BULLETIN 13 (1), 1970

A REMARK ON CONTRACTIVE MAPPINGS[1]

BY
KAI-WANG NG

Much current research is concerned with the fixed points of contractive mappings (mappings which shrink distance in some manner) from a metric space into itself. In this remark we shall point out that most mappings treated in the literature are very special in the sense that all these mappings satisfy a condition which is rather severe: every periodic point must necessarily be a fixed point.

We list some of these contractive conditions below.

(1) (Banach): There is a number α, $0 \le \alpha < 1$ such that $d(Tx, Ty) \le \alpha d(x, y)$, $x, y \in X$,

(2) (Rakotch [6]): There exists a decreasing function $\alpha(d(x, y))$ depending on the metric $d(x, y)$, $0 \le \alpha(d(x, y)) < 1$, such that $d(Tx, Ty) \le \alpha(d(x, y))d(x, y)$, $x, y \in X$,

(3) (Boyd and Wong [7]): For $x \ne y$, $d(Tx, Ty) \le \psi(d(x, y))$, where $\psi(d)$ is an upper semicontinuous function of the metric d and $\psi(d) < d$ for $d > 0$; furthermore $\liminf_{d\to\infty} \{d - \psi(d)\} > 0$,

(4) (Meir [5]): Given $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that $d(x, y) > \varepsilon$ implies $d(Tx, Ty) < d(x, y) - \lambda(\varepsilon)$,

(5) (Edelstein [3]): $d(Tx, Ty) < d(x, y)$ for all $x \ne y$,

(6) (Bailey [1]): For all $x \ne y$, there exists $n = n(x, y)$ such that $d(T^n x, T^n y) < d(x, y)$,

(7) (Belluce and Kirk [2], [4]): If $\delta(O(x)) > 0$ then $\lim_{n\to\infty} \delta(O(T^n x)) < \delta(O(x))$, where $O(x) = \{x, Tx, T^2x, \ldots, T^n x, \ldots\}$ and $\delta(A)$ is the diameter of a set A.

It is obvious that a mapping satisfying any one of (1), (2), (3), and (4) will satisfy (5) and in turn, condition (5) implies condition (6).

Definition. A mapping $T: X \to X$ is called *non-periodic* if $x \ne Tx$ implies $x \ne T^n x$ for all $n = 1, 2, \ldots$.

We now show that a mapping satisfying any one of the conditions (1) to (7) is a nonperiodic mapping. It is sufficient to show this for mappings satisfying condition (6) and (7).

Theorem 1. *A mapping $T: X \to X$ is nonperiodic if it satisfies condition* (6): *for $x \ne y$ there exist $n = n(x, y)$ such that*

$$d(T^n x, T^n y) < d(x, y).$$

Received by the editors May 1, 1969.

(1) This is an extract from the author's M.Sc. thesis. The author gratefully acknowledges the help given by his supervisor, Professor T. D. Rogers.

Proof. Suppose $x \neq Tx$ and there exists some positive integer K which is the smallest such that $T^K x = x$.

By hypothesis we can choose $n_1(x)$ which is the least positive integer such that $d(x, Tx) > d(T^{n_1}x, T^{n_1+1}x)$. Observe that $n_1 < K$ and $d(T^{n_1}x, T^{n_1+1}x) > 0$. Indeed, if $n_1 \geq K$, then $n_1 = rK + q$ where r, q are positive integers, $0 \leq q < K \leq n_1$; consequently

$$d(x, Tx) > d(T^{n_1}x, T^{n_1+1}x) = d(T^q x, T^{q+1}x),$$

contradicting minimality of n_1. Also, if $d(T^{n_1}x, T^{n_1+1}x) = 0$, then $T^{n_1}x = T^{n_1+1}x$, hence $T^{n_1+(K-n_1)}x = T^{n_1+1+(K-n_1)}x$, i.e. $x = Tx$, contradicting our assumption on x.

Now since $d(T^{n_1}x, T^{n_1+1}x) > 0$, we can select $n_2(x)$ as the smallest positive integer such that

$$d(T^{n_1}x, T^{n_1+1}x) > d(T^{n_2}x, T^{n_2+1}x).$$

The same argument as above is used to deduce that $n_2 < K$ and $d(T^{n_2}x, T^{n_2+1}x) > 0$.

Proceeding in this manner, we can find a sequence $\{n_i\}$ of positive integers such that $n_i < K$ and

$$d(x, y) > d(T^{n_1}x, T^{n_1+1}x) > d(T^{n_2}x, T^{n_2+1}) > \ldots .$$

But then there must be two indices, say $i > j$ such that $n_i = n_j$, since $n_i < K$, $i = 1, 2, \ldots$. This is a contradiction, for then $d(T^{n_j}x, T^{n_j+1}x) = d(T^{n_i}x, T^{n_i+1}x)$.

THEOREM 2. *A mapping T: $X \to X$ is nonperiodic if it satisfies condition* (7): *if* $\delta(O(x)) > 0$ *then*

$$\lim_{n\to\infty} \delta(O(T^n x)) < \delta(O(x)).$$

Proof. We first note that $\delta(O(x)) > 0$ if and only if $x \neq Tx$. Also, by definition of $O(x)$,

$$\delta(O(x)) \geq \delta(O(Tx)) \geq \cdots \geq \delta(O(T^n x)) \geq \cdots \geq \lim_{n\to\infty} \delta(O(T^n x)).$$

Suppose $x \neq Tx$, then by hypothesis we have

$$\delta(O(x)) > \lim_{n\to\infty} \delta(O(T^n x)).$$

Hence there is an N such that $\delta(O(x)) > \delta(O(T^N x))$, so we have $O(x) \neq O(T^N x)$. This implies that $x \notin O(T^N x)$, i.e. $x \neq T^N x, T^{N+1}x, \ldots$.

Finally, it is impossible that $x = T^m x$ for $m < N$. For if so, let $p > 0$ be an integer such that m divides $N + p$, then $x = T^{N+p}x$, contradicting the argument in the previous paragraph.

REFERENCES

1. D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. **41** (1966), 101–106.

https://doi.org/10.4153/CMB-1970-022-9 Published online by Cambridge University Press

2. L. P. Belluce and W. A. Kirk, *Fixed point theorems for certain class of non-expansive mappings* (to appear in Proc. Amer. Math. Soc.).

3. M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

4. W. A. Kirk, *On mappings with diminishing orbital diameters*, J. London Math. Soc. **44** (1969), 107–111.

5. A. Meir, *A theorem on contractive mappings*, (to appear).

6. E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459–465.

7. D. W. Boyd and J. S. W. Wong, *On non-linear contractions* (to appear).

UNIVERSITY OF ALBERTA,
EDMONTON, ALBERTA

https://doi.org/10.4153/CMB-1970-022-9 Published online by Cambridge University Press