Springer, 1965, 256–272.
[10] ——, On the classification of non-compact complex abelian Lie groups, Trans. Amer. Math. Soc., **123** (1966), 200–228.
[11] Vogt, Ch., Line bundles on toroidal groups, J. Reine Angew. Math., **335** (1982), 197–215.
[12] ——, Two remarks concerning toroidal groups, Manuscripta Math., **41** (1983), 217–232.

Department of Mathematics
Toyama University
Gofuku, Toyama 930, Japan