FINITE GROUPS WITH TWO \( p \)-REGULAR CONJUGACY CLASS LENGTHS II

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Abstract

Let \( G \) be a finite group. We prove that if the set of \( p \)-regular conjugacy class sizes of \( G \) has exactly two elements, then \( G \) has Abelian \( p \)-complement or \( G = P Q \times A \), with \( P \in \text{Syl}_p(G) \), \( Q \in \text{Syl}_q(G) \) and \( A \) Abelian.

Keywords and phrases: finite-groups, \( p \)-regular elements, conjugacy class sizes.

1. Introduction

Itô proved in [5] that if \( G \) is a finite group such that all its noncentral conjugacy classes have equal size, then \( G = Q \times A \), where \( Q \) is a Sylow \( q \)-subgroup of \( G \), for some prime \( q \), and \( A \) lies in \( Z(G) \). In [1], Beltrán and Felipe proved a generalization of this result for \( p \)-regular conjugacy class sizes and some prime \( p \), with the assumption that the group \( G \) is \( p \)-solvable. In the present paper, we improve this result by showing that the \( p \)-solvability condition is not necessary.

**Theorem A.** Let \( G \) be a finite group. If the set of \( p \)-regular conjugacy class sizes of \( G \) has exactly two elements, for some prime \( p \), then \( G \) has Abelian \( p \)-complement or \( G = P Q \times A \), with \( P \in \text{Syl}_p(G) \), \( Q \in \text{Syl}_q(G) \) and \( A \subseteq Z(G) \), with \( q \) a prime distinct from \( p \). As a consequence, if \( \{1, m\} \) are the \( p \)-regular conjugacy class sizes of \( G \), then \( m = p^a q^b \). In particular, if \( b = 0 \) then \( G \) has Abelian \( p \)-complements and if \( a = 0 \) then \( G = P \times Q \times A \) with \( A \subseteq Z(G) \).

The proof given in [1] for \( p \)-solvable groups is divided into two cases, when the centralizers of noncentral \( p \)-regular elements are all \( G \)-conjugated and when they are not. In the second case, it is easy to check that the hypothesis of \( p \)-solvability is not needed, so our study reduces then to the case in which all the centralizers of noncentral \( p \)-regular elements are conjugated. In order to solve this case, we are going to base

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our arguments on the proof of a theorem of Camina [2, Theorem 1]. We stress that while Camina used the classification obtained by Gorenstein and Walter [3] of those groups whose Sylow 2-subgroups are dihedral (this having been used to complete the classification of the simple finite groups), we present a more simple proof by making use of a well-known theorem of Kazarin which asserts that in any finite group the subgroup generated by an element of prime power class size is always solvable [4, Theorem 15.7].

Furthermore, we remark that it is not feasible that all the centralizers of noncentral elements of a group $G$ are conjugate, but it is easy to find examples where all the centralizers of noncentral $p$-regular elements are conjugate (consequently $G$ has exactly two $p$-regular conjugacy class sizes) for some prime $p$. For instance, the centralizers of all noncentral 2-elements of $\text{SL}(2, 3)$ are conjugate and the 3-regular class sizes are $\{1, 6\}$. Another example is $\text{Alt}(4)$, whose 2-regular class sizes are $\{1, 4\}$.

We shall assume that every group is finite and we shall denote by $G_{p'}$ the set of $p$-regular elements of $G$.

2. Preliminary results

We shall need some results on conjugacy classes of $p$-regular elements.

**Lemma 1.** Let $G$ be a finite group. Then all the conjugacy class sizes in $G_{p'}$ are $p$-numbers if and only if $G$ has Abelian $p$-complements.

**Proof.** See [1, Lemma 2].

The following is exactly [2, Lemma 1], but we present an easier proof. It generalizes [1, Lemma 3] by eliminating the hypothesis of $p$-solvability.

**Lemma 2.** Suppose that $G$ is a finite group and that $p$ is not a divisor of the sizes of $p$-regular conjugacy classes. Then $G = P \times H$ where $P$ is a Sylow $p$-subgroup and $H$ is a $p$-complement of $G$.

**Proof.** Let $g \in G$ and consider its $\{p, p'\}$-decomposition as $g = g_p g_{p'}$. Suppose that $g_{p'}$ is noncentral. As the class size of $g_{p'}$ is a $p'$-number, if we fix a Sylow $p$-subgroup $P$ of $G$, then there exists some $t \in G$ such that $g_p \in P^t \subseteq C_G(g_{p'})$. Therefore,

$$G = \bigcup_{t \in G} P^t C_G(P^t).$$

Then $G = PC_G(P)$ and so, $G = P \times H$ where $H$ is a $p$-complement of $G$.

**Lemma 3.** Let $P$ be an Abelian $p$-group, with $p$ a prime and let $K$ be a group of automorphisms of $P$ such that $|K|$ is divisible by $p$. Suppose that $C_P(x) = C_P(y)$ for all $x, y \in K - \{1\}$. Then $O_{p'}(K) = 1$.

**Proof.** Assume that $H = O_{p'}(K) > 1$ and we shall get a contradiction. Suppose first that $C_P(H) = 1$ and take some nontrivial $x \in H$. If there exists some element...
Let $G$ be a finite group such that all its Sylow subgroups are cyclic. If $r$ and $s$ are two distinct primes dividing $|G|$, then there exists a subgroup $U$ of $G$ such that $|U| = rs$.

**Proof.** We work by induction on the order of $G$. First, it is known that any finite group whose Sylow subgroups are all cyclic is solvable (see for example [6, 10.1.10]). Let $M$ be a maximal normal subgroup of $G$, so $|G : M| = p$ for some prime $p$. We can assume that $M$ is a $p'$-subgroup, otherwise we can apply the inductive hypothesis to $M$ and the result is obtained. Also, we only have to show that there exists a subgroup of order $pq$ for any prime $q \neq p$ dividing $|M|$, since the other cases are obtained by the inductive hypothesis as well. If $P$ is a Sylow $p$-subgroup of $G$, then $P$ acts coprimely on $M$, so if we fix a prime $q$ dividing $|M|$, we know (see for example [4, 14.3]) that there exists some $P$-invariant Sylow $q$-subgroup $Q$ of $G$, which is cyclic. Hence, if $x \in Q$ has order $q$, then $U = \langle x \rangle P$ has order $pq$, as required.

3. Proof of Theorem A

We shall prove by induction on the order of $G$ that either $G$ has Abelian $p$-complements or $G$ is a $\{p, q\}$-group for some prime $q \neq p$ without considering central factors. Likewise, we notice that when $G$ is solvable then the theorem is already proved by [1, Theorem A]. We shall assume then that the $p$-complements of $G$ are not Abelian and that there exist at least two prime divisors of the order of $G/Z(G)$ different from $p$, in order to get a contradiction.

As we have already pointed out in the introduction, we are also going to assume that all the centralizers of noncentral elements in $G_{p'}$ are conjugated in $G$. In the other case the theorem can be proved exactly the same as case 2 of [1, Theorem A], where the condition of $p$-solvability is not necessary. More precisely, the conjugation of the centralizers of all noncentral elements in $G_{p'}$ will be used from Step 4.

The first two steps are exactly Steps 1 and 4 of [1, Theorem A], so we shall omit their proofs.

**Step 1.** We can assume that $C_G(x) = P_x \times L_x$, with $P_x$ a Sylow $p$-subgroup of $C_G(x)$ and $L_x \leq Z(C_G(x))$, for any noncentral $x \in G_{p'}$.

**Step 2.** $C_G(x) < N_G(C_G(x))$ for every noncentral $x \in G_{p'}$. 
STEP 3. If $x \in G_{p'}$ is noncentral, then every Sylow subgroup of $N_G(C_G(x))/C_G(x)$ is cyclic or generalized quaternion. Furthermore, if $q \neq p$ is a prime divisor of the order of this group, then the Sylow $q$-subgroup has order $q$.

We fix some $x \in G_{p'}$ and write $W = N_G(C_G(x))/C_G(x)$. Let $Q$ be a Sylow $q$-subgroup of $W$ for some prime $q$ dividing $|W|$ (possibly $q = p$). By the assumptions we have made at the beginning of the proof there exists some prime $r$, divisor of $|G/Z(G)|$, distinct from $q$ and $p$. Clearly $r$ divides $|C_G(x)|$ since all these centralizers have the same size. Let $R_x$ be a Sylow $r$-subgroup of $C_G(x)$ and notice that $Q$ acts as a permutation group on $R_x$ since if $g \in Q$, then $C_{R_x}(g) = R_x \cap Z(G)$. Moreover, since this is a coprime action and $R_x$ is Abelian, we can write $R_x = [R_x, Q] \times C_{R_x}(Q)$.

Also, observe that $Q$ acts fixed-point-freely on $[R_x, Q]$, for if $t \in [R_x, Q] - \{1\}$, then $C_G(t) = C_G(x)$ by Step 1, so no element of $Q - \{1\}$ may fix $t$. Consequently, we can apply a well known result ([4, Theorem 16.12] for instance) to obtain that $Q$ is cyclic or generalized quaternion.

Assume now that $q \neq p$ and take $Q_x$ a Sylow $q$-subgroup of $C_G(x)$, which is normal by Step 1. Accordingly, $Q$ acts on $\overline{Q}_x = Q_x/Z(G)q$. If $M$ is the semidirect product defined by this action, we can take some element in $Z(M) \cap \overline{Q}_x$ which has exactly order $q$. If $t \in \overline{Q}_x$, with $t \in Q_x$ is such an element, we can construct the subgroup $T = \langle t \rangle Z(G)q \leq C_G(x)$.

Observe that $Q$ acts faithfully on $T$, that is, $C_Q(T) = 1$, since $C_G(t) = C_G(x)$ by Step 1. Furthermore, notice that $[T, Q] \subseteq Z(G)q$. We claim now that $Q$ is a $q$-elementary subgroup. Let $v \in Q$. As $t^q \in Z(G)$, then $1 = [t^q, v] = [t, v]^q$, where the last equality follows because $T$ is Abelian. Also, since $[t, v] \in Z(G)$ we have $[t, v]^q = [t, v]^q$, so we conclude that $v^q \in C_Q(T) = 1$ and thus $Q$ is elementary, as claimed. But this implies that $Q$ is cyclic of order $q$ by the above paragraph, and hence the step is proved.

STEP 4. For any $x \in G_{p'}$, we have $|N_G(C_G(x))/C_G(x)| = q$ for some fixed prime $q \neq p$.

First we are going to prove that $W = N_G(C_G(x))/C_G(x)$ is $q$-group for some prime $q$ (including the possibility $q = p$). Suppose that $|W|$ is divisible by at least two distinct primes and we shall prove that there exists a subgroup $U$ of $W$ such that $|U|$ is the product of two prime numbers. By Step 3, if every Sylow subgroup of $W$ is cyclic then there exists such subgroup $U$ by Lemma 4. We can assume then that $2$ divides $|W|$ and that the Sylow 2-subgroups of $W$ are generalized quaternion, so we can apply a classic theorem of Brauer and Suzuki (see [4, 45.1]) to obtain that $O_2(W)(\tau) \leq W$, where $\tau$ is an involution of $W$. Again by Step 3, the Sylow subgroups of $O_2(W)$ are cyclic, so if $|O_2(W)|$ is divisible by at least two distinct primes then the subgroup $U$ exists by Lemma 4 as well. So we can suppose that $O_2(W)$ is a cyclic $r$-group for some prime $r \neq 2$. Hence we can take $\alpha \in O_2(W)$ of order $r$ and we may construct the subgroup $U = \langle \alpha \rangle (\tau)$ of order $2r$. As a result, in all the cases we have a subgroup $U \leq W$ such that $|U| = rs$, for some primes $r$ and $s$, as we wanted to prove. We shall see now that this leads to a contradiction. If both primes are distinct from $p$, then either $U$ has a normal $r$-complement or has
a normal $s$-complement, and we shall assume without loss that the $r$-complement is normal. In the other case, that is, if $|U| = pr$, with $r \neq p$ then, arguing as in the first paragraph of Step 3, we get that $U$ operates as a permutation group and fixed-point-freely on $[S_r, U] − 1$, where $S_r$ is the Sylow $s$-subgroup of $C_G(x)$ for some prime $s \notin \{p, r\}$. Notice that such $s$ exists by the assumption we have made at the beginning. Furthermore, in this second case (by applying for instance [4, Lemma 16.12]) we get that $U$ is cyclic, so in particular, $U$ has nontrivial $r$-complement. Thus, in both cases, $U$ has a normal $r$-complement for some prime $r \neq p$. However, $U$ is an automorphism group of $R_x$, where $R_x$ is the Abelian Sylow $r$-subgroup of $C_G(x)$. Moreover, if $u, v \in U − \{1\}$, then $C_{R_x}(u) = C_{R_x}(v) = Z(G)_r$, so by Lemma 3, we get $O_{p'}(U) = 1$, which is a contradiction.

Take now a noncentral Sylow $r$-subgroup $R_x$ of $C_G(x)$, for some prime $r \neq p$. If $t \in R_x$ is noncentral, then by applying Step 1, we have $C_G(x) = C_G(t)$. If $w \in N_G(R_x)$, then by the same reason, $C_G(t^w) = C_G(t)$. Therefore, $C_G(x) = C_G(t^w) = C_G(x)^w$ and $w \in N_G(C_G(x))$. Thus $N_G(R_x) \leq N_G(C_G(x))$. Nevertheless, notice that if $R_x$ is not a Sylow $r$-subgroup of $G$, then $R_x \leq N_G(R_x)$, so $r$ divides $|N_G(R_x)/R_x|$, and this implies that $|W|$ is divisible by $r$, so $W$ cannot be a $p$-group. By taking into account Step 3, the step is proved.

The fact that all the centralizers are conjugated implies that we can assume for the rest of the proof that $|N_G(C_G(x))/C_G(x)| = q$, for a fixed prime $q \neq p$ and for any noncentral $x \in G_{p'}$.

**STEP 5.*** We can assume that $O_{p'}(G) = 1$ and that $|G : N_G(C_G(x))|$ is a $p$-number for any noncentral $x \in G_{p'}$.

We fix a noncentral $x \in G_{p'}$ and for any prime $r \neq p$ we take $R$ a Sylow $r$-subgroup of $G$. If $R$ is Abelian, as all the centralizers of noncentral elements in $G_{p'}$ have the same order, then the Sylow $r$-subgroup of $G$, $R_x$, is a Sylow $r$-subgroup of $G$ and $R$ is conjugated to $R_x$. Thus, $r$ does not divide $|G : N_G(C_G(x))|$. If $R$ is not Abelian, then it is an elementary fact that there exists some $t \in R − Z(R)$ such that $C_R(t) \triangleleft R$. As the centralizers of all noncentral $p$-regular elements are conjugate, we can assume without loss that $C_R(t) = C_G(x)$. In particular, $C_R(t) \leq C_G(x)$. On the other hand, is $g \in N_G(C_R(t))$, then $t^g \in C_R(t)$ and $C_R(t) = C_G(t^g)$ by Step 1. Consequently, $C_G(x) = C_G(t) = C_G(t^g) = C_G(x)^g$ and so $g \in N_G(C_G(x))$. Thus $R \leq N_G(C_R(t)) \leq N_G(C_G(x))$, and so $|G : N_G(C_G(x))|$ is an $r'$-number too. Accordingly, in both cases we have proved that $|G : N_G(C_G(x))|$ is a $p$-number.

Now we assume that $O_{p'}(G) \neq 1$ and we are going to see that $\overline{G} = G/O_{p'}(G)$ satisfies the hypotheses of the theorem. We fix some noncentral element $x \in G_{p'}$. Let $\overline{y} \in C_{G}(\overline{x})$ and notice that $[x, y] \in O_{p'}(G)$. Hence, we can write $x^y = xa$, with $a \in O_{p'}(G)$, so $x^y$ is a $p'$-element of $C_G(x)O_{p'}(G)$, and then $x^y \in L_x^t$, for some $t \in O_{p'}(G)$, where $L_x$ is the $p'$-subgroup appearing in Step 1. Therefore $x^{yt^{-1}} \in L_x$ and $C_G(x) = C_G(x^{yt^{-1}})$. As a consequence, $yt^{-1} \in N_G(C_G(x))$, so $y = wt$ with $w \in N_G(C_G(x))$. Thus, $\overline{y} = \overline{w}$ and $\overline{w} = \overline{x}\overline{w}$, that is, $[w, x] \in O_{p'}(G)$. On the other
hand, as \( w \in N_G(C_G(x)) \) and \( x \) is a \( p \)-regular element, this forces \( [w, x] \) to be a \( p \)-regular element, so \( [x, w] = 1 \). Therefore, \( C_{\overline{G}}(\overline{x}) = C_G(x) \) and we conclude that \( \overline{G} \) has two class sizes of \( p \)-regular elements. By the inductive hypothesis, either \( \overline{G} \) has an Abelian \( p \)-complement or \( \overline{G} = \overline{P} \overline{Q} \times \overline{A} \), with \( \overline{P} \in Syl_p(\overline{G}) \), \( \overline{Q} \in Syl_q(\overline{G}) \) and \( \overline{A} \leq Z(\overline{G}) \). In the first case, \( G \) has an Abelian \( p \)-complement, contradicting our first assumptions and in the second one, \( G \) is a solvable group, so the proof would be finished.

**Step 6.** \( O_r(G) \subseteq Z(G) \), for every prime \( r \neq p \).

Let \( r \) be any prime distinct from \( p \) and suppose that \( K = O_r(G) \) is noncentral. By Step 5, we have \( K \subseteq N_G(C_G(x)) \), for all \( x \in G_{p'} \). The hypothesis and Step 1 imply that there exists an Abelian noncentral normal Sylow \( s \)-subgroup of \( C_G(x) \), say \( S_x \), for some prime \( s \neq p, r \). Notice that \( S_x \) is normalized by \( K \) and thus \( [S_x, K] \subseteq S_x \cap K = 1 \), so \( K \subseteq C_G(S_x) = C_G(x) \), where the last equality follows as a consequence of Step 1. On the other hand, if \( t \in K - Z(G) \), then \( C_G(t) = C_G(x) \) again by Step 1. Moreover, if \( w \in N_G(K) \), then \( C_G(t^w) = C_G(x) \), hence \( C_G(x)^w = C_G(t)^w = C_G(t^w) = C_G(x) \). Thus, \( G = N_G(K) \subseteq N_G(C_G(x)) \) and \( C_G(x) \subseteq G \). By Step 4, we have \( |G : C_G(x)| = q \). This means that \( m = q \), so by applying Lemma 2 and Ito’s theorem on groups with two conjugacy class sizes (see for instance \([4, \text{Theorem } 33.6]\) ), we obtain \( G = P \times Q \times A \), with \( P \in Syl_p(G) \), \( Q \in Syl_q(G) \) and \( A \) Abelian, against our initial assumption.

**Step 7.** We can now derive the conclusion.

First, we notice that \( T(G)_q \neq 1 \), since any element lying in the centre of a Sylow \( q \)-subgroup of \( G \) must be central in \( G \) too because \( q \) divides \( m \) by Step 4. We write \( \overline{G} = G/Z(G)_q \) and we shall prove that \( \overline{G} \) has two \( p \)-regular conjugacy class sizes.

We can trivially assume that \( \overline{G} \) is not Abelian, otherwise \( G \) would be solvable and the proof is finished. If \( \overline{a} \in \overline{G} - Z(\overline{G}) \), we observe that \( C_{\overline{G}}(\overline{a}) \subseteq C_{\overline{G}}(\overline{a}) \). If \( C_{\overline{G}}(\overline{a}) = C_{\overline{G}}(\overline{a}) \) for all \( \overline{a} \in \overline{G} - Z(\overline{G}) \), it certainly follows that \( \overline{G} \) has two \( p \)-regular conjugacy class sizes, as we wanted. Suppose then that there is a \( p \)-regular element \( \overline{a} \in \overline{G} \) such that \( C_{\overline{G}}(\overline{a}) \neq C_{\overline{G}}(\overline{a}) \). It is easy to see that if \( \overline{w} \in C_{\overline{G}}(\overline{a}) \) then \( w \in N_G(C_G(a)) \), that is, \( C_{\overline{G}}(\overline{a}) \subseteq N_G(C_G(a)) \). As \( |N_G(C_G(a)) : C_G(a)| = q \) by Step 4, this implies that \( N_G(C_G(a)) = C_{\overline{G}}(\overline{a}) \) and so, by Step 5 we conclude that \( |G : C_{\overline{G}}(\overline{a})| \) is a \( p \)-number. Now, by a renowned theorem due to Kazarin (see for example \([4, 15.7]\) ), the subgroup \( \langle \overline{a} \overline{G} \rangle \) is a solvable normal subgroup of \( \overline{G} \). It is easy to see then that this implies that \( \langle a^G \rangle \) is a noncentral solvable normal subgroup of \( G \) too, but this is not possible in view of Steps 5 and 6.

Therefore, we have proved that \( \overline{G} \) has two \( p \)-regular conjugacy class sizes, and by induction we obtain that \( \overline{G} \) has an Abelian \( p \)-complement or \( \overline{G} = \overline{P} \overline{Q} \times \overline{A} \), with \( \overline{P} \in Syl_p(\overline{G}) \), \( \overline{Q} \in Syl_q(\overline{G}) \) and \( \overline{A} \subseteq Z(\overline{G}) \). Both cases lead to the solvability of \( G \), so the proof is finished.

The last assertions in the statement of the theorem will follow then by immediate application of Lemmas 1 and 2.
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