Third Meeting, January 13th, 1888.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

On Stirling's approximation to $n!$ when n is large.

By R. E. ALLARDICE, M.A.

The approximation is

$$n! = (n/e)^n \sqrt{2\pi n}.$$

We have

$$\left(\frac{n}{e}\right)^n = \left(\frac{n-1}{e}\right)^{n-1}.\frac{1}{e}.\left(\frac{n}{n-1}\right)^n.\left[n(n-1)\right]^{\frac{1}{2}}.\left(\frac{n-1}{n}\right)^{\frac{1}{2}}$$

$$= \left(\frac{n-1}{e}\right)^{n-1}.\frac{1}{e}.\left(\frac{n}{n-1}\right)^{n-\frac{1}{2}}.\left[n(n-1)\right]^{\frac{1}{2}}.$$

Now assume

$$\left(\frac{n}{n-1}\right)^{n-\frac{1}{2}} = e^x$$

$$\therefore\ x = (n-\tfrac{1}{2})\log\frac{n}{n-1}$$

$$= (-n+\tfrac{1}{2})\log\left(1-\frac{1}{n}\right)$$

$$= (-n+\tfrac{1}{2})\left(-\frac{1}{n} - \frac{1}{2}.\frac{1}{n^2} - \frac{1}{3}.\frac{1}{n^3} - \dots\dots\right)$$

$$= 1 + \frac{1}{12}.\frac{1}{n^2} + \frac{B}{n^3} + \dots\dots$$

$$= 1 + \frac{A}{n^2}$$

where A differs from 1/12 by a quantity of the order of $1/n$, since the above series for x is convergent; and since n is large, A may be taken to be 1/12.

Hence,
$$\left(\frac{n}{e}\right)^n = \left(\frac{n-1}{e}\right)^{n-1}.\frac{1}{e}.e^{1+A/n^2}.\left[n(n-1)\right]^{\frac{1}{2}}$$

$$= \left(\frac{n-1}{e}\right)^{n-1}.e^{A/n^2}.\left[n(n-1)\right]^{\frac{1}{2}}.$$

Substituting for n in succession the values, $2n$, $2n-1$ … … $n+1$, we get

$$\left(\frac{2n}{e}\right)^{2n}=\left[2n(2n-1)\ \ldots\ \ldots\ (n+1)(2n-1)\ \ldots\ \ldots\ n\right]^{\frac{1}{2}}.\left(\frac{n}{e}\right)^{n}.e^{k}$$

where $$k=\frac{\mathrm{A}}{(2n)^2}+\frac{\mathrm{A}}{(2n-1)^2}+\ \ldots\ \ldots\ +\frac{\mathrm{A}}{(n+1)^2}$$

$$\therefore\qquad \sqrt{2}\left(\frac{2n}{e}\right)^{2n}=\left(\frac{n}{e}\right)^{n}.\frac{(2n)\,!}{n\,!}.e^{k}.$$

$$\therefore\qquad \sqrt{2}\left(\frac{n}{e}\right)^{n}.2^{2n}=\frac{(2n)!}{n\,!}.e^{k}$$

$$\therefore\qquad n\,!=\left(\frac{n}{e}\right)^{n}.\sqrt{2}.2^{2n}.\frac{(n\,!)^2}{(2n)\,!}.e^{-k}.$$

Now, Wallis's formula for π is

$$\frac{\pi}{2}=\left[\frac{2.4.6\ \ldots\ \ldots\ 2n}{1.3.5\ \ldots\ \ldots\ 2n-1}\right]^{2}.\frac{1}{2n+1}$$

or $$\frac{\pi}{2}=\left[\frac{2.4.6\ \ldots\ \ldots\ 2n}{1.3.5\ \ldots\ \ldots\ 2n-1}\right]^{2}.\frac{1}{2n}.\left(1+\frac{1}{2n}\right)^{-1}$$

$$\therefore\qquad \sqrt{\pi}=\frac{2.4.6\ \ldots\ \ldots\ 2n}{1.3.5\ \ldots\ \ldots\ 2n-1}.\frac{1}{\sqrt{n}}.\left(1+\frac{1}{2n}\right)^{-\frac{1}{2}}$$

Hence we may write

$$\sqrt{\pi}=\frac{{}^{2n}(n\,!)^2}{(2n)\,!}.\frac{1}{\sqrt{n}}.\left(1-\frac{1}{4n}\right)$$

This result reduces the above equation for $n\,!$ to

$$n\,!=\left(\frac{n}{e}\right)^{n}.\sqrt{2}.\sqrt{\pi n}.e^{-k}.\left(1+\frac{1}{4n}\right)$$

Now, k lies between $n\mathrm{A}/(2n)^2$ and $n\mathrm{A}/(n+1)^2$. Hence if we write $n\,!=\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}$, the ratio of the error to the value found for $n\,!$ is of the order of $1/n$.

It may now be shown that in the above approximation to π, the error made bears to π a ratio of the order of $1/n$.

In the value assumed for π

$$\frac{\pi}{2}=\left[\frac{2.4.6\ \ldots\ \ldots\ 2n}{1.3.5\ \ldots\ \ldots\ 2n-1}\right]^{2}.\frac{1}{2n+1}$$

the factor u has been neglected, where

$$u=\frac{(2n+2)^2}{(2n+1)(2n+3)}.\frac{(2n+4)^2}{(2n+3)(2n+5)}\ \ldots\ \ldots.$$

Now, $$\frac{(2n+2)^2}{(2n+1)(2n+3)}=\frac{4n^2+8n+4}{4n^2+8n+3}=1+\frac{1}{(2n+1)(2n+3)}$$

$$\therefore \qquad u = \left[1 + \frac{1}{(2n+1)(2n+3)}\right]\left[1 + \frac{1}{(2n+3)(2n+5)}\right] \cdots \cdots$$

$$\therefore \qquad \log u = \frac{1}{(2n+1)(2n+3)} + \frac{1}{(2n+3)(2n+5)} + \cdots \cdots$$

$$- \tfrac{1}{2}.\left[\frac{1}{(2n+1)(2n+3)}\right]^2 - \tfrac{1}{2}\left[\frac{1}{(2n+3)(2n+5)}\right]^2 - .$$

Now $$\frac{1}{(2n+1)(2n+3)} = \tfrac{1}{2}\left[\frac{1}{2n+1} - \frac{1}{2n+3}\right].$$

Hence, neglecting small quantities of a higher order than $1/n$, we may write

$$\log u = 1/2(2n+1)$$

$$\therefore \qquad u = e^{1/2(2n+1)}$$

$$= 1 + 1/2(2n+1) + \ \ldots \ \ldots ;$$

which shows that the error made in the above approximation to π is of the order stated.

[Another elementary proof of Stirling's theorem, by Mr J. W. L. Glaisher, is given in the *Quarterly Journal of Mathematics*, vol. xv., p. 57.]

A Device for the Analysis of Intervals and Chords in Music.

By A. Y. Fraser, M.A., F.R.S.E.

§ 1. The device here described has been found to simplify greatly the "somewhat laborious discussion" of the different musical intervals as given, say, in Sedley Taylor's *Sound and Music* (chap. viii.) or in Helmholtz's *Sensations of Tone*. It has been found particularly helpful in giving an account, necessarily rapid, of the nature of harmony to classes studying sound as a part of physics, from whom much familiarity with musical terms and notation is not to be expected.

§ 2. The leading idea of the device is that the octave being what may be called a periodic phenomenon, should be represented on a circular or spiral curve, and not, as is usual, on a straight vertical line. Such a representation of four octaves is given in figure 14. (To read this and the other figures, begin at the extreme right and