PRODUCTS OF A C-MEASURE AND A LOCALLY INTEGRABLE MAPPING

MARSTON MORSE AND WILLIAM TRANSUE

1. Introduction. Let C be the field of complex numbers and E a locally compact topological space. The authors' theory of C-bimeasures Λ and their Λ-integrals in **(1**; **2)** leads to integral representation of bounded operators from A to B' where A and B are MT-spaces as defined in **(3)**. These MT-spaces include the $\mathfrak{L}$-spaces and Orlicz spaces as special cases. The object of this note is to present a theorem on integration necessary in completing the results on MT-spaces. Product measures $g \cdot \mu$, in which g is continuous on E and μ a measure, are introduced by Bourbaki in **(3**, p. 44**)**, and Bourbaki there indicates that products $g \cdot \mu$ in which g is not necessarily continuous will be studied in **(8)**. In this note α is a C-measure and y locally α-integrable in the sense defined below. We draw heavily upon the general theory of integration **(7)**.

Let $\mathfrak{X}$ be the aggregate of relatively compact open subsets X of E. Let ϕ_X be the characteristic function of X. A mapping $y \in \mathrm{C}^E$ will be said to be *locally α-integrable* if for each $X \in \mathfrak{X}$, $y\phi_X$ is α-integrable. The "product" $y \cdot \alpha$ is a C-measure with values

$$\int u\, d(y \cdot \alpha) = \int uy\, d\alpha \qquad [u \in \mathfrak{K}_{\mathrm{C}}(E)]. \tag{1.1}$$

The principal theorem of this note is as follows. Cf. **(3)** for definition of $|\alpha|$.

THEOREM 1.1. *Let α be a* C*-measure on E, and $y \in \mathrm{C}^E$ locally α-integrable. Let $x \in \mathrm{C}^E$ be such that*

$$\int^* |x|\, |y|\, d|\alpha| < \infty, \quad \int^* |x| d|y \cdot \alpha| < \infty. \tag{1.2}$$

Then (i) *these integrals are equal, and* (ii) *if either of the integrals*

$$\int xy\, d\alpha, \quad \int x\, d(y \cdot \alpha) \tag{1.3}$$

exists the other exists and

$$\int xy\, d\alpha = \int x\, d(y \cdot \alpha). \tag{1.4}$$

If the C*-measure α is replaced by a measure μ and y by a locally-integrable $g \in \overline{\mathrm{R}}^E$*, then $g \cdot \mu$ is defined as is $y \cdot \alpha$, replacing $\mathfrak{K}_{\mathrm{C}}$ by $\mathfrak{K}$.

Received January 7, 1957. The work of Dr. Transue on this paper was sponsored by the Office of Ordnance Research, U.S. Army, under contract No. DA-33-019-ORD-1265.

2. A lemma. Let μ be a positive measure on E (sense of Bourbaki). Recall that $\overline{\mathrm{R}}_+$ is the space of positive real numbers completed by the point $+\infty$ and that $\mathfrak{J}_+$ is the space of lower semi-continuous mappings in $\overline{\mathrm{R}}_+^E$.

LEMMA 2.1. *For h and λ in $\overline{\mathrm{R}}_+^E$, with λ locally μ-integrable,*

$$\int^* h\, d(\lambda\cdot\mu) = \int^* h\lambda\, d\mu \tag{2.1}$$

whenever both members of (2.1) *are finite.*

We shall establish the lemma by proving a sequence of statements. We say that (2.1) holds *finitely* if it holds, and if both members are finite.

(a) *If $h \in \mathfrak{J}_+$ is bounded with compact support* (2.1) *holds finitely.*

Set $\lambda\cdot\mu = \beta$. In accord with the definition of $\mu^*(h)$ there exists an increasing sequence (u_n) of $u_n \in \mathfrak{K}_+$ with $u_n \leqslant h$ such that

$$\lim_n \mu(u_n) = \mu^*(h).$$

But h is μ-integrable so that this can be written

$$\lim_n N_1(h - u_n, \mu) = 0.$$

Hence u_n converges to h (p.p. μ), that is, almost everywhere with respect to μ. By the definition of $\lambda\cdot\mu$ and of β^* respectively,

$$\int u_n\lambda\, d\mu = \int u_n\, d\beta \leqslant \int^* h\, d\beta < \infty.$$

Hence by the theorem of Lebesgue,

$$\lim_n \int u_n\lambda\, d\mu = \int h\lambda\, d\mu. \tag{2.2}$$

Let $_-\phi_h$ be the set of $u \in \mathfrak{K}_+$ with $u \leqslant h$, filtering for the relation $\leqslant$ (Cf. **3**, §6.) From (2.2), the definition of β and of β^*, respectively,

$$\int h\lambda\, d\mu = \sup_{u\in\Lambda\phi_h} \int u\lambda\, d\mu = \sup_{u\in\Lambda\phi_h} \int u\, d\beta = \int^* h\, d\beta.$$

Since $\int h\lambda\, d\mu = \int^* h\lambda\, d\mu$ (a) follows.

(b) *If h is bounded with compact support*

$$\int^* h\, d\beta \geqslant \int^* h\lambda\, d\mu. \tag{2.3}$$

Let $_+\phi_h$ be the set of $q \in \mathfrak{J}_+$ such that $q \geqslant h$, filtering for the relation $\geqslant$. There exists a $q \in {}_+\phi_h$ which is bounded, with compact support. Let S_q be the section of $_+\phi_h$ on which $q \geqslant p$. For $p \in S_q$ (a) implies that

$$\int^* p\, d\beta = \int^* p\lambda\, d\mu. \tag{2.4}$$

https://doi.org/10.4153/CJM-1957-056-4 Published online by Cambridge University Press

The definition of $\int^* h\, d\beta$ and (2.4) give

$$\int^* h\, d\beta = \inf_{p \in Sq} \int^* p\, d\beta = \inf_{p \in Sq} \int^* p\lambda\, d\mu \geqslant \int^* h\lambda\, d\mu.$$

(c) *If h is bounded with compact support and μ-integrable,* (2.1) *holds finitely.*

According to (**7**, p. 151) there exists a decreasing sequence of μ-integrable $p_n \in \mathfrak{J}_+$ such that for $t \in E$, $p_n(t) \geqslant h(t)$ and $\inf p_n(t) = h(t)$ (p.p. μ). We can suppose each p_n bounded with compact support. Both $p_n\lambda$ and $h\lambda$ are μ-integrable, so that, by the theorem of Lebesgue (noting that the p_n are uniformly bounded),

$$\lim_n \int p_n\lambda\, d\mu = \int h\lambda\, d\mu. \tag{2.5}$$

The definition of $\int^* p\, d\beta$ implies that

$$\int^* p\, d\beta \leqslant \int^* p\lambda\, d\mu. \tag{2.6}$$

Thus

$$\int^* h\, d\beta = \inf_{p \in {}_+\phi_h} \int^* p\, d\beta \leqslant \inf_{p \in {}_+\phi_h} \int^* p\lambda\, d\mu \leqslant \int p_n\lambda\, d\mu.$$

Taken with (2.5) this gives

$$\int^* h\, d\beta \leqslant \int^* h\lambda\, d\mu. \tag{2.7}$$

The inequality is excluded by (2.3), and (c) follows.

(d) *For h bounded with compact support* (2.1) *holds finitely.*

Let K be the compact support of h, and let M be a compact set containing K in its interior. For fixed $\epsilon > 0$ set $\lambda_\epsilon = \lambda + \epsilon\, \phi_M$ and $\beta_\epsilon = \lambda_\epsilon \cdot \mu$. Set $h\lambda_\epsilon = k$ and choose $q \in {}_+\phi_k$ so that the support of q is in M. Let S be the section of the filter ${}_+\phi_k$, of mappings $p \in \mathfrak{J}_+$ such that $k \leqslant p \leqslant q$. Noting that $\lambda_\epsilon(t) \geqslant \epsilon$ for $t \in M$, and $p(t) = 0$ for $t \in CM$ (the complement of M) and $p \in S$, set

$$h_p(t) = \frac{p(t)}{\lambda_\epsilon(t)},\, t \in M; \quad h_p(t) = 0,\, t \in CM.$$

Then $h_p \geqslant h$, since $\lambda_\epsilon h_p = p \geqslant \lambda_\epsilon h$. That $h_p(t) \geqslant h(t)$ is clear for $t \in M$, and it is trivial when $t \in CM$, since then $h_p(t) = h(t) = 0$. Let $h_p^{(n)}$ be h_p truncated at the level n. From (c)

$$\int^* h_p^{(n)}\, d\beta_\epsilon = \int^* h_p^{(n)}\, \lambda_\epsilon\, d\mu.$$

On letting $n \uparrow \infty$ it follows from (**7**, p. 111) that

$$\int^* h_p\, d\beta_\epsilon = \int^* h_p\, \lambda_\epsilon\, d\mu = \int^* p\, d\mu.$$

https://doi.org/10.4153/CJM-1957-056-4 Published online by Cambridge University Press

Since $h \leqslant h_p$ and $\beta \leqslant \beta_\epsilon$ this gives

$$\int^* h\, d\beta \leqslant \inf_{p \in S} \int^* p\, d\mu = \int^* h\lambda_\epsilon\, d\mu. \tag{2.8}$$

Now $\epsilon > 0$ is arbitrary in (2.8) so that

$$\int^* h\, d\beta \leqslant \int^* h\lambda\, d\mu < \infty.$$

The inequality is excluded by (2.3), and (d) follows.

(e) *If* h *has compact support* (2.1) *holds.*

For each integer $n > 0$, let $h^{(n)}$ be h, truncated at the level n. Then by (d)

$$\int^* h^{(n)}\, d\beta = \int^* h^{(n)}\lambda\, d\mu. \tag{2.9}$$

As $n \uparrow \infty$, $h^{(n)}(t)$ converges increasing to $h(t)$. It follows from **(**7, p. 111**)** that (2.1) holds.

(f) *If a member of* (2.1) *is finite the other is at least as great.*

Suppose the right member of (2.1) is finite. It then follows from **(**7, Lemma 2, p. 194**)** that E is a disjoint union $H \cup M$ in which $h\lambda\, \phi_M$ is μ-negligible and H is the increasing union of a sequence of compact sets K_n. Set

$$h\phi_{K_n} = h_n,\ h\phi_H = h',\ h\phi_M = h''.$$

By (e),

$$\int^* h_n\, d\beta = \int^* h_n\lambda\, d\mu.$$

Now $h_n(t)$, increasing, converges to $h'(t)$ as $n \uparrow \infty$, so that it follows as in the proof of (e) that

$$\int^* h'\, d\beta = \int^* h'\lambda\, d\mu. \tag{2.10}$$

Since $h = h' + h''$ and $\lambda h''$ is μ-negligible,

$$\mu^*(h\lambda) \leqslant \mu^*(h'\lambda) + \mu^*(h''\lambda) = \mu^*(h'\lambda) \leqslant \mu^*(h\lambda) \tag{2.11}$$

implying equalities throughout (2.11). Hence

$$\int^* h\, d\beta \geqslant \int^* h'\, d\beta = \int^* h'\lambda\, d\mu = \int^* h\lambda\, d\mu$$

so that (f) follows when $\int^* h\lambda\, d\mu < \infty$.

The proof of (f) when $\int^* h\, d\beta < \infty$ is similar, so that (f) is true.

Lemma 2.1 follows from (f).

Note. Since this paper reached the hands of the Editors, **(**8**)** has appeared. Our Lemma 2.1 should be compared with B. Prop. 2, p. 43, noting that

Bourbaki uses $\bar{\int}^*$ while we use $\int^*$. Prop. 2 of Bourbaki follows at once from (e) of this section. Conversely Prop. 2 of Bourbaki implies (e), and one may then continue, as we have done, with a proof of our Lemma 2.1.

One should also compare our Th. 1.1 with **(7**, Th. 1 p. 43**)**, noting that Bourbaki is concerned here with "essential" integrals of mappings into a Banach space with respect to positive measures, whereas we are concerned with integrals of mappings into C with respect to C-measures and product C-measures $y \cdot \alpha$. Our theorem depends on the fundamental formula $|y \cdot \alpha| = |y| \cdot |\alpha|$ of §3 and the lemmas of §4. The deeper connections between our theorem and that of Bourbaki will be brought out in a note **(5)**, presently to appear. The reader may also note the connection with the Radon-Nikodym Theorem as shown by Bourbaki.

3. The formula $|y \cdot \alpha| = |y| \cdot |\alpha|$. This formula is equivalent to the formula

$$\int f\, |y|\, d|\alpha| = \int f\, d|y \cdot \alpha| \qquad [f \in \mathfrak{K}_+]. \tag{3.1}$$

Set $\beta = y \cdot \alpha$. It follows from (1.1) and the definition of $|\beta|$ in **(3**, (3.3)**)** that

$$|\beta|(f) \leqslant \int f\, |y|\, d|\alpha|, \qquad [f \in \mathfrak{K}_+]. \tag{3.2}$$

It remains to show that the inequality is excluded. We shall need the following.

(i) *Let H be the set of points t at which $y(t) \neq 0$. The function σ with values*

$$\sigma(t) = \frac{y(t)}{|y(t)|}, \qquad t \in H; \sigma(t) = 0, t \in CH$$

is $|\alpha|$-measurable.

Since y is measurable, $|y|$ is measurable. The function τ with values $\tau(t) = |y(t)|^{-1}$ for $t \in H$, and $\tau(t) = 0$ when $t \in CH$, is measurable, as one sees with the aid of **(7**, Prop. 9, p. 192**)**. Since $\sigma = \tau y$, σ is also measurable. Thus (i) holds.

To show that the inequality is excluded in (3.2) we can suppose without loss of generality that $\max f(t) = 1$. Let K be the compact support of f. Let $\epsilon > 0$ be arbitrary. Set $\phi_K|y| = \lambda$ and $|\alpha| = \mu$. Since λ is μ-integrable there exists a $\lambda_0 \in \mathfrak{K}_+$ such that

$$\int |\lambda - \lambda_0|\, d\mu < \epsilon. \tag{3.3}$$

For our purposes a λ_0 is needed which is positive on K. To this end λ_0 is modified as follows. Let K_1 be compact and such that for some open set U, $K_1 \supset U \supset K$. Let $\lambda_1 \in \mathfrak{K}_+$ map E into [0, 1] with support K_1 and with $\lambda_1(t) = \phi_K(t)$ for $t \in K$. If $c > 0$ is sufficiently small, and if one sets $\lambda_2 = \lambda_0 + c\lambda_1$ then

$$\int |\lambda - \lambda_2|\, d\mu < \epsilon \tag{3.4}$$

while $\lambda_2(t) \geqslant c > 0$ for $t \in K$. From (3.4) and the assumption that max $f(t) = 1$ it follows that

$$\int f\lambda\, d\mu \leqslant \int f\lambda_2\, d\mu + \epsilon. \tag{3.5}$$

Now $f\lambda_2$ is in $\mathfrak{R}_+$. In accord with the definition of $|\alpha|$, and with u and $v \in \mathfrak{R}_C$

$$\int f\lambda_2\, d\mu = \sup_{|u| \leqslant f\lambda_2} \left| \int u\, d\alpha \right| = \sup_{|v| \leqslant f} \left| \int v\lambda_2\, d\alpha \right| . \tag{3.6$'$}$$

It follows from (3.4) that

$$\left| \int v\lambda_2\, d\alpha \right| \leqslant \left| \int v\lambda\, d\alpha \right| + \left| \int v(\lambda_2 - \lambda) d\alpha \right| \leqslant \left| \int v\lambda\, d\alpha \right| + \epsilon \tag{3.6$''$}$$

for $|v| \leqslant f$. From (3.5) and (3.6), for some $v \in \mathfrak{R}_C$ with $|v| \leqslant f$,

$$\int f\lambda\, d\mu \leqslant \left| \int v\lambda\, d\alpha \right| + 3\epsilon = \left| \int v\bar{\sigma}\, y\, d\alpha \right| + 3\epsilon \tag{3.7}$$

introducing σ as defined above.

Now $\bar{\sigma}$, as the conjugate of σ, is μ-measurable. There accordingly exists (**7**, Prop. 1, p. 180) a compact set $H \subset K$ such that $\bar{\sigma}|H$ is continuous and the μ-measure, say ω, of $K \cap CH = M$, is arbitrarily small. In particular one can suppose ω so small that $\int \phi_M f|y|\, d\mu < \epsilon$. Let g denote a continuous extension of $\bar{\sigma}|H$ to K with $\max|g(t)| < 1$ (see Note). Since $(\bar{\sigma} - g)|H = 0$ and $|vg| \leqslant |v| \leqslant f$

$$\left| \int vy(\bar{\sigma} - g) d\alpha \right| = \left| \int \phi_M\, vy(\bar{\sigma} - g) d\alpha \right| \leqslant 2 \int \phi_M f|y|\, d\mu \leqslant 2\epsilon$$

so that with $u \in \mathfrak{R}_C$

$$\left| \int vy\, \bar{\sigma}\, d\alpha \right| - 2\epsilon \leqslant \left| \int vgy\, d\alpha \right| \leqslant \sup_{|u| \leqslant f} \left| \int uy\, d\alpha \right| = |\beta|(f). \tag{3.8}$$

From (3.7) and (3.8)

$$\int f\lambda\, d\mu \leqslant |\beta|(f) + 5\epsilon.$$

Hence the inequality must be excluded in (3.2) and the relation $|y \cdot \alpha| = |y| \cdot |\alpha|$ follows.

The "product" $y \cdot \alpha$ is clearly doubly distributive. In the case of a real measure μ and real g, $|g \cdot \mu| = |g| \cdot |\mu|$, as the above proof shows after trivial notational modifications. It follows immediately that

$$(g \cdot \mu)^+ = g^+ \cdot \mu^+ + g^- \cdot \mu^-, \quad (g \cdot \mu)^- = g^- \cdot \mu^+ + g^+ \cdot \mu^-$$

Note. It follows from a theorem of Urysohn (**6**, p. 62) that a continuous extension f of $\bar{\sigma}|H$ over K exists. To obtain from f a continuous extension g of $\bar{\sigma}|H$ such that $\max|g(t)| < 1$, set $g(t) = f(t)$ when $|f(t)| \leqslant 1$, and at all other points of K set $g(t) = f(t)/|f(t)|$.

4. Proof of Theorem 1.1. We need a lemma on measurability. Recall that y is locally α-integrable.

LEMMA 4.1. *If*

$$\int^{*} |x||y|d|\alpha| = \int^{*} |x|d|y\cdot\alpha| < \infty \tag{4.1}$$

then (i) *for an arbitrary subset M of E then α-negligibility of $\phi_M xy$ is equivalent to the* $(y\cdot\alpha)$*-negligibility of $\phi_M x$, and* (ii) *the α-measurability of xy is equivalent to the* $(y\cdot\alpha)$*-measurability of x.*

One must first verify the fact that (4.1) has the form of the relation (2.1) if one sets $\mu = |\alpha|$ and $|y| = \lambda$. This follows from the formula $|y\cdot\alpha| = |\alpha|\cdot|y|$ just established. One can accordingly apply Lemma 2.1 as follows. For an arbitrary subset M of E

$$\int^{*} \phi_M|x||y|d|\alpha| = \int^{*} \phi_M|x|d|y\cdot\alpha| < \infty. \tag{4.2}$$

For the two members of (4.2) are finite and hence equal by Lemma 2.1. Statement (i) follows.

(α) *Set $\beta = y\cdot\alpha$. If x is β-measurable, xy is α-measurable.*

Let K be compact. According to the Bourbaki definition of measurability K is a disjoint union $K = H \cup M$, where H is a countable union of compact sets K_n on each of which x is continuous, and where M is β-negligible. Then $\phi_M x$ is β-negligible, and so by (i), $\phi_M xy$ is α-negligible and hence α-measurable (**7**, Prop. 6, p. 184). Further $\phi_H x$ is α-measurable and hence $\phi_H xy$. Finally $\phi_K xy = \phi_H xy + \phi_M xy$ is α-measurable, and hence xy. ("Principal of localization", **7**, p. 182.)

(β) *If x is α-measurable, x is β-measurable.* Let K, H, M be chosen as in the proof of (α) except that M shall here be α-negligible. Then $\phi_M x$ is β-negligible by (i), and hence β-measurable. Thus x is β-measurable since $\phi_K x = \phi_H x + \phi_M x$.

(γ) *If xy is α-measurable, x is β-measurable.* Let M be the subset of E on which $y(t) \neq 0$. Then M is α-measurable. Set $H = CM$. The relation

$$\phi_M x = \frac{\phi_M xy}{\phi_H + y} \tag{4.3}$$

shows that $\phi_M x$ is α-measurable, since both numerator and non-vanishing denominator are α-measurable. One can apply (β), replacing x by $\phi_M x$, since (4.1) holds with $\phi_M x$ replacing x. Hence $\phi_M x$ is β-measurable. Now $\phi_H xy$

https://doi.org/10.4153/CJM-1957-056-4 Published online by Cambridge University Press

vanishes, so that by (i) $\phi_H x$ is β-negligible and hence β-measurable. Hence $x = \phi_M x + \phi_H x$ is β-measurable.

Lemma 4.1 (ii) follows from (α) and (γ).

Statement (ii) is conditioned in Lemma 4.1 by (4.1). Actually this condition can be dropped.

COROLLARY 4.1. *For a locally α-integrable y, the α-measurability of xy is equivalent to the $(y\cdot\alpha)$-measurability of x.*

We commence with the following.

(a) *If xy is α-measurable $x^{(n)}y$ is α-measurable.* It is understood that $x^{(n)} = x_1^{(n)} + ix_2^{(n)}$ where $x_1 + ix_2$ is a Gaussian decomposition of x. If M is taken as in (γ) then $\phi_M x$ is α-measurable. Hence $\phi_M x^{(n)}$ and consequently $(\phi_M x^{(n)})y = x^{(n)}y$ is α-measurable.

(b) *If $x^{(n)}y$ is α-measurable for each positive integer $n > 0$, xy is α-measurable.* For $x^{(n)}y$ converges pointwise to xy as $n \uparrow \infty$.

From (a) and (b) we conclude that the Corollary is true if true for bounded x. We therefore prove the following.

(c) *The Corollary is true for bounded x.*

Let K be compact and set $z = \phi_K x$. Now z is bounded with compact support. Hence (cf. (d) of §2),

$$\int^* |z||y|d|\alpha| = \int^* |z|d|(y\cdot\alpha)|$$

and Lemma 4.1 applies, so that the Corollary is true if z replaces x. By the principle of the localization of measurability, (c) is true as stated.

The Corollary follows as indicated above.

Proof of Theorem 1.1. Statement (i) of the theorem follows from Lemma 2.1 since $|y\cdot\alpha| = |y|\cdot|\alpha|$ so that one can identify $|\alpha|$ with μ in Lemma 2.1. *Assuming then that* (4.1) *holds* we prove the following. Set $y\cdot\alpha = \beta$.

(a) *If x is β-integrable and if (x_n) is a sequence of β-integrable mappings such that $|x_n| \leqslant |x|$ and $x_n(t)$ converges to $x(t)$* (p.p. β), *then* (1.4) *holds provided*

$$\int x_n\, d\beta = \int x_n y\, d\alpha \qquad (n = 1,2,\ldots). \tag{4.4}$$

Let H be the set of points t on which $x_n(t)$ converges to $x(t)$. Set $M = CH$. Then M is β-negligible so that $\phi_M xy$ is α-negligible (Lemma 4.1). Since $|x_n| \leqslant |x|$, $\phi_M x_n y$ is α-negligible. Now $\phi_H x_n y$ converges pointwise to $\phi_H xy$. Moreover

$$|\phi_H x_n y| \leqslant \phi_H |x||y|.$$

Since $\alpha^*(\phi_H|x||y|) < \infty$ by hypothesis, the theorem of Lebesgue and the α-negligibility of $\phi_M xy$ imply that (**7**, p. 140, Th. 6)

$$\lim_{n\uparrow\infty} \int \phi_H x_n y d\alpha = \int \phi_H xy\, d\alpha = \int xy\, d\alpha.$$

Thus

$$\int x\,d\beta = \lim_{n\uparrow\infty}\int x_n\,d\beta = \lim_{n\uparrow\infty}\int x_n y\,d\alpha \qquad \text{from (4.4)}$$

$$= \lim_{n\uparrow\infty}\int \phi_H x_n y d\alpha = \int xy\,d\alpha.$$

(b) *If* $x \in \mathfrak{J}_+$ *is* β*-integrable then* (1.4) *holds*. For one can satisfy the conditions on (x_n) in (a) by proper choice of $x_n \in \mathfrak{K}_+$. The relation (4.4) holds by definition of β.

(c) *If* $x \in \overline{\mathrm{R}}_+^E$ *is bounded and upper semi-continuous with compact support, then* (1.4) *holds*. Let K be the compact support of x, and let $u \in \mathfrak{K}_+$ be such that $u(t) \geqslant x(t)$ on K. Then $u - x \in \mathfrak{J}_+$ and is β-integrable. From (b), (1.4) holds with $u - x$ replacing x. Hence (1.4) holds.

(d) *If* $x \in \overline{\mathrm{R}}_+^E$ *is* β*-integrable and upper semi-continuous with compact support, then* (1.4) *holds*. The truncation $x^{(n)}$ satisfies the conditions on x_n of (a), as follows from (c). Hence, from (a), (1.4) holds.

(e) *If* $x \in \overline{\mathrm{R}}_+^E$ *is* β*-integrable, then* (1.4) *holds*. For one can satisfy the conditions on (x_n) in (a), possibly excepting (4.4) by choice of x_n which are upper semi-continuous with compact support (**7**, p. 151). By virtue of (d), (4.4) will then hold, and (1.4) follows.

(f) *If x is* β*-integrable* (1.4) *holds*. Let h be any one of the Riesz components (**3**, §4) of x. Since (4.1) holds by hypothesis, (4.1) holds, h replacing x, since both members of (4.1) are then finite and hence equal by Lemma 2.1. Now h is β-integrable (**3**, Cor. 9.2). By (e), (1.4) holds, h replacing x. Hence (1.4) holds for x.

(g) *If xy is* α*-integrable* (1.4) *holds*. By Lemma 4.1, x is β-measurable, and since $\beta^*(x) < \infty$ by hypothesis, x is β-integrable. Hence (1.4) holds by (f).

Theorem 1.1 (ii) follows from (f) and (g).

5. The relation 2.1. We here present lemmas which facilitate the application of Theorem 1.1. We shall term a countable union of sets $X_n \in \mathfrak{X}$ an ω-set (cf. §1 for definition of $\mathfrak{X}$). We return to λ, h, μ of Lemma 2.1 and refer to the conditions

$$\int^* h\,d(\lambda\cdot\mu) \geqslant \int^* h\lambda\,d\mu, \tag{5.1}$$

$$\int^* h\,d(\lambda\cdot\mu) \leqslant \int^* h\lambda\,d\mu. \tag{5.2}$$

LEMMA 5.1. *If* $\lambda = \lambda' + \lambda''$, *where* $\lambda' \geqslant 0$, $\lambda'' \geqslant 0$, λ'' *is* μ*-negligible, and the support of* λ' *is contained in an* ω*-set* M, *then* (2.1) *holds*.

We shall need the following.

(a) *If ν is a positive measure with support H, then for $g \in \overline{\mathrm{R}}_+^E$*

$$\int^* g\, d\nu = \int^* g\phi_H\, d\nu. \tag{5.3}$$

To verify this relation let $p \in \mathfrak{F}_+$ be such that $p\phi_H = 0$ and $p(t) = +\infty$ when $t \in CH$. Let $g\phi_H = k$. Then $p \geqslant g - k$ so that $\nu^*(p) \geqslant \nu^*(g-k)$. However,

$$\nu^*(p) = \sup_{f \leqslant p} \nu(f) \qquad f \in \mathfrak{K}_+.$$

Since such $f \leqslant p$ vanish on H, and since H is the support of ν, $\nu^*(f) = 0$. Hence $0 = \nu^*(p) = \nu^*(g-k)$. Relation (5.3) follows.

To establish the lemma suppose first that $\lambda'' = 0$, so that $\lambda = \lambda'$. Suppose further that M is the countable union of increasing sets $X_n \in \mathfrak{X}$. Set

$$h_n = h\phi_{X_n}.$$

Since h_n has a compact support

$$\int^* h_n d(\lambda\cdot\mu) = \int^* h_n\lambda\, d\mu \qquad \text{by (e) of §2.} \tag{5.4}$$

Since $h_n(t)$ converges increasing to $\phi_M h(t)$ as $n \uparrow \infty$, it follows from (**7**, p. 111 Corollary) that

$$\int^* \phi_M h\, d(\lambda\cdot\mu) = \int^* \phi_M h\lambda\, d\mu = \int^* h\lambda\, d\mu. \tag{5.5}$$

The support H of $\lambda\cdot\mu$ is contained in the support of λ. Hence $H \subset M$. Two applications of (5.3) then give

$$\int^* h\, d(\lambda\cdot\mu) = \int^* \phi_H h\, d(\lambda\cdot\mu) = \int^* \phi_M h\, d(\lambda\cdot\mu). \tag{5.6}$$

Relation (2.1) follows from (5.5) and (5.6).

In the case of the general $\lambda = \lambda' + \lambda''$

$$\int^* h\, d(\lambda'\cdot\mu) = \int^* h\lambda'\, d\mu \tag{5.7}$$

as we have just seen. But $\lambda\cdot\mu = \lambda'\cdot\mu$ and $h\lambda''$ is μ-negligible. Hence (5.7) implies (2.1).

A mapping function which vanishes in the complement of an ω-set M will be termed an *ω-function with ω-set M.*

Lemma 5.2 (i). *If $h\lambda$ is μ-equivalent to an ω-function with ω-set M then* (5.1) *holds.*

(ii) *If h is $(\lambda\cdot\mu)$-equivalent to an ω-function with ω-set M then* (5.2) *holds.*

Proof of (i). Let $\lambda h = \lambda h' + h''$ where h' is the ω-function $h\phi_M$ with ω-set M, and h'' is μ-negligible. Suppose that M is the countable union of sets $X_n \in \mathfrak{X}$ and set $h_n = h\ \phi_{X_n}$. Then h_n converges to $\phi_M h$ so that

$$\int^* \phi_M h\, d(\lambda \cdot \mu) = \int^* \phi_M h\ \lambda\, d\mu = \int^* h\lambda\, d\mu$$

as in the proof of (5.5). Relation (5.1) follows.

The proof of (ii) is similar.

Lemma 5.2 (i) applies to a product λh in $\mathfrak{L}^p(\mu)$, since such a λh is μ-equivalent to an ω-function by **(7**, Lemma 2, p. 194**)**. An $h \in L^p(\lambda \cdot \mu)$ is similarly relevant to Lemma 5.2 (ii).

A space E which is "countable at infinity" is an ω-set by definition, so that on such a space (2.1) holds by virtue of Lemma 5.1.

If μ is a bounded measure, E is the union of an ω-set and a μ-negligible set, so that each mapping in C^E is μ-equivalent to an ω-function. A similar remark applies to a bounded measure $\lambda \cdot \mu$.

There are many other combinations of special conditions relevant to these lemmas.

6. Two counter-examples. The fact that inequalities appear in (i) and (ii) of Lemma 5.2 raises the question as to whether or not there are examples in which the equality is excluded. The question also arises in connection with Lemma 2.1. The following two examples show that the inequality may occur in either (i) or (ii).

Example 6.1. Let $H \subset E$ be locally μ-negligible, but not μ-negligible. Such a μ and set exist **(7**, p. 184**)**. Set $h = \phi_H$, $\lambda = \phi_{CH}$. Then h and λ are bounded and μ-measurable, so that λ is locally μ-integrable. Moreover h is not μ-integrable since it is not μ-negligible **(7**, p. 195**)**. Hence $\mu^*(h) = \infty$. For $u \in \mathfrak{K}_+$, hu is μ-negligible so that

$$\int u\, d\mu = \int hu\, d\mu + \int \lambda u\, d\mu = \int \lambda u\, d\mu.$$

It follows that $\mu = \lambda \cdot \mu$. Since $h\lambda = 0$, $\mu^*(h\lambda) = 0$, while

$$(\lambda \cdot \mu)^*(h) = \mu^*(h) = \infty.$$

Thus $\mu^*(h\lambda) < (\lambda \cdot \mu)^*(h)$. Note finally that $h\lambda$ is μ-equivalent to an ω-function, the null function.

Example 6.2. Take H as in Example 6.1. Set $\lambda = \phi_H$ and $h = \phi_E$. Then $\lambda \cdot \mu$ is a null measure so that $(\lambda \cdot \mu)^*(h) = 0$. Moreover $\mu^*(h\lambda) = \mu^*(\lambda) = \infty$. Note also that h is $(\lambda \cdot \mu)$-equivalent to a null function.

References

1. M. Morse and W. Transue, C-*bimeasures* Λ *and their superior integrals*, Rend. Circ. Mat. Palermo, *4* (1955), 270–300.
2. ———, C-*bimeasures* Λ *and their integral extensions*, Ann. Maths., *64* (1956), 480–504.
3. ———, *Semi-normed vector spaces with duals of integral type*, J. d'Analyse Math., *4* (1955), 149–186.
4. ———, *Vector subspaces A of* C^E *with duals of integral type*, J. Math. Pures Appl., *00* (1957), to be published.
5. ———, *A comparison of two theorems on integration*, Proc. Nat. Acad. Sci., *00* (1957), to be published.
6. N. Bourbaki, *Éléments de Mathématique*, *8*, *Topologie Générale*, Chap. IX (Paris, 1948).
7. ———, *Éléments de Mathématique*, *13*, *Intégration*, Chap. I-IV (Paris, 1952).
8. ———, *Éléments de Mathématique*, *21*, *Intégration*, Chap. V (Paris, 1956)

Institute for Advanced Study
Princeton, New Jersey

Kenyon College
Gambier, Ohio