Can. J. Math. Vol. 47 (6), 1995 pp. 1219–1239

ANALYTIC PROPERTIES OF POWER PRODUCT EXPANSIONS

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ABSTRACT. Let $f(z)$ be a complex function analytic in some neighbourhood of the origin with $f(0) = 1$. It is known that $f(z)$ admits a unique "power product" expansion of the form

$$f(z) = \prod_{n=1}^{\infty}(1 + g_n z^n)$$

convergent near zero. We derive a simple direct bound for the radius of convergence of this product expansion in terms of the coefficients of $f(z)$. In addition we show that the same bound holds in the case of "inverse power product" expansions

$$f(z) = \prod_{n=1}^{\infty}(1 - h_n z^n)^{-1}.$$

Examples are given for which these bounds are sharp. We show also that products with nonnegative coefficients have the same radius of convergence as their corresponding series.

1. **Introduction.** The idea of representing a formal power series

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

in the form of a formal product expansion

$$f(z) = \prod_{n=1}^{\infty}(1 + g_n z^n) \tag{1.2}$$

with $g_n \in \mathbb{C}$ for $n \in \mathbb{N}$, appears to have first been studied in the 1930's by Ritt [R] and Feld [F]. More recently, Knopfmacher and Lucht [KL] established a sharper domain of convergence: Consider the Maclaurin series

$$f'(z)/f(z) = \sum_{n=1}^{\infty} d_n z^{n-1} \tag{1.3}$$

and let

$$r = \sup_{n\geq 1} |d_n|^{1/n}. \tag{1.4}$$

(1.5) Then the expansion (1.2) converges for $|z| < 1/r$.

Received by the editors February 9, 1994.
AMS subject classification: 41A10, 30E10.

This result has been established independently also by Indlekofer and Warlimont [IW]. In practice, for functions $f(z)$ given in the form (1.1), the determination of r can prove difficult.

The main aim of this paper is to prove a simple direct bound for the radius of convergence of (1.2) in terms of the known coefficients of (1.1). For this purpose define

$$s = \sup\{|a_n|^{1/n}, n \in \mathbb{N}\}. \tag{1.6}$$

THEOREM 1. *If $f(z)$ given by (1.1) is analytic in some neighbourhood of the origin then the expansion (1.2), termed the power product expansion of $f(z)$ converges for $|z| < \frac{1}{2s}$.*

In particular, the power product expansion of the function $f(z) = \frac{1-2z}{1-z}$ studied in Section 2 shows that this bound can be sharp.

The same bound is also shown to hold in the case of inverse power product expansions,

$$f(z) = \prod_{n=1}^{\infty}(1 - h_n z^n)^{-1} \tag{1.7}$$

which are studied in Section 3. Inverse power product expansions are known to play an important role in the theory of Witt vectors. (See *e.g.* Dress and Siebeneicher [DS] and Borwein and Lou [BL]).

Finally in Section 4 we look at analytic properties of product expansions arising in combinatorial applications.

2. **Power product expansions.**

It is easy to see that if $f(z)$ has representations in both forms (1.1) and (1.2) then

$$f(z) = 1 + g_1 z + g_2 z^2 + (g_3 + g_1 g_2) z^3 + \cdots$$

where in general for $n \geq 1$,

$$a_n = \sum_{\substack{j_1+j_2+\cdots+j_r=n \\ 1\leq j_1<j_2<\cdots<j_r\leq n}} g_{j_1} g_{j_2} \cdots g_{j_r}, \tag{2.1}$$

the summation being over all partitions of n into distinct parts. Various concrete examples of (2.1) are given in Gingold, Gould and Mays [GGM]. As two further concrete examples of (2.1) arising in combinatorics we mention $\prod_{n=1}^{\infty}(1 + \frac{z^n}{n})$, treated by Greene and Knuth [GK, p. 48] and $\prod_{n=1}^{\infty}(1 + \frac{z^n}{n!})$ treated by Knopfmacher et al [KORSW].

Of more interest for our purposes is the expression of the coefficients g_n in terms of the coefficients a_i, $1 \leq i \leq n$. From (2.1),

$$g_n = a_n - \sum_{\substack{n=i_1+i_2+\cdots+i_r \\ i_1<i_2<\cdots<i_r \\ r\geq 2}} g_{i_1} g_{i_2} \cdots g_{i_r}. \tag{2.2}$$

Now we may successively substitute for each g_{i_j} using (2.2) and continue this process until all the $g's$ are eliminated. This leads to the following representations

$$
\begin{aligned}
g_1 &= a_1 \\
g_2 &= a_2 \\
g_3 &= a_3 - a_2a_1 \\
g_4 &= a_4 - a_3a_1 + a_2a_1^2 \\
g_5 &= a_5 - a_4a_1 - a_3a_2 + a_3a_1^2 + a_2^2a_1 - a_2a_1^3 \\
g_6 &= a_6 - a_5a_1 - a_4a_2 + a_4a_1^2 + a_3a_2a_1 - a_3a_1^3 - a_2^2a_1^2 + a_2a_1^4 \\
g_7 &= a_7 - a_6a_1 - a_5a_2 + a_5a_1^2 - a_4a_3 + 2a_4a_2a_1 - a_4a_1^3 + a_3^2a_1 + a_3a_2^2 \\
&\quad - 3a_3a_2a_1^2 + a_3a_1^4 - a_2^3a_1 + 2a_2^2a_1^3 - a_2a_1^5 \\
g_8 &= a_8 - a_7a_1 - a_6a_2 + a_6a_1^2 - a_5a_3 + 2a_5a_2a_1 - a_5a_1^3 + a_4a_3a_1 + a_4a_2^2 \\
&\quad - 2a_4a_2a_1^2 + a_4a_1^4 + a_3^2a_2 - a_3^2a_1^2 - 3a_3a_2^2a_1 + 3a_3a_2a_1^3 - a_3a_1^5 \\
&\quad + 2a_2^3a_1^2 - 2a_2^2a_1^4 + a_2a_1^6 \\
g_9 &= a_9 - a_8a_1 - a_7a_2 + a_7a_1^2 - a_6a_3 + 2a_6a_2a_1 - a_6a_1^3 - a_5a_4 + 2a_5a_3a_1 \\
&\quad + a_5a_2^2 - 3a_5a_2a_1^2 + a_5a_1^4 + a_4^2a_1 + 2a_4a_3a_2 - 3a_4a_3a_1^2 - 3a_4a_2^2a_1 \\
&\quad + 4a_4a_2a_1^3 - a_4a_1^5 - 2a_3^2a_2a_1 + 2a_3^2a_1^3 - a_3a_2^3 + 5a_3a_2^2a_1^2 - 5a_3a_2a_1^4 \\
&\quad + a_3a_1^6 + a_2^4a_1 - 3a_2^3a_1^3 + 3a_2^2a_1^5 - a_2a_1^7 \\
g_{10} &= a_{10} - a_9a_1 - a_8a_2 + a_8a_1^2 - a_7a_3 + 2a_7a_2a_1 - a_7a_1^3 - a_6a_4 + 2a_6a_3a_1 \\
&\quad + a_6a_2^2 - 3a_6a_2a_1^2 + a_6a_1^4 + a_5a_4a_1 + a_5a_3a_2 - 2a_5a_3a_1^2 - 2a_5a_2^2a_1 \\
&\quad + 3a_5a_2a_1^3 - a_5a_1^5 + a_4^2a_2 - a_4^2a_1^2 + a_4a_3^2 - 5a_4a_3a_2a_1 + 3a_4a_3a_1^3 \\
&\quad - a_4a_2^3 + 5a_4a_2^2a_1^2 - 4a_4a_2a_1^4 + a_4a_1^6 - a_3^3a_1 - a_3^2a_2^2 + 5a_3^2a_2a_1^2 - 2a_3^2a_1^4 \\
&\quad + 3a_3a_2^3a_1 - 8a_3a_2^2a_1^3 + 5a_3a_2a_1^5 - a_3a_1^7 - 2a_2^4a_1^2 + 4a_2^3a_1^4 - 3a_2^2a_1^6 + a_2a_1^8.
\end{aligned}
$$

At first glance these formulas would not appear to be useful in trying to estimate the size of growth of the g_n's in terms of that of the a_n's. However, it turns out that there is quite a lot of structure to these seemingly haphazard expressions and by exploiting this we are lead ultimately to the proof of Theorem 1.

We begin by introducing some further notation that we will use throughout. Let $\lambda = (1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ denote the partition

$$\lambda_1 \cdot 1 + \lambda_2 \cdot 2 + \cdots + \lambda_n \cdot n = n, \quad \lambda_i \geq 0,\ 1 \leq i \leq n,$$

and let $|\lambda|$ denote the sum $\sum_{i=1}^n \lambda_i$. Furthermore we use $c(\lambda)$ to denote a constant $\in \mathbb{Z}$ which depends on λ and use a^λ for the product $a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}$. Where necessary we write $\lambda = \lambda(n)$ in order to make the size of the partition explicit.

LEMMA 2.1. *For* $n \geq 1$

$$g_n = \sum_{\lambda(n)} c(\lambda)a^\lambda \tag{2.3}$$

https://doi.org/10.4153/CJM-1995-062-9 Published online by Cambridge University Press

where the sum is over all partitions λ of n.

PROOF. For $n = 1$ we have $g_1 = a_1$. Now we proceed inductively using the relation (2.2),

$$g_n = a_n - \sum_{\substack{n=i_1+i_2+\cdots+i_r \\ i_1<i_2\cdots<i_r \\ r\geq 2}} g_{i_1} g_{i_2} \cdots g_{i_r}.$$

By assumption an arbitrary term in the expansion of g_{i_j} is of the form $c\big(\lambda(i_j)\big)a^{\lambda(i_j)}$. Thus an arbitrary term in the product $g_{i_1}g_{i_2}\cdots g_{i_r}$ has the form $c\big(\lambda(i_1)\big)c\big(\lambda(i_2)\big)\cdots c\big(\lambda(i_r)\big)a^{\lambda(i_1)}a^{\lambda(i_2)}\cdots a^{\lambda(i_r)} = d\big(\lambda(m)\big)a^{\lambda(m)}$, where the constant $d(\lambda) \in \mathbb{Z}$ since each $c\big(\lambda(i_j)\big) \in \mathbb{Z}$, and $a^{\lambda(m)}$ represents a product which corresponds to a partition of size

$$\begin{aligned} m &= 1 \cdot \sum_{j=1}^{r} \lambda_1(i_j) + 2\sum_{j=1}^{r} \lambda_2(i_j) + \cdots + n \cdot \sum_{j=1}^{r} \lambda_n(i_r) \\ &= \sum_{k=1}^{i_1} k\lambda_k(i_1) + \sum_{k=1}^{i_2} k\lambda_k(i_2) + \sum_{k=1}^{i_r} k\lambda_k(i_r) \\ &= i_1 + i_2 + \cdots + i_r = n. \end{aligned}$$

■

THEOREM 2.2. *The sign of the term $c(\lambda)a^\lambda$ ($c(\lambda) \neq 0$) in the expansion of g_n is*

$$(-1)^{|\lambda|+1}. \tag{2.4}$$

PROOF. To prove this we make use of a three term recurrence for g_n established in [GGM, Theorem 3]. Let

$$\begin{aligned} A_{1,n} &= a_n \\ A_{m,n} &= 0 \quad \text{if } m > n \quad \text{else} \\ A_{m,n} &= A_{m-1,n} - A_{m-1,m-1}A_{m,n-m+1}. \end{aligned} \tag{2.5}$$

Then

$$A_{n,n} = g_n.$$

Firstly by iteration of (2.5),

$$\begin{aligned} A_{m,n} &= A_{m-1,n} - A_{m-1,m-1}[A_{m-1,n-m+1} - A_{m-1,m-1}A_{m,n-2(m-1)}] \\ &= A_{m-1,n} - A_{m-1,m-1}A_{m-1,n-m+1} + A^2_{m-1,m-1}A_{m,n-2(m-1)} = \cdots \\ &= \sum_{j=0}^{r} A^j_{m-1,m-1}A_{m-1,n-j(m-1)} + (-1)^{r+1}A^{r+1}_{m-1,m-1}A_{m,n-(r+1)(m-1)}. \end{aligned}$$

Now choosing r large enough so that $m > n - (r+1)(m-1)$ gives

$$A_{m,n} = \sum_{j=0}^{[\frac{n-m}{m-1}]} (-1)^j A^j_{m-1,m-1}A_{m-1,n-j(m-1)}. \tag{2.6}$$

From this we deduce in the same way as in Lemma 2.1 that $A_{m,n} = \sum_{|\lambda|=n} c_m(\lambda)a^\lambda$ where $c_m(\lambda) \in \mathbb{Z}$ represents a constant depending on λ and m. For $m = 1$, $A_{1,n} = a_n$ satisfies (2.4) since $|\lambda| = 1$.

Suppose now by induction that for $\ell = 1, 2, \ldots, m-1$ $A_{\ell,n}$ has terms whose signs obey (2.4). We consider now the sign in $A_{m,n}$ of the term corresponding to an arbitrary partition $\lambda(n)$ with $c_m(\lambda) \neq 0$. By (2.6) terms having the form $a^{\lambda(n)}$ will arise from one or more products

$$(-1)^j A^j_{m-1,m-1} A_{m-1,n-j(m-1)}, \quad j = 1, 2, \ldots, \left[\frac{n-m}{m-1}\right].$$

Suppose the term from $A^j_{m-1,m-1}$ corresponds to the product $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_r^{\alpha_r}$, this being made up of the product of j terms from $A_{m-1,m-1}$. For each of these j terms the sign is determined according to our inductive hypothesis by (-1) raised to the power 1 plus the sum of the exponents of the corresponding term. Thus the sign of the product term $a_1^{\alpha_1} \cdots a_r^{\alpha_r}$ in $A^j_{m-1,m-1}$ will be just $(-1)^{\Sigma\alpha_i + j}$.

On the other hand the sign of the term from $A_{m-1,n-j(m-1)}$, say $a_1^{\beta_1} a_2^{\beta_2} \cdots a_{n-j(m-1)}^{\beta_{n-j(m-1)}}$ will be $(-1)^{1+\Sigma\beta_i}$.

Combining these implies that the sign of $a^{\lambda(n)}$ from $(-1)^j A^j_{m-1,m-1} A_{m-1,n-j(m-1)}$ will be $(-1)^{2j+1+\Sigma\alpha_i+\Sigma\beta_i} = (-1)^{1+\Sigma\lambda_i}$ as required, since we have $\lambda_i = \alpha_i + \beta_i$, $1 \leq i \leq n$. ■

REMARK. We have

$$\begin{aligned}(-1)^{|\lambda|+1} &= (-1)^{\Sigma\lambda_{2i-1}+\Sigma\lambda_{2i}+1} = (-1)^{-\Sigma\lambda_{2i-1}+\Sigma\lambda_{2i}+1} \\ &= (-1)^{n+1+\Sigma\lambda_{2i}} = \begin{cases} (-1)^{1+\Sigma\lambda_{2i}}, & n \text{ even}, \\ (-1)^{\Sigma\lambda_{2i}}, & n \text{ odd}, \end{cases}\end{aligned}$$

where we have used the fact that $n = \sum_{2\geq 1} 2i\lambda_{2i} + \sum_{i\geq 1}(2i-1)\lambda_{2i-1}$. This gives an alternative rule for determining the sign of the term in $a^{\lambda(n)}$ that depends only on the even exponents λ_{2i} as well as the parity of n.

In order to estimate the rate of growth of the coefficients g_n in terms of the size of the coefficients a_i, $1 \leq i \leq n$, we will require estimates for the function

$$B(n) = \sum_{\lambda(n)} \left| c\big(\lambda(n)\big) \right|. \tag{2.7}$$

Thus $B(n)$ denotes the sum of the absolute values of the coefficients occuring in the expansion (2.3) of g_n.

We first remark that the similar sum

$$\sum c\big(\lambda(n)\big) = \begin{cases} 0, & n \neq 2^k, \\ 1, & n = 2^k \end{cases} \quad k = 0, 1, 2, \ldots. \tag{2.8}$$

This follows by setting $a_n = 1$, $n \geq 1$ in (1.1) which corresponds to the function $\frac{1}{1-z}$ with the well known power product expansion $\prod_{k=0}^{\infty}(1 + z^{2^k})$.

Suppose instead we choose $a_n = -1$, $n \geq 1$ in (1.1) with corresponding function $1 + \sum_{n=1}^{\infty} a_n z^n = 1 - \frac{z}{1-z} = \frac{1-2z}{1-z}$.

In this case by Theorem 2.2 each term $c\big((n)\big)a^{\lambda(n)}$ of the representation (2.3) becomes $(-1)^{\Sigma\lambda_i+1}\big|c\big(\lambda(n)\big)\big|(-1)^{\Sigma\lambda_i} = -\big|c\big(\lambda(n)\big)\big|$.

It follows that the coefficients g_n in the power product expansion of $\frac{1-2z}{1-z}$ are equal to $-B(n)$. We use this to prove

THEOREM 2.3. *(a) For $n = p$ prime,*

$$B(p) = \frac{2^p - 2}{p}. \tag{2.9}$$

(b)

$$\frac{2^{n-1}}{n} \leq B(n) < \frac{2^n}{n}, \quad n \geq 1. \tag{2.10}$$

(c) As $n \longrightarrow \infty$,

$$B(n) = \frac{2^n}{n}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{2.11}$$

PROOF. By taking logarithmic derivatives of the formal power product expansion (1.2) and using (1.3), we obtain the recurrence relation (see *e.g.* [KL]),

$$g_n = \frac{d_n}{n} + \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(-g_{n/d})^d, \quad n \geq 1. \tag{2.12}$$

We use this recurrence to estimate $g_n = -B(n)$ in the particular case $f(z) = \frac{1-2z}{1-z}$ for which

$$\frac{f'(z)}{f(z)} = \frac{-1}{(1-2z)(1-z)} = \sum_{n=1}^{\infty}(1-2^n)z^{n-1}.$$

Firstly $g_1 = d_1 = -1$ so that $B(1) = 1$. Next for $n = p$ prime,

$$g_p = \frac{d_p}{p} + \frac{1}{p}(-g_1)^p = \frac{2-2^p}{p},$$

which proves a).

b) Suppose now inductively that

$$\frac{-2^j}{j} < g_j \leq \frac{-2^{j-i}}{j}, \quad 1 \leq j \leq n.$$

It is straightforward to verify these inequalities for $1 \leq j \leq 8$. Hence we may assume $n \geq 9$. By (2.12),

$$g_n = \frac{-2^n}{n} + \frac{1}{n} + \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(-g_{n/d})^d.$$

Now

$$\begin{aligned} 0 < \frac{1}{n} + \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(-g_{n/d})^d &\leq \frac{2}{n} + \sum_{d|n} \frac{1}{d} |g_{n/d}|^d \leq \frac{2}{n} + \frac{2^n}{n} \sum_{\substack{d|n \\ 1<d<n}} \left(\frac{d}{n}\right)^{d-1} \\ &\leq \frac{2}{n} + \frac{2^n}{n} \left\{ \frac{2}{n} + \sum_{d=3}^{n/3} \left(\frac{d}{n}\right)^{d-1} + \left(\frac{1}{2}\right)^{\frac{n}{2}-1} \right\} \\ &\leq \frac{2^n}{n} \left\{ \frac{1}{2^8} + \frac{2}{9} + \sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k + \left(\frac{1}{2}\right)^{\frac{7}{2}} \right\} \quad \text{(since } n \geq 9) \\ &< \frac{1}{2} \cdot \frac{2^n}{n}, \end{aligned}$$

and the result follows.

c) As shown above

$$0 < \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(-g_{n/d})^d \leq \frac{1}{n} + \frac{2^n}{n} \left\{ \frac{2}{n} + \sum_{d=3}^{n/2} \left(\frac{d}{n}\right)^{d-1} \right\}.$$

Now by dissecting the sum $\sum_{d=3}^{n/2} (\frac{d}{n})^{d-1}$ into the three parts

$$\left\{ \sum_{3\leq d\leq 8} + \sum_{8\leq d\leq \sqrt{n}} + \sum_{\sqrt{n}\leq d\leq n/2} \right\} \left(\frac{d}{n}\right)^{d-1}$$

as in Greene and Knuth [GK] we find that

$$\sum_{d=3}^{n/2} \left(\frac{d}{n}\right)^{d-1} = O(n^{-2}), \quad n \longrightarrow \infty.$$

It follows that

$$g_n = -B(n) = \frac{-2^n}{n} \left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \longrightarrow \infty. \qquad \blacksquare$$

REMARK. We can improve this asymptotic estimate by bootstrapping. Since

$$\sum_{\substack{d|n \\ d>1}} \frac{1}{d}(-g_{n/d})^d = \frac{1}{2}(g_{n/2})^2 + O(n^{-3}2^n),$$

as shown above, and since (2.11) gives

$$\frac{1}{2}(g_{n/2})^2 = \frac{2^{n+1}}{n^2} \left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \longrightarrow \infty$$

we deduce that as $n \longrightarrow \infty$,

$$B(n) = \begin{cases} \frac{2^n}{n}\left(1 + O(n^{-2})\right), & n \text{ odd} \\ \frac{2^n}{n}\left(1 + \frac{2}{n} + O(n^{-2})\right), & n \text{ even} \end{cases}. \tag{2.13}$$

https://doi.org/10.4153/CJM-1995-062-9 Published online by Cambridge University Press

In view of the significant role played by the sequence $\{B(n)\}$ in the relationship between $\{g_n\}$ and $\{a_n\}$ it seems worthwhile to note a few arithmetic properties of these numbers. The first few values appear below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$B(n)$	1	1	2	3	6	8	18	27	54	84	186	296	630	1008	2106	3711

From this table we notice some interesting divisibility properties of $B(n)$ which are easily established.

PROPOSITION 2.4. *(a)*

$$B(n) \equiv \begin{cases} 0 \pmod 2, & n \neq 2^k, \\ 1 \pmod 2, & n = 2^k, \end{cases} \quad k = 0, 1, 2, \ldots.$$

(b) For $n \geq 3$,

$$B(n) \equiv \begin{cases} 0 \pmod 3, & n \neq 3 \cdot 2^k \\ 2 \pmod 3, & n = 3 \cdot 2^k \end{cases} \quad k = 0, 1, 2, \ldots.$$

PROOF. (a) $1 - \frac{z}{1-z} \equiv \frac{1}{1-z} \pmod 2$ and thus the power product for $1 - \frac{z}{1-z}$ modulo 2 equals that of $\frac{1}{1-z} = \prod_{k=0}^{\infty}(1 + z^{2^k})$.

(b) This follows from the identity

$$(1-z)(1-z^2)\prod_{k=0}^{\infty}(1 + z^{3.2^k}) = (1-z)(1-z^2)/(1-z^3)r = \frac{1-z^2}{1+z+z^2}.$$

Now modulo 3,

$$\frac{1-z^2}{1+2+2^2} \equiv \frac{1-z^2}{(1-z)^2} \equiv \frac{1+z}{1-z} \equiv 1 - \frac{z}{1-z} \equiv \prod_{n=1}^{\infty}\left(1 - B(n)z^n\right). \qquad \blacksquare$$

Theorem 1 of the introduction is now one part of the following main result.

THEOREM 2.5. *If*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = \prod_{n=1}^{\infty}(1 + g_n z^n) \tag{2.14}$$

and if the associated function

$$\hat{f}(z) = 1 - \sum_{n=1}^{\infty} |a_n| z^n = \prod_{n=1}^{\infty}(1 - G_n z^n) \tag{2.15}$$

then for $n \geq 1$,

$$|g_n| \leq G_n \leq B(n) s^n \tag{2.16}$$

where $s = \sup_{n \geq 1} |a_n|^{1/n}$.

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Furthermore the power products (2.14) and (2.15) converge at least for

$$|z| < \frac{1}{2s}. \tag{2.17}$$

PROOF. Let $|a|^\lambda$ denote the product $|a_1|^{\lambda_1}|a_2|^{\lambda_2}\cdots|a_n|^{\lambda_n}$. We have by (2.3)

$$|g_n| = \Big|\sum_{\lambda(n)} c(\lambda)a^\lambda\Big| \leq \sum_{\lambda(n)} |c(\lambda)|\,|a|^\lambda = G_n,$$

since $-G_n = \Sigma_{\lambda(n)}(-1)^{|\lambda|+1}|c(\lambda)|(-1)^{|\lambda|}|a|^\lambda$ using (2.4). Furthermore by definitions (1.6) and (2.7)

$$G_n \leq \sum_{\lambda(n)} |c(\lambda)|s^n = B(n)s^n.$$

It follows that

$$\limsup_{n\geq 1} |g_n|^{1/n} \leq \limsup_{n\geq 1} |G_n|^{1/n} \leq \limsup_{n\to\infty} B(n)^{1/n}s = 2s$$

by our estimate (2.11) for $B(n)$. The bounds for the power product radii of convergence then follow. ∎

REMARKS. 1) The example $\frac{1-2z}{1-z}$ whose power product converges for $|z| < \frac{1}{2}$ shows that this bound for the radius of convergence is in general best possible.

2) We are indebted to Lutz Lucht (Clausthal) for showing us that the bound (2.17) of Theorem 2.5 can be deduced from the result (1.5): If we assume for convenience that $s = 1$, then from the recurrence relation

$$na_n = d_n + \sum_{k=1}^{n-1} d_k a_{n-k}, \quad n \geq 1,$$

it is easy to show by induction that $|d_n| \leq 2^n - 1$ which with (1.5) implies the bound (2.17). However the proofs of (1.5) in [KL, IW] depend heavily on complex analysis whereas the direct proof of (2.17) given above requires only elementary techniques and leads also to the more precise estimates (2.16).

Although the bound (2.17) is in general best possible, sharper bounds can be derived for classes of functions satisfying addition restrictions. The following proposition provides a useful tool for deducing such improved estimates.

PROPOSITION 2.6. *If*

$$\hat{f}(z) = 1 - \sum_{n=1}^{\infty} |a_n|z^n = \prod_{n=1}^{\infty}(1 - G_n z^n) \tag{2.18}$$

then for $n \geq 1$,

$$0 \leq G_n < -\frac{\hat{d}_n}{n}, \tag{2.19}$$

where $\frac{\hat{f}'(z)}{\hat{f}(z)} = \sum_{n=1}^{\infty} \hat{d}_n z^{n-1}$.

Furthermore the radius of convergence of the product in (2.18) is greater than or equal to that of $\frac{\hat{f}'(z)}{\hat{f}(z)}$.

PROOF. The inequality $G_n \geq 0$ is immediate from equation (2.16) of the Theorem. Next, from the recurrence (2.12),

$$G_n = -\frac{\hat{d}_n}{n} - \sum_{\substack{d|n \\ d>1}} \frac{1}{d} G_{n/d}^d \leq -\frac{\hat{d}_n}{n}.$$

Thus $\limsup |G_n|^{1/n} \leq \limsup |\hat{d}_n|^{1/n}$ which implies that the radius of convergence of the product in (2.18) is greater than or equal to that of $\frac{\hat{f}'(z)}{\hat{f}(z)}$. ∎

REMARK. The radius of convergence of (2.18) can be strictly greater than that of $\frac{\hat{f}'(z)}{\hat{f}(z)}$, for example if $\hat{f}(z) = (1 - z)$.

Now the use of the above result in conjunction with Theorem 2.5 leads easily to numerous stronger corollaries.

COROLLARY 2.7. *a) If* $f(z) - 1$ *is an odd function then the power product (1.2) converges at least for* $|z| < \frac{1}{\phi s}$, *where* $\phi = \frac{\sqrt{5}+1}{2}$.

b) If $f(z)$ *is an even function then the power product (1.2) converges at least for* $|z| < \frac{1}{\sqrt{2}s}$.

PROOF. a) Let $g(z)$ denote the "odd" function $1 - \sum_{k=1}^{\infty} z^{2k-1} = \frac{1-z-z^2}{1-z^2}$ and define the sequence $\{E(n)\}_{n=1}^{\infty}$ by

$$g(z) = \prod_{n=1}^{\infty} \left(1 - E(n) z^n\right). \tag{2.20}$$

Now as in the proof of (2.16) in Theorem 2.5, we have the inequalities

$$|g_n| \leq G_n \leq E(n) s^n. \tag{2.21}$$

Now since $g'(z)$ converges for $|z| < 1$ and $\frac{1}{g(z)}$ has simple poles on the circles $|z| = \phi$ and $|z| = \frac{1}{\phi}$, we deduce that $\frac{g'(z)}{g(z)}$ has radius of convergence $\frac{1}{\phi}$. Thus by Proposition 2.6 the product representation (2.20) converges for $|z| < \frac{1}{\phi}$. We deduce that $\limsup_{n\to\infty} E(n)^{1/n} \leq \phi$ and the stated result then follows by using (2.21).

b) This follows in the same way as a) above by considering the power product expansion for the function $g(z) = 1 - \sum_{k=1}^{\infty} z^{2k} = \frac{1-2z^2}{1-z^2}$. ■

REMARK. We can generalise part b) above by considering functions which have the form $h(z) = f(z^k)$, $k = 2, 3, 4$, for which the radius of convergence is at least $\frac{1}{2^{1/k}s}$.

Similarly if we impose growth restrictions on all the coefficients a_n of $f(z)$, better bounds for the radius of convergence can be obtained. The next corollary is merely a sample of the large number of results of this type that can also be deduced.

COROLLARY 2.8. *a) If $f(z)$ given by (1.1) has coefficients which satisfy the growth condition $|a_n| \leq \frac{1}{2^{n-1}}$, $n \geq 1$, then the product (1.2) converges at least for $|z| < (2/3)s^{-1}$.*

b) If the coefficient satisfy $|a_n| \leq \frac{1}{n!}$, $n \geq 1$, then the product (1.2) converges at least for $|x| < (\ln 2)s^{-1}$.

The proofs follow as in the previous corollary by considering the poles of $\frac{g'(z)}{g(z)}$ for the respective functions

$$g(z) = 1 - \sum_{n=1}^{\infty} \frac{z^n}{2^{n-1}} = \frac{2-3z}{2-z}$$

and $g(z) = 1 - \sum_{n=1}^{\infty} \frac{z^n}{n!} = 2 - e^z$.

We remark finally that the functions $g(z)$ themselves of Corollaries 2.7 and 2.8 can be used to show that the respective bounds are sharp.

3. **Inverse power product expansions.**

If $f(z)$ has representations in both forms (1.1) and (1.7) then

$$f(z) = 1 + h_1 z + (h_2 + h_1^2)z^2 + (h_3 + h_2 h_1 + h_1^3)z^3 + \cdots$$

where in general for $n \geq 1$,

$$a_n = \sum_{\substack{j_1+j_2+\cdots+j_r=n \\ 1\leq j_1\leq j_2\leq\cdots\leq j_r\leq n}} h_{j_1} h_{j_2} \cdots h_{j_r} \tag{3.1}$$

where the summation is over all partitions of n. Concrete examples of (3.1) with $h_n \in \{0, 1\}$ arise in the theory of partitions. A further concrete case is the product $\prod_{n=1}^{\infty}(1 - \frac{z^n}{n})^{-1}$, treated recently by Knopfmacher and Ridley [KR].

By separating out the term h_n from the right hand side of (3.1) and successively sub-

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stituting for each h_{j_i} that occurs, we are lead to the representations

$$h_1 = a_1$$
$$h_2 = a_2 - a_1^2$$
$$h_3 = a_3 - a_2a_1$$
$$h_4 = a_4 - a_3a_1 - a_2^2 + 2a_2a_1^2 - a_1^4$$
$$h_5 = a_5 - a_4a_1 - a_3a_2 + a_3a_1^2 + a_2^2a_1 - a_2a_1^3$$
$$h_6 = a_6 - a_5a_1 - a_4a_2 + a_4a_1^2 - a_3^2 + 3a_3a_2a_1 - a_3a_1^3 - a_2^2a_1^2$$
$$h_7 = a_7 - a_6a_1 - a_5a_2 + a_5a_1^2 - a_4a_3 + 2a_4a_2a_1 - a_4a_1^3 + a_3^2a_1 + a_3a_2^2 - 3a_3a_2a_1^2 + a_3a_1^4 - a_2^3a_1 + 2a_2^2a_1^3 - a_2a_1^5$$
$$h_8 = a_8 - a_7a_1 - a_6a_2 + a_6a_1^2 - a_5a_3 + 2a_5a_2a_1 - a_5a_1^3 - a_4^2 + 3a_4a_3a_2 + 2a_4a_2^2 - 5a_4a_2a_1^2 + 2a_4a_1^4 + a_3^2a_2 - 2a_3^2a_1^2 - 4a_3a_2^2a_1 + 6a_3a_2a_1^3 - 2a_3a_1^5 - a_2^4 + 5a_2^3a_1^2 - 7a_2^2a_1^4 + 4a_2a_1^6 - a_1^8$$
$$h_9 = a_9 - a_8a_1 - a_7a_2 + a_7a_1^2 - a_6a_3 + 2a_6a_2a_1 - a_6a_1^3 - a_5a_4 + 2a_5a_3a_1 + a_5a_2^2 - 3a_5a_2a_1^2 + a_5a_1^4 + a_4^2a_1 + 2a_4a_3a_2 - 3a_4a_3a_1^2 - 3a_4a_2^2a_1 + 4a_4a_2a_1^3 - a_4a_1^5 - 2a_3^2a_2a_1 + 2a_3^2a_1^3 - a_3a_2^3 + 5a_3a_2^2a_1^2 - 5a_3a_2a_1^4 + a_3a_1^6 + a_2^4a_1 - 3a_2^3a_1^3 + 3a_2^2a_1^5 - a_2a_1^7$$
$$h_{10} = a_{10} - a_9a_1 - a_8a_2 + a_8a_1^2 - a_7a_3 + 2a_7a_2a_1 - a_7a_1^3 - a_6a_4 + 2a_6a_3a_1 + a_6a_2^2 - 3a_6a_2a_1^2 + a_6a_1^4 - a_5^2 + 3a_5a_4a_1 + 3a_5a_3a_2 - 4a_5a_3a_1^2 - 4a_5a_2^2a_1 + 5a_5a_2a_1^3 - a_5a_1^5 + a_4^2a_2 - 2a_4^2a_1^2 + a_4a_3^2 - 7a_4a_3a_2a_1 + 5a_4a_3a_1^3 - a_4a_2^3 + 7a_4a_2^2a_1^2 - 6a_4a_2a_1^4 + a_4a_1^6 - a_3^3a_1 - 2a_3^2a_2^2 + 7a_3^2a_2a_1^2 - 3a_3^2a_1^4 + 5a_3a_2^3a_1 - 12a_3a_2^2a_1^3 + 7a_3a_2a_1^5 - a_3a_1^7 - 2a_2^4a_1^2 + 4a_2^3a_1^4 - 2a_2^2a_1^6.$$

To begin our study of inverse power product representations we note that formally (1.7) corresponds precisely to the power product expansion (abbreviated PPE),

$$\frac{1}{f(z)} = \prod_{n=1}^{\infty}(1 - h_n z^n)$$

Now

$$\frac{(\frac{1}{f(z)})'}{\frac{1}{f(z)}} = -\frac{f'(z)}{f(z)} = -\sum_{n=1}^{\infty} d_n z^{n-1}.$$

Thus by substituting into the recurrence relation for the power product (2.12) we obtain

$$-h_n = -\frac{d_n}{n} + \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(h_{n/d})^d$$

which gives for the inverse power product the recurrence

$$h_n = \frac{d_n}{n} - \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(h_{n/d})^d. \tag{3.2}$$

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If we compare this with the corresponding recurrence for the power product of $f(z)$,

$$g_n = \frac{d_n}{n} + \sum_{\substack{d|n \\ d>1}} \frac{1}{d}(-g_{n/d})^d,$$

we see that in the case of n odd, since $h_1 = d_1 = g_1$ and all divisors d of n are also odd, that $h_n = g_n, n = 1, 3, 5, \ldots$. Thus we have shown

PROPOSITION 3.1. *If $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ has formal product expansions*

$$f(z) = \prod_{n=1}^{\infty} (1 + g_n z^n)$$

and

$$f(z) = \prod_{n=1}^{\infty} (1 - h_n z^n)^{-1}$$

then for n odd, $h_n = g_n$.

REMARK. If for example we consider the power product for the partition generating function $1 + \sum_{n=1}^{\infty} p(n)z^n$, then the proposition tells us that $g_n = 1$ for odd n. The PPE for this particular function is given a detailed treatment in [IW].

It follows that in considering further properties of inverse PPE's we may restrict our attention to the coefficients $\{h_{2n}\}_{n=1}^{\infty}$. Corresponding results for $\{h_{2n-1}\}_{n=1}^{\infty}$ follow immediately from those previously established for $\{g_{2n-1}\}_{n-1}^{\infty}$.

Furthermore since $|d_n| = |-d_n|$, all the results in the power product literature which depend on estimates for $|d_n|$ apply with trivial modification to inverse PPE's as well. In particular if $r = \sup_{n \geq 1} |d_n|^{1/n}$ then the result of [KL] and [IW] implies that the inverse PPE (1.7) of $f(z)$ converges at least for

$$|z| < \frac{1}{r}. \tag{3.3}$$

Similarly, Theorem 3 of [GKL] gives conditions on the sequence $\{|d_n|\}_{n=1}^{\infty}$ which when applied to inverse PPE's ensure that the conclusion $\lim_{n\to\infty} h_n/(d_n/n) = 1$ holds. Furthermore by considering $\frac{1}{f(z)}$ in place of $f(z)$, the results of [GKL] on the distribution of the zeros of partial products become theorems on the distribution of the poles of the partial products $Q_m(z) = \prod_{n=1}^{m}(1 - h_n z^n)^{-1}$, $m \geq 1$, of inverse PPE'S.

NOTE. If we eliminate $\frac{d_n}{n}$ from each recurrence (3.2) and (2.12) then we have the relationship

$$\sum_{d|n} \frac{1}{d} h_{n/d}^d = \sum_{d|n} \frac{(-1)^{d+1}}{d} g_{n/d}^d. \tag{3.4}$$

This can be used to recursively calculate h_n, n even, if the sequence $\{g_n\}_{n=1}^{\infty}$ is known and vice versa. In particular in the case of partition generating function considered above, we have for even n the recurrence

$$\sum_{d|n} (-1)^{\frac{n}{d}+1} d g_d^{n/d} = \sigma(n)$$

where $\sigma(n) = \sum_{d|n} d$ is the divisor sum function.

By analogy with our results for PPE's we show

LEMMA 3.2. *For $n \geq 1$,*

$$h_n = \sum_{\lambda(n)} b(\lambda)a^{\lambda} \tag{3.5}$$

where the sum is over all partitions λ of n.

PROOF. The proof is analogous to that of Lemma 2.1 using instead the relation

$$h_n = a_n - \sum_{\substack{n=i_1+i_2+\cdots+i_r \\ i_1 \leq i_2 \leq \cdots \leq i_r \\ r \geq 2}} h_{i_1} h_{i_2} \cdots h_{i_r}. \qquad \blacksquare$$

NOTE. Proposition 3.1 implies that $b\big(\lambda(2n-1)\big) = c\big(\lambda(2n-1)\big)$ for $n \geq 1$. Also since $\frac{1}{1-z}$ is its own inverse PPE we deduce immediately the identity

$$\sum_{\lambda(n)} b\big(\lambda(n)\big) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}. \tag{3.6}$$

(Compare with (2.8) for $\sum c\big(\lambda(n)\big)$).

LEMMA 3.3. *The sign of the term $b(\lambda)a^{\lambda}$ ($b(\lambda) \neq 0$) in the expansion of h_n is*

$$(-1)^{|\lambda|+1}. \tag{3.7}$$

PROOF. As in the case of PPE's we use a three term recurrence relation for h_n established in [GGM]. Let

$$\begin{aligned} B_{1,n} &= a_n \\ B_{m,n} &= 0 \quad \text{if } m > n \quad \text{else} \\ B_{m,n} &= B_{m-1,n} - B_{m-1,m-1}B_{m-1,n-m+1}, \end{aligned} \tag{3.8}$$

then $B_{nn} = h_n$.

The proof by induction on m of the signs of $B_{m,n}$ now follows from (3.8) in a similar but simpler way to that of Lemma 2.2. ■

We see that precisely the same rule of signs applies here as applies in the case of PPE's. Thus in this case too, the choice $a_n = -1$, $n \geq 1$, leads to $h_n = -\sum_{\lambda} \big| b\big(\lambda(n)\big) \big|$. For convenience we denote this sum by

$$\tilde{B}(n) = \sum_{\lambda} \big| b\big(\lambda(n)\big) \big|. \tag{3.9}$$

THEOREM 3.4. *(a) For n odd,*

$$\tilde{B}(n) = B(n).$$

(b) For n even,

$$\beta \frac{2^n}{n} \leq \tilde{B}(n) \leq \alpha \frac{2^n}{n}, \quad n \geq 2, \tag{3.10}$$

where $\alpha = \frac{54}{32}$ *and* $\beta = -\alpha + 2 = \frac{10}{32}$.

(c) As $n \longrightarrow \infty$,

$$\tilde{B}(n) = \frac{2^n}{n}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{3.11}$$

PROOF. (a) This is immediate since the coefficients h_{2n-1} and g_{2n-1} in the inverse PPE and PPE, respectively for the function $f(z) = 1 - \frac{z}{1-z}$ are equal.

(b) Suppose now that for $f(z) = \frac{1-2z}{1-z}$ for which $d_n = 1 - 2^n, n \geq 1$, we have inductively $-\alpha\frac{2^j}{j} < h_j \leq -\beta\frac{2^j}{j}$, $1 \leq j < n$. The recurrence (3.2) gives $h_1 = d_1 = -1$ and $h_n = -\frac{2^n}{n} + \frac{1}{n} - \sum_{\substack{d|n\\ d>1}} \frac{1}{d}(h_{n/d})^d$, $n > 1$.

Now since $|h_j| < \alpha 2^j/j$, $1 \leq j < n$,

$$\begin{aligned}
&\left|\frac{1}{n} + \sum_{\substack{d|n\\ d>1}} \frac{1}{d}(h_{n/d})^d\right| \\
&\quad \leq \frac{2}{n} + \sum_{\substack{d|n\\ 1<d<n}} \frac{1}{d}|h_{n/d}|^d \\
&\quad \leq \frac{2}{n} + \alpha\frac{2^n}{n} \sum_{\substack{d|n\\ 1<d<n}} \left(\frac{d}{n}\right)^{d-1} \alpha^{d-1} \\
&\quad \leq \alpha\frac{2^n}{n}\left\{\frac{1}{\alpha 2^{n-1}} + \frac{2\alpha}{n} + \left(\frac{3\alpha}{n}\right)^2 + \sum_{k=3}^{\infty}\left(\frac{\alpha}{4}\right)^k + \left(\frac{\alpha}{2}\right)^{\frac{n}{2}-1} + \left(\frac{\alpha}{3}\right)^{\frac{n}{3}-1} - 1\right\} \\
&\quad \leq \alpha\frac{2^n}{n}\left\{\frac{1}{\alpha 2^{23}} + \frac{\alpha}{12} + \left(\frac{\alpha}{8}\right)^2 + \frac{(\frac{\alpha}{4})^3}{1-\frac{\alpha}{4}} + \left(\frac{\alpha}{2}\right)^{11} + \left(\frac{\alpha}{3}\right)^7\right\} \quad (\text{if } n \geq 24) \\
&\quad \leq \alpha\frac{2^n}{n}\left(1 - \frac{1}{\alpha}\right).
\end{aligned}$$

Hence

$$|h_n| = \tilde{B}(n) \leq \alpha\frac{2^n}{n} + \left\{\frac{1}{\alpha} + 1 - \frac{1}{\alpha}\right\} = \alpha\frac{2^n}{n}$$

and

$$h_n \leq -\frac{2^n}{n} + (\alpha - 1)\frac{2^n}{n} = (-\beta)\frac{2^n}{n}.$$

The cases $2 \leq n \leq 22$ for even n are easily verified by computer. We remark that the value $\alpha = \frac{54}{32}$ is the smallest value for which the upper bound is valid for every n, with the equality $h_n = -\alpha\frac{2^n}{n}$ holding for $n = 8$. With more work an improved value for β could be found: The computational results suggest that the largest possible value of β is $\frac{60}{64}$ which is achieved in the case $n = 6$.

(c) Now $|\sum_{\substack{d|n\\ d>1}} \frac{1}{d}|(h_{n/d})^d| \leq \frac{1}{n} + \alpha\frac{2^n}{n}\{\frac{2\alpha}{n} + \sum_{d=3}^{n/2}(\frac{\alpha d}{n})^{d-1}\}$. Now applying the same dissection used for $B(n)$ results in $\sum_{d=3}^{n/2}(\frac{\alpha d}{n})^{d-1} = O(n^{-2}), n \longrightarrow \infty$. Hence

$$h_n = -\tilde{B}(n) = -\frac{2^n}{n}\left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \longrightarrow \infty. \qquad \blacksquare$$

REMARK. We may bootstrap in precisely the same manner as before to obtain as $n \longrightarrow \infty$,

$$\tilde{B}(n) = \frac{2^n}{n}\left(1 + \frac{2}{n} + O(n^{-2})\right), \quad n \text{ even}.$$

The first few values of the sequence $\{\tilde{B}(n))\}$ appear below

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\tilde{B}(n)$	1	2	2	6	6	10	18	54	54	114	186	334	630	1314	2106	5910

Corresponding to Proposition 2.4 for $B(n)$ we have

PROPOSITION 3.5. *a)* $\tilde{B}(n) \equiv 0 \pmod 2$ *for* $n > 1$.
b) For $n \geq 4$,

$$B(n) \equiv \begin{cases} 0 \pmod 3, & n \neq 3.2^k, \\ 1 \pmod 3, & n = 3.2^k \end{cases} \quad k = 1, 2, 3, \ldots.$$

PROOF. a) This is immediate since $1 - \frac{z}{1-z} \equiv \frac{1}{1-z}$ modulo 2.
b) Here we have the identity

$$(1+z)^{-1}(1-z^2)^{-1}(1-z^3)^{-1}\prod_{k=1}^{\infty}(1+z^{3.2^k})^{-1}$$

Now modulo 3,

$$\frac{1+z^3}{(1+z)(1-z^2)} \equiv \frac{(1+z)^2}{1-z^2} \equiv \frac{1+z}{1-z} \equiv 1 - \frac{z}{1-z}$$

as required.

THEOREM 3.6. *If*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = \prod_{n=1}^{\infty}(1 - h_n z^n)^{-1} \tag{3.12}$$

and if the associated function

$$\hat{f}(z) = 1 - \sum_{n=1}^{\infty} |a_n| z^n = \prod_{n=1}^{\infty}(1 + H_n z^n)^{-1} \tag{3.13}$$

then for $n \geq 1$,

$$|h_n| \leq H_n \leq \tilde{B}(n) s^n \tag{3.14}$$

where $s = \sup_{n \geq 1} |a_n|^{1/n}$. *Furthermore the inverse* PPE*'s (3.12) and (3.13) converge at least for* $|z| < \frac{1}{2s}$.

PROOF. We have $|h_n| = |\sum_{\lambda(n)} b(\lambda) a^{\lambda}| \leq \sum_{\lambda(n)} |b(\lambda)|\, |a|^{\lambda} = H_n$.

The last equality follows as in the PPE case from the rule for signs. Furthermore $H_n \leq \sum_{\lambda(n)} |b(\lambda)| s^n = \tilde{B}(n) s^n \leq (2s)^n$ from our estimates for $\tilde{B}(n)$.

Now

$$\prod_{n=1}^{\infty}(1-h_nz^n)^{-1}=\prod_{n=1}^{\infty}\left(1+\frac{h_nz^n}{1-h_nz^n}\right)$$

and

$$\prod_{n=1}^{\infty}(1-H_nz^n)^{-1}=\prod_{n=1}^{\infty}\left(1+\frac{H_nz^n}{1-H_nz^n}\right).$$

From the above estimates, each of the functions $\frac{h_nz^n}{1-h_nz^n}$ and $\frac{H_nz^n}{1-H_nz^n}$ is analytic within the circle $|z|<\frac{1}{2s}$.

Consider $|z|\leq\rho<\frac{1}{2s}$ then

$$\sum_{n=1}^{\infty}\left|\frac{h_nz^n}{1-h_nz^n}\right|\leq\sum_{n=1}^{\infty}\left|\frac{H_nz^n}{1-H_nz^n}\right|\leq\sum_{n=1}^{\infty}\frac{(2sz)^n}{1-(2sz)^n}\leq\frac{1}{1-2s\rho}\sum_{n=1}^{\infty}(2sz)^n.$$

Thus the series converge uniformly for such z, and by standard results on infinite products (see *e.g.* Knopp [p. 437]), the inverse PPE's converge for $|z|<\frac{1}{2s}$. ■

REMARKS. If $\hat{f}(z)=1-\sum_{n=1}^{\infty}|a_n|z^n$ has product representation (2.15) and (3.13) then we know already that $H_n=G_n$ for n odd. Furthermore by induction using (2.12) and (3.2) is is easy to show that $H_{2^k}>G_{2^k}$, $k=1,2,3,\ldots$. However, such inequalities need not hold for arbitrary even n. For example if $n=6$ we have $H_6-G_6=a_3^2+2a_3a_2a_1-a_2a_1^4$. Now choosing $a_1=a_2=0$, $a_3\neq 0$ gives $H_6-G_6=a_3^2>0$, while the choice $a_3=0$, $a_1\neq 0$ and $a_2>0$ gives $H_6-G_6=a_2a_1a_1^4<0$.

Next since $H_n=-\frac{\hat{a}_n}{n}+\sum_{\substack{d|n\\ d>1}}\frac{(-1)^d}{d}H^d_{n/d}$ it need not be true that $H_n>-\frac{\hat{a}_n}{n}$, so that the analogue of Proposition 2.6 does not hold for inverse PPE's. However, it is still possible to deduce sharper corollaries from Theorem 3.6 in the manner used below.

COROLLARY 3.7. *a) If $f(z)$ is an odd function then the inverse* PPE *(1.7) converges at least for $|z|<\frac{1}{\phi s}$, where $\phi=\frac{\sqrt{5}-1}{2}$.*

b) If $f(z)$ is an even function then the inverse PPE *(1.7) converges at least for $|z|<\frac{1}{\sqrt{2s}}$.*

PROOF. Suppose that

$$g(z)=\frac{1-z-z^2}{1-z^2}=\prod_{n=1}^{\infty}\left(1+\tilde{E}(n)z^n\right)^{-1} \tag{3.15}$$

Then as in Theorem 3.6, for $f(z)$ an odd function, we have,

$$|h_n|\leq H_n\leq\tilde{E}(n)S^n.$$

Now using the result (3.3) we find that (3.15) converges at least for $|z|<\frac{1}{\phi}$ from which we can deduce as in Theorem 3.6 that $f(z)$ converges for $|z|<\frac{1}{\phi s}$. The proof of b) using (3.3) and $g(z)=\frac{1-2z^2}{1-z^2}$ is similar. ■

We conclude this section by briefly considering some simple upper bounds for the radius of convergence (which we shall denote by ρ) of an inverse PPE (1.7) for $f(z)$.

PROPOSITION 3.8. *a) ρ is less than or equal to the radius of convergence of $f'(z)/f(z)$.*
b) ρ is less than or equal to the radius of convergence of the product $\prod_{n=1}^{\infty}(1-h_n z^n)$ (which corresponds formally to $\frac{1}{f(z)}$).

PROOF. a) The convergence of the product (1.7) for $|z| < \rho$ implies that the power series (1.1) for $f(z)$ converges and has no zeros within $|z| < \rho$. Hence the power series for $f'(z)$, $\frac{1}{f(z)}$ and their Cauchy product must also converge for $|z| < \rho$.

b) In order for $\prod_{n=1}^{\infty}(1-h_n z^n)^{-1}$ to converge in some domain, it is necessary that each term $(1-h_n z^n)^{-1}$ converge within this domain. Hence we require that $|z| < \frac{1}{|h_n|^{1/n}}$, $n = 1, 2, 3, \ldots$. Thus $\rho \leq \inf_{n \in \mathbb{N}} 1/|h_n|^{1/n} \leq 1/\limsup_{n\to\infty} |h_n|^{1/n}$, from which the result follows. ■

Let us denote the radius of convergence of the PPE (1.2) by R. In view of Proposition 3.8b) one may be tempted to believe that the inequality $\rho \leq R$ must also be true. Proposition 2.6 in conjunction with Proposition 2.8a) shows that this is indeed the case for the class of functions denoted by $\hat{f}(z)$. However the following example shows that it need not hold for arbitrary functions:

Let

$$f(z) = 1 + \sum_{n=1}^{\infty}\left[\frac{n+1}{2}\right] z^n.$$

Then it is easily verified that $f(z) = (1-z)^{-1}(1-z^2)^{-1}$, with $\rho = 1$. On the other hand since $g_1 = a_1$ and $g_2 = a_2$ the PPE for $f(z)$ has the form $(1+z)(1+2z^2)\cdots$. In view of the zeros of the second term we see that $R \leq \frac{1}{\sqrt{2}} < \rho$.

Nevertheless, we show that for functions $f(z)$ whose PPE coefficients are "nicely behaved" we do have $\rho \leq R$. In particular this holds in the case that $\lim_{n\to\infty} |g_n|^{1/n}$ exists.

PROPOSITION 3.9. *Let $f(z)$ be a function whose* PPE *(1.2) satisfies*

$$\limsup_{n\to\infty} |g_{2n}|^{\frac{1}{2n}} \leq \limsup_{n\to\infty} |g_{2n-1}|^{\frac{1}{2n-1}}. \tag{3.16}$$

Then $\rho \leq R$.

PROOF. We have

$$\begin{aligned} R^{-1} &= \limsup_{n\to\infty} |g_n|^{1/n} \geq \limsup_{n\to\infty} |g_{2n-1}|^{1/2n-1} \\ &= \limsup_{n\to\infty} |h_{2n-1}|^{\frac{1}{2n-1}} \equiv R_o^{-1}. \end{aligned}$$

Now as in the proof of Proposition 3.8b) we require that $\rho \leq \inf_{n\in\mathbb{N}} 1/|h_{2n-1}|^{\frac{1}{2n-1}} \leq R_o$.

Now our assumption (3.16) implies that $R_o = R$ and hence that $\rho \leq R$. ■

https://doi.org/10.4153/CJM-1995-062-9 Published online by Cambridge University Press

4. **Combinatorial products.** Various product expansions of types (1.2) and (1.7) arise as generating functions in combinatorial problems. In such cases it is usually the product and its coefficients which are known explicitly, the reverse of the situation studied in the previous sections. In such counting problems the product coefficients g_n or h_n are normally non-negative real or ever rational numbers. For product expansions such as these with non-negative coefficients, the following result provides a simple but useful tool for deducing the analytic behaviour of the corresponding power series $f(z)$.

THEOREM 4.1. *a) If*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = \prod_{n=1}^{\infty} (1 + g_n z^n)$$

where $g_n \geq 0$ for $n \geq 1$, then the series and the product have the same radius of convergence.

b) Similarly if

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = \prod_{n=1}^{\infty} (1 - h_n z^n)^{-1}$$

where $h_n \geq 0$ for $n \geq 1$, then the series and the product have the same radius of convergence.

In fact we will deduce Theorem 4.1 from the following result which applies also to much more general classes of product expansions, such as those treated for example in [KKR, K].

THEOREM 4.2. *Let $g_\ell(z) = \sum_{n=\ell}^{\infty} g_{\ell n} z^n$ be an infinite sequence of formal power series with $g_{\ell n} \geq 0$ for each ℓ and n in $\mathbb{N}$. Suppose that the formal product expansion*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = \prod_{\ell=1}^{\infty} \big(1 + g_\ell(z)\big) \tag{4.1}$$

holds in the sense that

$$a_n = \sum g_{\ell_1 n_1} g_{\ell_2 n_2} \cdots g_{\ell_r n_r} \tag{4.2}$$

where the summation is over all $(n_1, n_2, \ldots, n_r)$ with $n = n_1 + n_2 + \cdots + n_r$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$.

(i) If $\sum_{\ell=1}^{\infty} g_\ell(z)$ is absolutely convergent for $0 \leq z < R$, then $1 + \sum_{n=1}^{\infty} a_n z^n$ with coefficients satisfying (4.2) converges in $|z| < R$.

(ii) If $f(z)$ is an analytic function in $|z| < R$ then for each ℓ, $g_\ell(z)$ converges absolutely in $|z| < R$ and $\prod_{\ell=1}^{\infty} \big(1 + g_\ell(z)\big)$ converges absolutely in $|z| < R$ to $f(z)$.

PROOF. (i) Under these conditions the product expansion on the right hand side is absolutely convergent and thus defines an analytic function in $|z| < R$. The Taylor series expansion of this analytic function (see *e.g.* Knopp [Kn]) coincides with that of $f(z)$ and the result follows.

(ii) Firstly the sum in (4.2) for a_n consists of finitely many terms since $g_{\ell n} = 0$ for $\ell > n$. Moreover, (4.2) implies that

$$a_n \geq \sum_{\ell=1}^{n} g_{\ell n}.$$

Therefore,

$$\sum_{n=1}^{\infty} a_n z^n \geq \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} g_{\ell n} z^n.$$

The absolute convergence of the right hand side for $0 \leq z < R$ implies that the left hand side is absolutely convergent. Hence we may interchange the order of summation on the right to obtain

$$\infty > \sum_{n=1}^{\infty} a_n z^n \geq \sum_{\ell=1}^{\infty} \sum_{n=\ell}^{\infty} g_{\ell n} z^n = \sum_{\ell=1}^{\infty} g_\ell(z).$$

This then implies that $\prod_{\ell=1}^{\infty}\bigl(1 + g_\ell(z)\bigr)$ is absolutely convergent for $|z| < R$. ∎

Using Theorem 4.1 we deduce for example that the generating functions for partitions and distinct partitions each have radius of convergence 1. Such results are frequently given independent derivations in textbooks on number theory and combinatorics. More generally we can deduce the following Theorem of Groswald [G, p. 114] concerning partitions with parts belonging to any finite of infinite set $A = \{a_1, a_2, \ldots\}$ of positive integers:

THEOREM. *The generating functions of the partition functions converge inside the unit circle.*

We remark finally that results analogous to the above hold also for products whose coefficients alternate in sign. For example setting $z = -t$ in Theorem 4.1a) leads to the result:

If

$$1 + \sum_{n=1}^{\infty} a_n z^n = \prod_{n=1}^{\infty}\bigl(1 + (-1)^n g_n z^n\bigr)$$

where $g_n \geq 0$, $n \geq 1$ then the series and product have the same radius of convergence.

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