§ 0. Introduction

A connected complex Lie group without non-constant holomorphic functions is called a toroidal group ([5]) or an \((H, C)\)-group ([9]). Let \(X\) be an \(n\)-dimensional toroidal group. Since a toroidal group is commutative ([5], [9] and [10]), \(X\) is isomorphic to the quotient group \(C^*/\Gamma\) by a lattice of \(C^*\). A complex torus is a compact toroidal group. Cousin first studied a non-compact toroidal group ([2]).

Let \(L\) be a holomorphic line bundle over \(X\). \(L\) is said to be homogeneous if \(T^*_xL\) is isomorphic to \(L\) for all \(x \in X\), where \(T_x\) is the translation defined by \(x \in X\). It is well-known that if \(X\) is a complex torus, then the following assertions are equivalent:

1. \(L\) is topologically trivial.
2. \(L\) is given by a representation of \(\Gamma\).
3. \(L\) is homogeneous.

But this is not always true for a toroidal group. Vogt showed in [11] that every topologically trivial holomorphic line bundle over \(X\) is homogeneous if and only if \(\dim H^r(X, \mathcal{O}) < \infty\) ([6]). The cohomology groups \(H^r(X, \mathcal{O})\) were classified by Kazama [3] and Kazama-Umeno [4].

In this paper we shall show the equivalence of conditions (2) and (3). In the case that \(X\) is a complex torus, a similar equivalence was proved for a vector bundle ([7] and [8]). We state our theorem.

**Theorem.** Let \(X = C^*/\Gamma\) be a toroidal group. Then every homogeneous line bundle over \(X\) is given by a 1-dimensional representation of \(\Gamma\).

The converse of the above theorem is easily seen by the definitions ([11, Proposition 6]). We shall prove the theorem by virtue of the following proposition.

**Proposition.** Every homogeneous line bundle over a toroidal group is topologically trivial.

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§ 1. Preliminaries

We state some results concerning toroidal groups and fix the notations used in this paper.

If $X = C^n/\Gamma$ is a toroidal group, then the rank of $\Gamma$ is $n + m$, $0 < m \leq n$. Let $p^i = (p_{1i}, \ldots, p_{ni}), \ldots, p^{n+m} = (p_{1, n+m}, \ldots, p_{n, n+m}) \in C^n$ be generators of $\Gamma$. The $n \times (n + m)$ matrix

$$P = (\{p^i, \ldots, p^{n+m}\})$$

is called a period matrix of $\Gamma$. We may assume by Proposition 2 in [11] that $\Gamma$ has a period matrix $P$ as follows

$$(1.1) \quad P = \begin{pmatrix} 0 & T \\ I_{n-m} & R \end{pmatrix},$$

where $I_{n-m}$ is the $(n - m) \times (n - m)$ unit matrix, $T$ is a period matrix of an $m$-dimensional complex torus and $R$ is a real $(n - m) \times 2m$ matrix with

$$\sigma R \notin 0 \mod Z^{2m} \quad \text{for all } \sigma \in Z^{n-m}\setminus\{0\}. \quad \tag{1.2}$$

Let $R_{p;m}$ be the real-linear subspace of $C^n$ spanned by $\Gamma$. We denote by $C_{p;m}$ the maximal complex-linear subspace contained in $R_{p;m}$. When a period matrix $P$ of $\Gamma$ has the form as (1.1), $C_{p;m}$ is the space of the first $m$ variables. Then we take the coordinates of $C^n = C_{p;m} \times C^{n-m}$ as $(z, w)$ with $z \in C_{p;m}$, $w \in C^{n-m}$.

We refer the reader to [11] for the definitions of factors of automorphy and summands of automorphy.

**Lemma 1** ([11, Proposition 8]). Let $X = C^n/\Gamma$ be a toroidal group. Then every summand of automorphy $b: \Gamma \times C^n \to C$ is equivalent to a summand of automorphy $a: \Gamma \times C^n \to C$ with the following properties:

a) $a(\gamma; z, w) = a(\gamma, w)$ for all $\gamma \in \Gamma$.

b) $a(\gamma; z, w) = 0$ for $\gamma \in (0 \, Z^{n-m})$.

c) For all $\gamma \in \Gamma$ the holomorphic function $a(\gamma; w) := a(\gamma, w)$ is $Z^{n-m}$-periodic.

A homomorphism $\alpha: \Gamma \to C^*$ is called a (1-dimensional) representation of $\Gamma$. Two representations $\alpha$ and $\beta$ of $\Gamma$ are equivalent if there exists a holomorphic function $g: C^n \to C^*$ such that

$$g(x + \gamma)\alpha(\gamma)g(x)^{-1} = \beta(\gamma).$$
for all \( \gamma \in \Gamma \) and \( x = (z, w) \in C^n \).

**Lemma 2.** Let \( X = C^n/\Gamma \) be a toroidal group and let \( \alpha: \Gamma \to C_1^\times = \{ \zeta \in C; |\zeta| = 1 \} \) be a homomorphism. If \( \alpha \) is equivalent to the constant map 1, then there exists a \( C \)-linear form \( L \) on \( C^n \) depending only on \( w \) such that
\[
\alpha(\gamma) = e(L(\gamma)) \quad \text{for all} \quad \gamma \in \Gamma,
\]
where \( e(x) = \exp(2\pi\sqrt{-1}x) \).

**Proof.** By the assumption, there exists a holomorphic function \( g: C^n \to C^* \) with \( g(x) = e(h(x)) \). All first order derivatives of \( h \) are \( \Gamma \)-periodic by (1.3). Then we can write \( h(x) = -\mathcal{L}(x) + c \), where \( \mathcal{L}(x) \) is a \( C \)-linear form on \( C^n \) and \( c \) is a complex number. Using (1.3) again, we have \( \alpha(\gamma) = e(\mathcal{L}(\gamma)) \). Since \( |\alpha(\gamma)| = 1 \) for all \( \gamma \in \Gamma \), \( L \) is real-valued on \( \mathfrak{g}^- + \mathfrak{m} \). Then \( L \) is constant on \( C^n \).

A factor of automorphy \( \alpha(\gamma; z, w) \) is called a theta factor if it is expressed by a linear polynomial \( \ell(\gamma; z, w) \) on \( (z, w) \), so \( \alpha(\gamma; z, w) = e(\ell(\gamma; z, w)) \).

**Lemma 3** ([5]). Let \( \rho(\gamma; z, w) \) be a theta factor for \( \Gamma \) on \( C^n \). Then there exist a hermitian form \( \mathcal{H} \) on \( C^n \times C^n \) with \( \mathcal{H} := \text{Im} \mathcal{H} \) \( \mathbb{Z} \)-valued on \( \Gamma \times \Gamma \), a \( C \)-bilinear symmetric form \( \mathcal{J} \), a \( C \)-linear form \( \mathcal{L} \) and a semi-character \( \psi \) of \( \Gamma \) associated with \( \mathcal{H}^\Gamma \) such that
\[
\rho(\gamma; z, w) = \psi(\gamma)e\left[\frac{1}{2\sqrt{-1}}(\mathcal{H} + \mathcal{J})(\gamma; z, w) + \frac{1}{4\sqrt{-1}}(\mathcal{H} + \mathcal{J})(\gamma, \gamma) + \mathcal{L}(\gamma)\right]
\]
for all \( \gamma \in \Gamma \) and \( (z, w) \in C^n \). We say that \( \rho \) is of type \((\mathcal{H}, \psi, \mathcal{J}, \mathcal{L})\) when it has an expression as the above.

**Remark.** If rank \( \Gamma = 2n \), then \( \rho \) is of the unique type. But in general, a type of \( \rho \) is not uniquely decided. Let \( \mathcal{R}_F^{+m} = C_F^m \oplus V \), where \( V \) is a real-linear subspace of \( \mathcal{R}_F^{+m} \). Then \( C^n = C_F^n \oplus V \oplus \sqrt{-1}V \). A hermitian form \( \mathcal{H} \) changes according to the choice of \( \mathcal{H}^\Gamma \). We may assume that \( \mathcal{H}^\Gamma_{\mathbb{R} \times \sqrt{-1} \mathbb{R}} = 0 \).

§ 2. Proof of the proposition

Let \( L \) be a homogeneous line bundle over a toroidal group \( X = C^n/\Gamma \). We may assume by a result of Vogt ([12], see also [1]) that \( L = L_\alpha \otimes L_\psi \),

\[\text{https://www.cambridge.org/core/terms} \]
\[\text{https://doi.org/10.1017/S0027763000001665} \]
where \( L_a \) is a topologically trivial holomorphic line bundle given by a factor of automorphy \( \alpha \) and \( L_p \) is a theta bundle given by a theta factor \( \rho \). Furthermore we may assume that \( \rho \) is reduced, i.e. \( \rho \) is of type \((\mathscr{H}, \psi) = (\mathscr{H}, \psi, 0, 0)\), and \( \alpha \) has the properties in Lemma 1.

Let \( \pi : C^n \to X \) be the projection. Take any \( x^* = (x_1^*, x_2^*) \in C^n \times Z^{n-m} \), and set \( x = \pi(x^*) \). The pull-back \( T^x_*L \) of \( L \) by a translation \( T_x \) is given by a factor of automorphy \( \alpha(\gamma, w - x_1^*)\rho(\gamma; z - x_1^*, w - x_2^*) \). Since \( \alpha(\gamma, w) \) is \( Z^{n-m} \)-periodic, we have \( T^x_*L_a = L_a \). Then \( T^x_*L_p \cong L_p \). We set \( \alpha := - x^* \) and \( \rho_1(\gamma; z, w) := \rho(\gamma; z - x_1^*, w - x_2^*) \). Then \( \rho_1 \) is of type \((\mathscr{H}, \psi, 0, \mathscr{L})\), where

\[
\psi_1(\gamma) := \psi(\gamma) e(-\mathscr{A}(a, \gamma)), \\
\mathscr{L}(z, w) := \frac{1}{2\sqrt{-1}} \mathscr{H}(a; z, w).
\]

We define a homomorphism \( \beta : \Gamma \to C^n \) by

\[
\beta(\gamma) := \psi(\gamma) e\left(-\mathscr{A}(a, \gamma)\right) = e(\mathscr{A}(a, \gamma)).
\]

Since \( \rho \cdot \rho_1 \) is equivalent to 1, \( \beta \) is also equivalent to 1. By Lemma 2 there exists a \( C \)-linear form \( \mathscr{L} \) on \( C^n \) depending only on \( w \) such that

\[
\beta(\gamma) = e(\mathscr{L}(\gamma)) \quad \text{for all } \gamma \in \Gamma.
\]

It follows immediately from the above equality that

\[
\mathscr{A}(a, x) = \mathscr{L}(x) \quad \text{for all } x \in R^{n+m}_r.
\]

Since \( a \in C^n \times Z^{n-m} \) is arbitrary, have

\[
\mathscr{A}(x, y) = 0 \quad \text{for all } x \in R^{n+m}_r \text{ and } y \in C^n.
\]

By Remark below Lemma 3 we may assume that \( \mathscr{A}|_{\Gamma \times \mathbb{R}^n} = 0 \). Then we have

\[
(2.1) \quad \mathscr{A}|_{C^n \times C^n} = 0 \text{ and } \mathscr{A}|_{C^n \times C^n} = 0,
\]

because \( \mathscr{A} \) is the imaginary part of a hermitian form \( \mathscr{H} \). By (2.1) a hermitian form \( \mathscr{H} \) is regarded as a hermitian form on \( C^{n-m} \times C^{n-m} \).

We set \( (r_{n-m}, R) = (e_1, \ldots, e_{n-m}, r_1, \ldots, r_{m}) \) in the period matrix (1.1). Every \( r_k \) is represented as

\[
r_k = \sum_{j=1}^{n-m} r_{j,k} e_j, \quad r_{j,k} \in \mathbb{R}.
\]

For any \( i \) and \( k \) we have
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\[ J^*(e^i, r) = \sum_{j=1}^{n-m} r_j, e_j \in \mathbb{Z}. \]

Since \( X = C^n/\Gamma \) is a toroidal group, we obtain by (1.2) that

(2.2) \[ J^*(e^i, e_j) = 0 \quad \text{for all } i, j = 1, \ldots, n - m. \]

By (2.1) and (2.2) we conclude

(2.3) \[ J^* = 0 \quad \text{on } C^n \times C^n, \]

hence \( J^* = 0 \) on \( C^n \times C^n \). This means that \( L^*_1 \) is given by a representation of \( \Gamma \), therefore \( L^*_1 \) is topologically trivial.

§ 3. Proof of the theorem

Let \( L \) be a homogeneous line bundle over a toroidal group \( X = C^n/\Gamma \). By Proposition \( L \) is topologically trivial. Then \( L \) is given by a factor of automorphy \( \alpha(\gamma, w) = \exp(a(\gamma, w)), \) where a summand of automorphy \( a(\gamma, w) \) has the properties in Lemma 1. Since \( L \) is homogeneous, \( a(\gamma, w + x) \) and \( a(\gamma, w) \) are equivalent for all \( x \in C^{n-m} \). That is, there exist a holomorphic function \( g_x : C^n \to C \) and a homomorphism \( n_x : \Gamma \to \mathbb{Z} \) for any \( x \) such that

(3.1) \[ g_x(z + \gamma, w + \gamma) - g_x(z, w) = a(\gamma, w + x) - a(\gamma, w) + 2\pi \sqrt{-1} n_x(\gamma) \]

for all \( \gamma \in \Gamma \) and \( z, w \in C^n \). We see by (3.1) that all first order derivatives of \( g_x \) with respect to \( z \) are \( \Gamma \)-periodic. Then \( g_x \) is expressed as

\[ g_x(z, w) = \ell_x(z) + h_x(w), \]

where \( \ell_x(z) \) is a \( C \)-linear form on \( C^n \) and \( h_x(w) \) is a holomorphic function on \( C^{n-m} \). By (3.1) it holds that

(3.2) \[ h_x(w + \gamma) - h_x(w) = a(\gamma, w + x) - a(\gamma, w) + 2\pi \sqrt{-1} n_x(\gamma) - \ell_x(\gamma) \]

\[ = a(\gamma, w + x) - a(\gamma, w) + c_x(\gamma) \]

for all \( \gamma \in \Gamma \) and \( w \in C^{n-m} \), where we set \( c_x(\gamma) = 2\pi \sqrt{-1} n_x(\gamma) - \ell_x(\gamma) \).

Let \( p^l = (p^l_1, p^l_2) \in C^n \times C^{n-m} \). We define a \( C \)-linear form \( L_x(w) \) on \( C^{n-m} \) by

\[ L_x(w) := \sum_{j=1}^{n-m} c_x(p^l_j) w_j. \]

Putting \( \tilde{g}_x(w) := h_x(w) - L_x(w) \), we have by (3.2) that

\[ \tilde{g}_x(w + \gamma) - \tilde{g}_x(w) = a(\gamma, w + x) - a(\gamma, w) + c_x(\gamma) - L_x(\gamma). \]
for all $\gamma \in \Gamma$ and $w \in \mathcal{C}^{n-m}$. We set newly $g_x(w) = \tilde{g}_x(w)$ and $c_x(\gamma) = c_x(\gamma) - \mathcal{L}_x(\gamma)$. Then we have

$$(3.1') \quad g_x(w + \gamma) - g_x(w) = a(\gamma, w + x) - a(\gamma, w) + c_x(\gamma)$$

for all $\gamma \in \Gamma$ and $w \in \mathcal{C}^{n-m}$, where $c_x(\gamma) = 0$ for $\gamma \in (0 \mathcal{Z}^{n-m})$ and $g_x(w)$ is a $\mathcal{Z}^{n-m}$-periodic holomorphic function on $\mathcal{C}^{n-m}$.

We set $(I_{n-m} R) = (s_1, \ldots, s_{n+m})$, i.e. $s_j = p_j$ and define

$$b_j^i(w) := a(p', w + x) - a(p', w) + c_x(p').$$

Then $b_j^i(w)$ is a $\mathcal{Z}^{n-m}$-periodic holomorphic function on $\mathcal{C}^{n-m}$. We obtain by (3.1') that

$$(3.3) \quad g_x(w + s_j) - g_x(w) = b_j^i(w), \quad j = 1, \ldots, n + m.$$  

We put

$$a(p', w) = \sum_{\sigma' \in \mathcal{Z}^{n-m}} a_{j,\sigma} e(\sigma'w),$$

$$b_j^i(w) = \sum_{\sigma' \in \mathcal{Z}^{n-m}} b_{i,\sigma}^j e(\sigma'w)$$

and

$$g_x(w) = \sum_{\sigma' \in \mathcal{Z}^{n-m}} g_{x,\sigma} e(\sigma'w).$$

Since $g_x(w)$ is a solution of the system of difference equations (3.3), we have

$$b_{i,0}^j = c_x(p') = 0$$

and

$$g_{x,\sigma} = \frac{b_{j,\sigma}^i}{e(\sigma's_j) - 1}, \quad \sigma \neq 0$$

for $j$ with $\sigma's_j \in \mathcal{Z}$ ([11, Lemma 2]). The system of difference equations (3.3) is independent of $g_{x,0}$. So we may assume that $g_{x,0} = 0$. It follows from the definition of $b_j^i$ that

$$(3.4) \quad g_{x,\sigma} = a_{j,\sigma} \frac{e(\sigma'x) - 1}{e(\sigma's_j) - 1}, \quad \sigma \neq 0.$$  

For any $\gamma \in \Gamma$ we have

$$e(\sigma'(x + \gamma)) - 1 = e(\sigma'\gamma)(e(\sigma'x) - 1) + e(\sigma'\gamma) - 1.$$  

Using (3.1'), (3.4) and the above equality, we get

$$(3.5) \quad g_{x+\gamma}(w) - g_x(w) = a(\gamma, w + x) - a(\gamma, w) + g_{\gamma}(w)$$
for all $r \in \Gamma$ and $w \in C^{n-m}$.

The series $\sum_{s \in \mathbb{Z}^{n-m}} g_{z,s}$ is absolutely convergent at each point $x \in C^{n-m}$. We shall show that this series is uniformly absolutely convergent in the wider sense on $C^{n-m}$. Let

$$A_\sigma := \begin{cases} \frac{a_{j,s}}{e(\sigma_i s_j) - 1} & \text{if } \sigma \neq 0 \\ 0 & \text{if } \sigma = 0. \end{cases}$$

Then

$$g_{z,s} = A_\sigma (e(\sigma_i x) - 1) \quad \text{for } \sigma \neq 0.$$ 

It suffices to show that $\sum_{s \in \mathbb{Z}^{n-m}} A_\sigma X^\sigma$ is uniformly absolutely convergent in the wider sense of $C^{n-m}$. Now we set

$$r_\sigma(x) := \exp(- 2\pi \sigma \text{Im } x).$$

Then we have

$$|g_{z,s}| \geq |A_\sigma||r_\sigma(x) - 1|.$$ 

We can write $r_\sigma(x) = r_i(x_i)^{\sigma_1 + \cdots + \sigma_{n-m}(x_{n-m})}$, where $r_i(x_i) := \exp(- 2\pi \text{Im } x_i)$, $i = 1, \ldots, n - m$. There exists a positive number $C$ such that for sufficiently large $r_i(x_i), \ldots, r_{n-m}(x_{n-m})$

$$|r_\sigma(x) - 1| \geq C r(x)^{\sum_{i=1}^{n-m}(x_{n-m})}$$

for all $\sigma_1 > 0, \ldots, \sigma_{n-m} > 0$. Thus we have

(3.6) $\sum_{\sigma_1 \geq 0, \ldots, \sigma_{n-m} \geq 0} |A_\sigma||r_\sigma(x) - 1| \geq C \sum_{\sigma_1 > 0, \ldots, \sigma_{n-m} \geq 0} |A_\sigma||r_\sigma(x)^{\sum_{i=1}^{n-m}(x_{n-m})}.$

This implies that the series $\sum_{s \in \mathbb{Z}^{n-m}} A_\sigma X^\sigma$ is absolutely convergent in the wider sense on $C^{n-m}$. Also we have

(3.7) $\sum_{s \in \mathbb{Z}^{n-m}} A_\sigma X^\sigma = \sum_{\sigma_1 \geq 0, \ldots, \sigma_{n-m} \geq 0} A_\sigma X^\sigma + \sum_{\sigma_1 < 0, \sigma_2 \geq 0, \ldots, \sigma_{n-m} \geq 0} A_\sigma X^\sigma + \cdots + \sum_{\sigma_1 < 0, \ldots, \sigma_{n-m} < 0} A_\sigma X^\sigma.$

Since we can write $r_i(x_i)^{\epsilon_i} = r_i(- x_i)^{-\epsilon_i}$ when $\sigma_i < 0$, we obtain similar inequalities as (3.6) and each term in the right side of (3.7) is uniformly absolutely convergent in the wider sense on $C^{n-m}$. Hence $\sum_{s \in \mathbb{Z}^{n-m}} g_{z,s}$ is
uniformly absolutely convergent in the wider sense on $C^{n-m}$. Let $G(x) := g_x(0)$. Since each $g_x$ is holomorphic, $G(X)$ is a holomorphic function on $C^{n-m}$. It follows from (3.5) that

\[(3.8) \quad G(x + \tau) - G(x) = a(\tau, x) - a(\tau, 0) + G(\tau)\]

for all $\tau \in \Gamma$. This implies that a factor of automorphy $a(\tau, x) = \exp(a(\tau, x))$ is equivalent to a representation $\exp(\phi(\tau))$ of $\Gamma$, where $\phi(\tau) := a(\tau, 0) - G(\tau)$.

**References**


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