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RELATIVE PERTURBATION BOUNDS FOR THE JOINT SPECTRUM OF COMMUTING TUPLES OF MATRICES

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Abstract

In this paper, we study the relative perturbation bounds for joint eigenvalues of commuting tuples of normal $n \times n$ matrices. Some Hoffman–Wielandt-type relative perturbation bounds are proved using the Clifford algebra technique. We also extend a result for diagonalisable matrices which improves a relative perturbation bound for single matrices.

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1. Introduction

Let $A = (A^{(1)}, A^{(2)}, \dots, A^{(m)})$ be an m-tuple of commuting $n \times n$ matrices acting on $\mathbb{C}^n$. A joint eigenvalue of A is an element $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}) \in \mathbb{C}^m$ such that

$$A^{(j)}x = \lambda^{(j)}x \quad \text{for } j = 1, 2, \dots, m$$

holds for some nonzero vector $x \in \mathbb{C}^n$. The vector x is called a joint eigenvector. The set of all joint eigenvalues of A is called the joint spectrum of A.

The main concern of perturbation theory of matrix eigenvalues is to estimate the error when the eigenvalues of a matrix are approximated by the eigenvalues of a perturbed matrix. Let A and B be two $n \times n$ matrices with respective eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\{\mu_1, \mu_2, \dots, \mu_n\}$. An important result in the direction of an absolute perturbation bound is given by the Hoffman–Wielandt theorem [4], which states: if A and B are normal matrices, then there exists a permutation π of $\{1, 2, \dots, n\}$ such that

$$\left(\sum_{i=1}^{n} |\lambda_i - \mu_{\pi(i)}|^2\right)^{1/2} \le \|A - B\|_F,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. This result has been generalised in several directions. In 1993, Bhatia and Bhattacharyya [1] extended the Hoffman–Wielandt theorem to joint eigenvalues of m-tuples of commuting normal matrices. More results on absolute perturbation bounds for joint eigenvalues may be found in [3, 8].

In 1998, Eisenstat and Ipsen [2] studied relative perturbation bounds for eigenvalues of diagonalisable matrices. They proved that, if A and B are both diagonalisable and A is nonsingular, then there exists a permutation π of $\{1, 2, \ldots, n\}$ such that

$$\Big(\sum_{i=1}^{n}\Big|\frac{\lambda_i - \mu_{\pi(i)}}{\lambda_i}\Big|^2\Big)^{1/2} \leq \kappa(X)\kappa(\tilde{X})\|A^{-1}(A-B)\|_F,$$

where X and $\tilde{X}$ are invertible matrices which diagonalise A and B respectively and $\kappa(X) = \|X\|\,\|X^{-1}\|$ is the condition number of the matrix X. These results were further extended by Li and Sun [6] for a nonsingular normal matrix A and an arbitrary matrix B and by Li and Chen [5] for diagonalisable matrices.

To the best of our knowledge, there are no investigations of the relative perturbation bounds for the joint spectrum of commuting tuples of matrices. The present work is an attempt in this direction. We derive some relative perturbation bounds for joint eigenvalues of m-tuples of commuting normal and diagonalisable matrices using the Clifford algebra technique proposed by McIntosh and Pryde [7]. For the convenience of the reader we briefly discuss the Clifford algebra technique in Section 2.

2. The Clifford algebra technique

Let $\mathbb{R}^m$ be the real vector space of dimension m and let $e_1, e_2, \ldots, e_m$ be a basis. The Clifford algebra $\mathbb{R}_{(m)}$ is an algebra generated by $e_1, e_2, \ldots, e_m$ with the relations

$$e_ie_j = -e_je_i \quad \text{for } i \neq j \quad \text{and} \quad e_i^2 = -1 \quad \text{for all } i.$$

Then $\mathbb{R}_{(m)}$ is an algebra over $\mathbb{R}$ of dimension 2^m. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a subset of $\{1, \ldots, m\}$ such that $1 \leq s_1 < s_2 < \cdots < s_k \leq m$. Then the elements $e_S = e_{s_1}e_{s_2}\ldots e_{s_k}$ form a basis of $\mathbb{R}_{(m)}$, where S runs over all subsets of $\{1, \ldots, m\}$ and $e_\emptyset = 1$. An element α of $\mathbb{R}_{(m)}$ is of the form $\alpha = \sum_S \alpha_S e_S$, where $\alpha_S \in \mathbb{R}$. If $\beta = \sum_S \beta_S e_S$, $\beta_S \in \mathbb{R}$, is another element of $\mathbb{R}_{(m)}$, the inner product of α and β is

$$\langle\alpha, \beta\rangle = \sum_S \alpha_S\beta_S.$$

Under this inner product $\mathbb{R}_{(m)}$ becomes a Hilbert space with the orthonormal basis e_S. The tensor product $\mathbb{C}^n \otimes \mathbb{R}_{(m)}$, where

$$\mathbb{C}^n \otimes \mathbb{R}_{(m)} = \Big\{\sum_S x_S \otimes e_S : x_S \in \mathbb{C}^n\Big\},$$

is a Hilbert space under the inner product

$$\langle x, y\rangle = \Big\langle\sum_S x_S \otimes e_S, \sum_S y_S \otimes e_S\Big\rangle = \sum_S \langle x_S, y_S\rangle,$$

where $x_S, y_S \in \mathbb{C}^n$ and the inner product on the right-hand side is the usual inner product in $\mathbb{C}^n$. Therefore, the norm on $\mathbb{C}^n \otimes \mathbb{R}_{(m)}$ is defined by

$$\Big\|\sum_S x_S \otimes e_S\Big\| = \Big(\sum_S \|x_S\|^2\Big)^{1/2},$$

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where the norm on the right-hand side is the usual norm in $\mathbb{C}^n$. Let M_n be the space of $n \times n$ matrices with complex entries. Then $M_n \otimes \mathbb{R}_{(m)}$ is a linear space and an element $A \in M_n \otimes \mathbb{R}_{(m)}$ has the form $A = \sum_S A_S \otimes e_S$, where $A_S \in M_n$. Each element $A = \sum_S A_S \otimes e_S \in M_n \otimes \mathbb{R}_{(m)}$ acts on the elements $x = \sum_T x_T \otimes e_T \in \mathbb{C}^n \otimes \mathbb{R}_{(m)}$ by

$$Ax = \Big(\sum_S A_S \otimes e_S\Big)\Big(\sum_T x_T \otimes e_T\Big) = \sum_{S,T} A_S x_T \otimes e_S e_T.$$

For an m-tuple $A = (A^{(1)}, A^{(2)}, \ldots, A^{(m)})$ of $n \times n$ complex matrices, the corresponding Clifford operator $\mathrm{Cliff}(A) \in M_n \otimes \mathbb{R}_{(m)}$ acting on $\mathbb{C}^n \otimes \mathbb{R}_{(m)}$ is defined by

$$\mathrm{Cliff}(A) = i \sum_{j=1}^{m} A^{(j)} \otimes e_j. \tag{2.1}$$

3. Relative perturbation bounds

Throughout this paper, S_n denotes the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$, $\|\cdot\|_F$ and $\|\cdot\|$ denote the Frobenius norm and the usual operator norm respectively and $\Re(z)$ denotes the real part of a complex number z. A square matrix of nonnegative real numbers is said to be a *doubly stochastic* matrix if the sum of each row and each column is 1. A *permutation matrix* is a square matrix in which each row and each column contains exactly one nonzero entry 1 and 0 entries everywhere else.

Lemma 3.1 [1, Lemma 1]. *Let $A = (A^{(1)}, A^{(2)}, \ldots, A^{(m)})$ be any m-tuple of operators in $\mathbb{C}^n$ and let* $\mathrm{Cliff}(A)$ *be the corresponding Clifford operator. Then*

$$\|\mathrm{Cliff}(A)\|_F^2 = 2^m \sum_{k=1}^{m} \|A^{(k)}\|_F^2.$$

Lemma 3.2 [1]. *If P is any operator of $\mathbb{C}^n$, then:*

(i) $\mathrm{trace}(P \otimes e_T) = 0$ *for any nonempty subset T of* $\{1, 2, \ldots, m\}$;

(ii) $\mathrm{trace}(P \otimes e_\emptyset) = 2^m\,\mathrm{trace} P$.

Now we prove some results on relative perturbation bounds for tuples of matrices. Let $A = (A^{(1)}, A^{(2)}, \ldots, A^{(m)})$ and $B = (B^{(1)}, B^{(2)}, \ldots, B^{(m)})$ be two m-tuples of normal $n \times n$ matrices such that $B = A + E$, where $E = (E^{(1)}, E^{(2)}, \ldots, E^{(m)})$ is the perturbation given to A. Let $\alpha_i = (\alpha_i^{(1)}, \alpha_i^{(2)}, \ldots, \alpha_i^{(m)})$ and $\beta_i = (\beta_i^{(1)}, \beta_i^{(2)}, \ldots, \beta_i^{(m)})$ be the joint eigenvalues of A and B, respectively.

Theorem 3.3. *If A and $B = A + E$ are m-tuples of commuting normal matrices as defined above and each $A^{(k)}$ is nonsingular for $k = 1, 2, \ldots, m$, then there exists a permutation π of S_n such that*

$$\sum_{j=1}^{n} \sum_{k=1}^{m} \left| \frac{\alpha_j^{(k)} - \beta_{\pi(j)}^{(k)}}{\alpha_j^{(k)}} \right|^2 \le \sum_{k=1}^{m} \|A^{(k)^{-1}} E^{(k)}\|_F^2.$$

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PROOF. Since $E = B - A$,

$$E^{(k)} = B^{(k)} - A^{(k)} \Rightarrow {A^{(k)}}^{-1}E^{(k)} = {A^{(k)}}^{-1}B^{(k)} - I \quad \text{for } k = 1, 2, \ldots, m. \tag{3.1}$$

Let $C = ({A^{(1)}}^{-1}B^{(1)}, \ldots, {A^{(m)}}^{-1}B^{(m)})$ and $D = ({A^{(1)}}^{-1}E^{(1)}, \ldots, {A^{(m)}}^{-1}E^{(m)})$. In addition, let $\tilde{I} = (I, I, \ldots, I)$ be the m-tuple of identity matrices of order $n \times n$. From (3.1),

$$C - \tilde{I} = D. \tag{3.2}$$

Since A and B are m-tuples of normal matrices, there exist orthonormal bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ such that

$$A^{(k)}u_j = \alpha_j^{(k)}u_j, \quad B^{(k)}v_j = \beta_j^{(k)}v_j, \quad \text{for } j = 1, 2, \ldots, n \text{ and } k = 1, 2, \ldots, m.$$

Let P_j and Q_j respectively denote the orthogonal projection operators to the spaces spanned by the vectors u_j and v_j. For $k = 1, 2, \ldots, m$,

$$A^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)}P_j, \quad B^{(k)} = \sum_{l=1}^{n} \beta_l^{(k)}Q_l.$$

From (2.1) and the above relations,

$$\begin{aligned}\mathrm{Cliff}(C) &= i\sum_{k=1}^{m} {A^{(k)}}^{-1}B^{(k)} \otimes e_k = i\sum_{k=1}^{m}\Big(\sum_{j=1}^{n}\sum_{l=1}^{n} {\alpha_j^{(k)}}^{-1}P_j\beta_l^{(k)}Q_l\Big)\otimes e_k \\ &= i\sum_{j,l=1}^{n}\Big(\sum_{k=1}^{m} {\alpha_j^{(k)}}^{-1}\beta_l^{(k)} I \otimes e_k\Big)(P_jQ_l \otimes e_\emptyset).\end{aligned} \tag{3.3}$$

Similarly,

$$\mathrm{Cliff}(\tilde{I}) = i\sum_{r=1}^{n}\Big(\sum_{t=1}^{m} I \otimes e_t\Big)(Q_r \otimes e_\emptyset). \tag{3.4}$$

Let $\mathrm{Cliff}(C)^*$ denote the adjoint of $\mathrm{Cliff}(C)$. From (3.3),

$$\begin{aligned}&\mathrm{trace}[\mathrm{Cliff}(\tilde{I})\mathrm{Cliff}(C)^*] \\ &\quad= -\mathrm{trace}\Big[\sum_{j,l,r=1}^{n}\Big(\sum_{t=1}^{m} I \otimes e_t\Big)\Big(\sum_{k=1}^{m} \overline{{\alpha_j^{(k)}}^{-1}\beta_l^{(k)}} I \otimes e_k\Big)(Q_r \otimes e_\phi)(Q_lP_j \otimes e_\emptyset)\Big] \\ &\quad= -\mathrm{trace}\Big[\sum_{j,l,r=1}^{n}\Big(\sum_{t=1}^{m} I \otimes e_t\Big)\Big(\sum_{k=1}^{m} \overline{{\alpha_j^{(k)}}^{-1}\beta_l^{(k)}} I \otimes e_k\Big)(Q_rQ_lP_j \otimes e_\emptyset)\Big] \\ &\quad= -\mathrm{trace}\sum_{j,l,r=1}^{n}\Big[-\Big(\sum_{k=1}^{m} \overline{{\alpha_j^{(k)}}^{-1}\beta_l^{(k)}}\Big)(Q_rQ_lP_j \otimes e_\emptyset)\Big] \\ &\qquad -\mathrm{trace}\sum_{j,l,r=1}^{n}\Big[\sum_{\substack{k,t=1\\ k\neq t}}^{m}\Big(\overline{{\alpha_j^{(k)}}^{-1}\beta_l^{(k)}} - \overline{{\alpha_j^{(t)}}^{-1}\beta_l^{(t)}}\Big)(Q_rQ_lP_j \otimes e_te_k)\Big].\end{aligned}$$

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Applying Lemma 3.2(ii) to the above relation,

$$\begin{aligned}\operatorname{trace}[\operatorname{Cliff}(\tilde{I})\operatorname{Cliff}(C)^*] &= -\operatorname{trace}\sum_{j,l,r=1}^{n}\Big[-\Big(\sum_{k=1}^{m}\overline{\alpha_j^{(k)^{-1}}\beta_l^{(k)}}\Big)(Q_rQ_lP_j\otimes e_\emptyset)\Big]\\ &= \sum_{j,l,r=1}^{n}\sum_{k=1}^{m}\overline{\alpha_j^{(k)^{-1}}\beta_l^{(k)}}\operatorname{trace}(Q_rQ_lP_j\otimes e_\emptyset)\\ &= 2^m\sum_{j,l=1}^{n}\sum_{k=1}^{m}\overline{\alpha_j^{(k)^{-1}}\beta_l^{(k)}}\operatorname{trace}(Q_lP_j).\end{aligned}$$

Since $\operatorname{trace}(Q_lP_j) = \operatorname{trace}(P_jQ_l)$,

$$\operatorname{trace}[\operatorname{Cliff}(\tilde{I})\operatorname{Cliff}(C)^*] = 2^m\sum_{j,l=1}^{n}\sum_{k=1}^{m}\overline{\alpha_j^{(k)^{-1}}\beta_l^{(k)}}\operatorname{trace}(P_jQ_l). \tag{3.5}$$

Also,

$$\begin{aligned}\|\operatorname{Cliff}(C)\|_F^2 &= 2^m\sum_{k=1}^{m}\|A^{(k)^{-1}}B^{(k)}\|_F^2\\ &= 2^m\sum_{k=1}^{m}\Big\|\sum_{j=1}^{n}\alpha_j^{(k)^{-1}}P_j\sum_{l=1}^{n}\beta_l^{(k)}Q_l\Big\|_F^2 = 2^m\sum_{k=1}^{m}\Big\|\sum_{j,l=1}^{n}\alpha_j^{(k)^{-1}}\beta_l^{(k)}P_jQ_l\Big\|_F^2\\ &= 2^m\sum_{k=1}^{m}\operatorname{trace}\Big[\Big(\sum_{j,l=1}^{n}\alpha_j^{(k)^{-1}}\beta_l^{(k)}P_jQ_l\Big)\Big(\sum_{r,t=1}^{n}\alpha_r^{(k)^{-1}}\beta_t^{(k)}P_rQ_t\Big)^*\Big]\\ &= 2^m\sum_{k=1}^{m}\operatorname{trace}\Big[\sum_{j,l,r=1}^{n}\alpha_j^{(k)^{-1}}\beta_l^{(k)}\overline{\alpha_r^{(k)^{-1}}\beta_l^{(k)}}P_jQ_lP_r\Big]\\ &= 2^m\sum_{k=1}^{m}\sum_{j,l,r=1}^{n}\alpha_j^{(k)^{-1}}\beta_l^{(k)}\overline{\alpha_r^{(k)^{-1}}\beta_l^{(k)}}\operatorname{trace}(P_jQ_lP_r)\\ &= 2^m\sum_{k=1}^{m}\sum_{j,l=1}^{n}\Big|\alpha_j^{(k)^{-1}}\beta_l^{(k)}\Big|^2\operatorname{trace}(P_jQ_l).\end{aligned} \tag{3.6}$$

Let $W = (w_{ij})$, where $w_{ij} = \operatorname{trace}(P_iQ_j)$. It can be easily verified that W is a doubly stochastic matrix. Hence, by Birkhoff's theorem, W is a convex combination of permutation matrices. Therefore,

$$W = \sum_{s=1}^{n!}t_sP_s, \quad \text{where } t_s \ge 0 \text{ and } \sum_{s=1}^{n!}t_s = 1$$

and P_s is the permutation matrix corresponding to the permutation π_s. Finally, from (3.2), (3.4), (3.5) and (3.6),

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$$\begin{aligned}
\|\mathrm{Cliff}(D)\|_F^2 &= \|\mathrm{Cliff}(C-\tilde{I})\|_F^2 \\
&= \|\mathrm{Cliff}(C)\|_F^2 + \|\mathrm{Cliff}(\tilde{I})\|_F^2 - 2\Re(\mathrm{trace}(\mathrm{Cliff}(C)^*\mathrm{Cliff}(\tilde{I}))) \\
&= 2^m \sum_{s=1}^{n!} t_s \sum_{k=1}^{m}\sum_{j=1}^{n}\Big[1 + \Big|\alpha_j^{(k)^{-1}}\beta_{\pi_s(j)}^{(k)}\Big|^2 - 2\Re\Big(\overline{\alpha_j^{(k)^{-1}}\beta_{\pi_s(j)}^{(k)}}\Big)\Big] \\
&\geq 2^m \min_s \sum_{k=1}^{m}\sum_{j=1}^{n}\Big[1 + \Big|\alpha_j^{(k)^{-1}}\beta_{\pi_s(j)}^{(k)}\Big|^2 - 2\Re\Big(\overline{\alpha_j^{(k)^{-1}}\beta_{\pi_s(j)}^{(k)}}\Big)\Big] \\
&= 2^m \sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{\alpha_j^{(k)} - \beta_{\pi(j)}^{(k)}}{\alpha_j^{(k)}}\right|^2.
\end{aligned}$$

Hence, the result is proved. □

Remark 3.4. Sun [10] has generalised the Hoffman–Wielandt inequality for the case when one matrix is normal and the other is arbitrary. Similarly, Theorem 3.3 can be extended to the case when one of the commuting tuples of matrices is arbitrary in the following way. Let $A = (A^{(1)}, A^{(2)}, \dots, A^{(m)})$ and $B = (B^{(1)}, B^{(2)}, \dots, B^{(m)})$ be two m-tuples of commuting matrices in M_n with joint eigenvalues $\alpha_i = (\alpha_i^{(1)}, \alpha_i^{(2)}, \dots, \alpha_i^{(m)})$ and $\beta_i = (\beta_i^{(1)}, \beta_i^{(2)}, \dots, \beta_i^{(m)})$, respectively, such that each $A^{(k)}$ is normal and nonsingular. Since $B^{(1)}, B^{(2)}, \dots, B^{(m)}$ are commuting matrices, they can be reduced to upper triangular form by a single unitary matrix. From Pryde [8], it follows that the m-tuple B has n joint eigenvalues as described above. The following theorem can then be established by using Theorem 3.3 and the proof of Theorem 1.1 of [10].

Theorem 3.5. *If A and B $(= A + E)$ are two m-tuples of commuting matrices as described above, then there exists a permutation σ in S_n such that*

$$\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{\alpha_j^{(k)} - \beta_{\sigma(j)}^{(k)}}{\alpha_j^{(k)}}\right|^2 \leq n\sum_{k=1}^{m}\|A^{(k)^{-1}}\|^2\|E^{(k)}\|_F^2. \tag{3.7}$$

Remark 3.6. When we relax the normality condition on each $B^{(k)}$, the constant n which appears on the right-hand side of the above relation (3.7) is best possible. This can be verified by considering the $n \times n$ matrices

$$A^{(k)} = \begin{bmatrix} 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k \\ k & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B^{(k)} = \begin{bmatrix} 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where k runs over $1, 2, \dots, m$.

Now we prove the following theorem, which is the diagonalisable analogue of Theorem 3.3.

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THEOREM 3.7. *If A and B are m-tuples of commuting diagonalisable matrices and each $A^{(k)}$ is nonsingular for $k = 1, 2, \dots, m$, then there exists a permutation π of S_n such that*

$$\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{\alpha_j^{(k)}-\beta_{\pi(j)}^{(k)}}{\alpha_j^{(k)}}\right|^2 \le \kappa(P)^2\kappa(Q)^2\sum_{k=1}^{m}\|A^{(k)^{-1}}(B^{(k)}-A^{(k)})\|_F^2,$$

where $\kappa(P) = \|P\|\,\|P^{-1}\|$ is the condition number of P.

To prove this theorem, we need the following lemma, which is a slight variation on the result proved in [9, page 216].

LEMMA 3.8. *If M and N are normal matrices and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$, then*

$$\|M\Sigma N - \Sigma\|_F \ge \sigma_n\|MN - I\|_F.$$

PROOF. Set $\Omega = \Sigma - \sigma_n I$. Clearly, the diagonal elements of Ω are nonnegative. Now

$$\begin{aligned}\|M\Sigma N - \Sigma\|_F^2 - \sigma_n^2\|MN - I\|_F^2 &= \|M(\Omega + \sigma_n I)N - (\Omega + \sigma_n I)\|_F^2 - \sigma_n^2\|MN - I\|_F^2\\ &= \|(M\Omega N - \Omega) + \sigma_n(MN - I)\|_F^2 - \sigma_n^2\|MN - I\|_F^2\\ &= \|(M\Omega N - \Omega)\|_F^2 + 2\sigma_n\Re\{\mathrm{trace}[(M\Omega N - \Omega)^*(MN - I)]\}\\ &= \|(M\Omega N - \Omega)\|_F^2 + \sigma_n\mathrm{trace}\{\Omega[(MN - I)^*(MN - I) + (MN - I)(MN - I)^*]\}\\ &\ge 0.\end{aligned}$$

□

PROOF OF THEOREM 3.7. Let $A = (A^{(1)}, A^{(2)}, \dots, A^{(m)})$ and $B = (B^{(1)}, B^{(2)}, \dots, B^{(m)})$ be two m-tuples of commuting diagonalisable matrices having the joint eigenvalues $\alpha_i = (\alpha_i^{(1)}, \alpha_i^{(2)}, \dots, \alpha_i^{(m)})$ and $\beta_i = (\beta_i^{(1)}, \beta_i^{(2)}, \dots, \beta_i^{(m)})$, respectively. Since the $A^{(k)}$ and $B^{(k)}$ are diagonalisable, there are two nonsingular matrices P and Q such that

$$\begin{aligned}PA^{(k)}P^{-1} &= D_1^{(k)} = \mathrm{diag}(\alpha_1^{(k)}, \alpha_2^{(k)}, \dots, \alpha_n^{(k)}),\\ QB^{(k)}Q^{-1} &= D_2^{(k)} = \mathrm{diag}(\beta_1^{(k)}, \beta_2^{(k)}, \dots, \beta_n^{(k)})\end{aligned}$$

for $k = 1, 2, \dots, m$. Consider

$$\begin{aligned}\|A^{(k)^{-1}}(B^{(k)} - A^{(k)})\|_F^2 &= \|A^{(k)^{-1}}B^{(k)} - I\|_F^2\\ &= \|P^{-1}D_1^{(k)^{-1}}PQ^{-1}D_2^{(k)}Q - I\|_F^2\\ &\ge \|P\|^{-2}\|Q^{-1}\|^{-2}\|D_1^{(k)^{-1}}PQ^{-1}D_2^{(k)} - PQ^{-1}\|_F^2.\end{aligned}$$

Let $U\Sigma V^*$ be the singular value decomposition of PQ^{-1} and σ_n be the smallest diagonal element of Σ. From the above relation,

$$\begin{aligned}\|A^{(k)^{-1}}(B^{(k)} - A^{(k)})\|_F^2 &\ge \|P\|^{-2}\|Q^{-1}\|^{-2}\|D_1^{(k)^{-1}}U\Sigma V^*D_2^{(k)} - U\Sigma V^*\|_F^2\\ &\ge \|P\|^{-2}\|Q^{-1}\|^{-2}\|(U^*D_1^{(k)^{-1}}U)\Sigma(V^*D_2^{(k)}V) - \Sigma\|_F^2\\ &= \|P\|^{-2}\|Q^{-1}\|^{-2}\|M^{(k)^{-1}}\Sigma N^{(k)} - \Sigma\|_F^2,\end{aligned}$$

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where $M^{(k)} = U^* D_1^{(k)} U$ and $N^{(k)} = V^* D_2^{(k)} V$ are normal for each k. From Lemma 3.8,

$$\|A^{(k)^{-1}}(B^{(k)} - A^{(k)})\|_F^2 \geq \sigma_n^2 \|P\|^{-2} \|Q^{-1}\|^{-2} \|M^{(k)^{-1}} N^{(k)} - I\|_F^2.$$

Finally,

$$\|M^{(k)^{-1}}(N^{(k)} - M^{(k)})\|_F^2 \leq \kappa(P)^2 \kappa(Q)^2 \|A^{(k)^{-1}}(B^{(k)} - A^{(k)})\|_F^2.$$

The required result follows by applying Theorem 3.3 to the commuting tuples of normal matrices $M = (M^{(1)}, M^{(2)}, \ldots, M^{(m)})$ and $N = (N^{(1)}, N^{(2)}, \ldots, N^{(m)})$ and using the above relations. □

REMARK 3.9. Corollary 5.2 of [2] is a special case of Theorem 3.7 for $m = 1$.

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