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On the Neumann Problem for the Schrödinger Equations with Singular Potentials in Lipschitz Domains

Xiangxing Tao and Henggeng Wang

Abstract. We consider the Neumann problem for the Schrödinger equations $-\Delta u + Vu = 0$, with singular nonnegative potentials V belonging to the reverse Hölder class $\mathcal{B}_n$, in a connected Lipschitz domain $\Omega \subset \mathbf{R}^n$. Given boundary data g in H^p or L^p for $1 - \epsilon < p \leq 2$, where $0 < \epsilon < \frac{1}{n}$, it is shown that there is a unique solution, u, that solves the Neumann problem for the given data and such that the nontangential maximal function of ∇u is in $L^p(\partial\Omega)$. Moreover, the uniform estimates are found.

1 Introduction

There has been increasing interest in boundary value problems with noncontinuous data in Lipschitz domains (see [1, 3, 5, 10]). In particular, Shen [10] considered the L^p, $1 < p \leq 2$, Neumann problem for operator $-\Delta + V$ with positive potential V belonging to the class $\mathcal{B}_\infty$. Brown [1] studied the H^p Neumann problem for Laplace operator $-\Delta$ with $1 - \epsilon < p < 1$ for some $0 < \epsilon < \frac{1}{n}$.

The purpose of this paper is to extend these results in several directions without additional assumption, and give optimal results for the solvability of the Neumann problem for Schrödinger operators in Lipschitz domains with data in L^p and H^p. We consider in this paper the Schrödinger equation $-\Delta u(X) + V(X)u(X) = 0$, where V is a nonnegative potential satisfying the reverse Hölder condition $\mathcal{B}_n$. As it is known, a nonnegative locally L^q integrable function $V(X)$ on $\mathbf{R}^n$ is said to belong to $\mathcal{B}_q$ $(1 < q \leq \infty)$ if there exists a positive constant C_q such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V(X)^q\,dX\Big)^{\frac{1}{q}} \leq \frac{C_q}{|B|}\int_B V(X)\,dX$$

holds for every ball B in $\mathbf{R}^n$ ([9]). One remarkable feature about the $\mathcal{B}_q$ class is that, if $V \in \mathcal{B}_q$ for some $q > 1$, then there exists $\epsilon > 0$, which depends only on n and the constant C_q, such that $V \in \mathcal{B}_{q+\epsilon}$. On the other hand, $\mathcal{B}_q \subset \mathcal{B}_p$ if $1 < p < q \leq \infty$.

Throughout this paper, $\Omega \subset \mathbf{R}^n$, $n \geq 3$, denotes the Lipschitz domain with connected boundary. Our results will be proven for the regions above some Lipschitz graphs, but it is not difficult to extend these results to general Lipschitz domains. For

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a point Q on the boundary $\partial\Omega$, let $\Gamma(Q)$ denote the nontangential approach region interior to Ω

$$\Gamma(Q) = \{X \in \Omega : |X - Q| < 2\,\delta(X)\}$$

and $\delta(X)$ denotes the distance from X to the boundary of Ω. For a function w which is continuous in Ω, we defined the nontangential maximal function w^* on $\partial\Omega$ by

$$w^*(Q) = \sup_{X\in\Gamma(Q)} |w(X)|.$$

One of the main results is the following

Theorem 1.1 *Let $1 < p \leq 2$ and $V \in \mathcal{B}_n$, and let $g \in L^p(\partial\Omega)$ be given. Then there exists a unique solution u to the following problem*

$$\text{(NLP)}\qquad \begin{aligned} &-\Delta u + Vu = 0, \ \textit{in}\ \Omega \\ &\frac{\partial u}{\partial v} = g, \ \textit{on}\ \partial\Omega \\ &\|(\nabla u)^*\|_{L^p(\partial\Omega)} < \infty, \end{aligned}$$

where $\partial u/\partial v = g$ on $\partial\Omega$ means that $\lim_{X\to Q, X\in\Gamma(Q)} \nabla u(X)\cdot\vec{v}(Q) = g(Q)$ for almost every $Q \in \partial\Omega$, and where $\vec{v}(Q)$ is the outward unit normal to $\partial\Omega$.

Moreover, we have the following uniform estimates

$$(1.1)\qquad \int_{\partial\Omega} |(\nabla u)^*|^p\, dQ + \int_\Omega |u|^p V^{sp} m(V,X)^{1+p-2sp}\, dX \leq C_{s,p} \int_{\partial\Omega} |g|^p\, dQ$$

with each $s \in [0,1]$, where

$$m(V,X) = \inf\{\frac{1}{r} > 0 : \frac{1}{r^{n-2}} \int_{B(X,r)} V(Y)\, dY \leq 1\}.$$

In order to pass to the H^p theory, we need to recall some definitions. Let $\Lambda(Q,r) = Z(Q,r) \cap \partial\Omega$ for $Q \in \partial\Omega$ and $r < \operatorname{diam}(\partial\Omega)$, where

$$Z(Q,r) = \{(X', X_n) : |X' - Q'| < r, |X_n - Q_n| < (1+2m)r\}$$

is the coordinate cylinder, and m is the Lipschitz character of the boundary. We say that a is an atom for $H^p(\partial\Omega)$, $\frac{n-1}{n} < p \leq 1$, if for some Q_0 and r we have $\operatorname{supp} a \subset \Lambda(Q_0, r)$, $\int_{\Lambda(Q_0,r)} a\, dQ = 0$ and $\|a\|_{L^2(\Lambda(Q_0,r))} \leq Cr^{-(n-1)(1/p-1/2)}$.

The space $H^p(\partial\Omega)$ is defined as the collection

$$\{g : g = \sum \lambda_j a_j \text{ with } \sum \lambda_j^p < \infty\}$$

for some sequence of atoms a_j, and the quasi-normal for $H^p(\partial\Omega)$ given by

$$\|g\|^p_{H^p(\partial\Omega)} = \inf\{\sum \lambda_j^p : g = \sum \lambda_j a_j\}.$$

We note that the dual space of $H^p(\partial\Omega)$, $\frac{n-1}{n} < p < 1$, is the space of Hölder continuous functions of exponent $\alpha(p) = (n-1)(1-p)/p$, $C^{\alpha(p)}(\partial\Omega)$. Thus the pairing between an element of $H^p(\partial\Omega)$ and $C^{\alpha(p)}(\partial\Omega)$ is defined. We will say $\partial u/\partial v = g$ in the H^p-sense, if for each coordinate cylinder Z and compactly supported function $\psi \in C^{\alpha(p)}(\partial\Omega \cap Z)$ we have

$$\lim_{\tau\to 0+} \int_{\partial\Omega\cap Z} \psi(Q) \frac{\partial u_\tau}{\partial v}(Q)\, dQ = \int_{\partial\Omega} \psi(Q) g(Q)\, dQ$$

where $u_\tau(X) = u(X + \tau(0,1))$.

Theorem 1.2 *Let $V \in \mathcal{B}_n$ and $1-\epsilon < p \le 1$, where $0 < \epsilon < \frac{1}{n}$ depends on the Lipschitz character. Given data $g \in H^p(\partial\Omega)$, then there exists a unique function u satisfying*

$$\text{(NHP)}\qquad \begin{aligned} -\Delta u + Vu &= 0, \text{ in } \Omega \\ \frac{\partial u}{\partial v} &= g, \text{ on } \partial\Omega \text{ in } H^p\text{-sense} \\ \|(\nabla u)^*\|_{L^p(\partial\Omega)} &< \infty, \end{aligned}$$

Moreover,

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \le C\|g\|_{H^p(\partial\Omega)}. \tag{1.2}$$

The work in this paper can be viewed as the continuation of the work in [10], where Z. Shen solve the L^p-Neumann problem (NLP) with $\mathcal{B}_\infty$ potential V and $1 < p \le 2$, and get the uniform estimate (1. 1) for $s = \frac{1+p}{2p}$. A major difference between $\mathcal{B}_\infty$ potential and $\mathcal{B}_n$ potential is the following:

$$V(X) \le Cm(V,X) \quad \text{for } V \in \mathcal{B}_\infty, \tag{1.3}$$

while the $\mathcal{B}_n$ potential V does not satisfy the above property. The property (1.3) is important in the priori estimates, see for example Lemma 1.16 and 2.6 in [10] for the Rellich estimates of solution to the L^p-Neumann problem. Instead of using this property, we establish some integral estimates for V, see Lemma 2.2, 2.3 and 3.4 below, for L^p-Neumann problem. Moreover, we get over the non-integrability of boundary data to solve the H^p-Neumann problem.

The paper is organized in following way. We consider the case $p = 2$ of Theorem 1.1 in Section 3 (Theorem 3.6), by establishing a variant of the Rellich type identity (Theorem 3.1). As preliminary, we will observe the L^2 Neumann problem for the bounded domains (Theorem 2.8) in Section 2. The extension to $1 < p < 2$ in Theorem 1.1 and the case $1 - \epsilon < p \le 1$ in Theorem 1.2 is done by two steps. We first prove the uniqueness (Theorem 4.2) in Section 4. This is accomplished by showing that for solutions u of the Schrödinger equation, we can control $\|u^*\|_{L^q(\partial\Omega)}$ for some $q > 1$ by $\|(\nabla u)^*\|_{L^p(\partial\Omega)}$, when $p > 1 - \epsilon$. Finally, We prove the existence and regularity in Section 5 by estimating the L^2-solutions with atomic data and found the uniform estimates by harmonic techniques.

The letter C always denotes a positive constant which is not necessarily the same at each occurrence, which depend at most on n, the Lipschitz character m and the constant C_n for the class $\mathcal{B}_n$. Capital letters X and Y will denote points in Ω or $\mathbf{R}^n$, while Q and P will be reserved for points on $\partial\Omega$. We always assume that $V \neq 0$ in this paper, and let $D(X,r) = B(X,r) \cap \Omega$ for $X \in \bar{\Omega}$.

2 Preliminaries

Observing that $V \in \mathcal{B}_n$ implies that $V(X)dX$ is a doubling measure, and $V \in \mathcal{B}_{n+\epsilon}$ for some $\epsilon > 0$. Let $\Psi(V,X,r) = \frac{1}{r^{n-2}} \int_{B(X,r)} V(Y)\,dY$, then, by Hölder's inequality, $\Psi(V,X,r) \leq C(\frac{r}{s})^{1+\frac{\epsilon}{n+\epsilon}} \Psi(V,X,s)$ for any $0 < r < s < \infty$. We define the auxiliary function $m(V,X)$ by

$$\frac{1}{m(V,X)} = \sup_{r>0}\{r : \Psi(V,X,r) \leq 1\},$$

which appears in [11]. We will use the following property of $m(V,X)$.

Lemma 2.1 [11] *There exist two constants $C > 0$ and $k_0 > 0$ such that*

1. $m(V,X) \sim m(V,Y)$, *if* $|X - Y| \leq Cm(V,X)^{-1}$;
2. $m(V,Y) \leq C\{1 + |X - Y|m(V,X)\}^{k_0} m(V,X)$;
3. $m(V,Y) \geq C^{-1}\{1 + |X - Y|m(V,X)\}^{k_0/(k_0+1)} m(V,X)$

for every X, Y in $\mathbf{R}^n$.

Lemma 2.2 *Let $q > s \geq 0$, $q \geq \max\{1, sn/\alpha\}$, $\alpha > 0$, and k sufficiently large, then there are positive constants k_0, C and C_k such that*

$$\int_{|X-Y|<r} \frac{V(Y)^s}{|X-Y|^{n-\alpha}}\,dY \leq Cr^{\alpha-2s}\{1 + rm(V,X)\}^{sk_0}$$

and

$$\int_{R^n} \frac{V(Y)^s\,dY}{\{1 + m(V,X)|X-Y|\}^k |X-Y|^{n-\alpha}} \leq C_k m(V,X)^{2s-\alpha}$$

for any $r > 0$, $X \in \mathbf{R}^n$ and $V \in \mathcal{B}_q$.

Proof Noting that $V \in \mathcal{B}_{q_0}$ for some $q_0 > q$, it then follows from Hölder's inequality

$$\int_{|X-Y|<r} \frac{V(Y)^s}{|X-Y|^{n-\alpha}}\,dY \leq Cr^{\alpha-2s}\Psi(V,X,r)^s.$$

Put $r_0 = m(V,X)^{-1}$, it is clear that $\Psi(V,X,r) \leq C$ if $r \leq r_0$. So we let $2^j r_0 \leq r < 2^{j+1} r_0$, $j \geq 0$, one can see from the doubling property of $V(Y)dY$ that

$$\Psi(V,X,r) \leq C^{j+1}\frac{r_0^{n-2}}{r^{n-2}} \leq C(2^{2-n}C)^j \leq C\big(rm(V,X)\big)^{k_0},$$

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where $k_0 = \log_2 C + 2 - n > 0$, thus the first formula holds.

Moreover, we have

$$\int_{R^n} \frac{V(Y)^s\,dY}{\{1+m(V,X)|X-Y|\}^k |X-Y|^{n-\alpha}} \le \int_{|X-Y|<\frac{1}{m(V,X)}} \frac{V^s\,dY}{|X-Y|^{n-\alpha}}$$
$$+ C\sum_{j=1}^{\infty} 2^{-k(j-1)} \int_{|X-Y|<\frac{2^j}{m(V,X)}} \frac{V^s dY}{|X-Y|^{n-\alpha}} \le C_k m(V,X)^{2s-\alpha}.$$

The lemma then follows. ∎

Lemma 2.3 *Let $V \in \mathcal{B}_q$ for some $q > n$. Then there exists a dimensional constant $C_n > 0$ such that for every $u \in C^1(\mathbf{R}^n)$ and $r > 0$,*

$$\int_{D(X_0,r)} V^2 u^2\,dX \le C_n \eta(X_0,r)\Big[\frac{1}{r}\int_{\partial D(X_0,r)} u^2\,dQ + \int_{D(X_0,r)} |\nabla u|^2\,dX\Big]$$

where $\eta(X_0,r) = \sup_{X\in B_r(X_0)} \Psi(V,X,r)^2/r^2$.

Proof We consider the solution ψ to the Neumann problem

$$\triangle\psi = V^2 \qquad \text{in } D(X_0,r)$$
$$\frac{\partial\psi}{\partial\upsilon} = \frac{1}{|\partial D(X_0,r)|}\int_{D(X_0,r)} V^2\,dY \qquad \text{on } \partial D(X_0,r),$$

where $|\partial D(X_0,r)|$ denotes the surface measure of $\partial D(X_0,r)$. Then, with an analogous argument as Lemma 1.1 in [4], we can get $\|\psi\|_{L^\infty(D(X_0,r))} \le C_n\eta(X_0,r)$, and then the proof follows the lines of the proof of Lemma 1.1 in [4]. ∎

Lemma 2.4 [10] *Suppose $V \in \mathcal{B}_n$, and $-\Delta u + Vu = 0$ in $Z(X_0,2r)\cap\Omega$ for some $X_0 \in \bar\Omega$ and $r > 0$. Also assume $(\nabla u)^* \in L^2\big(Z(X_0,2r)\cap\partial\Omega\big)$ and that $\partial u/\partial\upsilon = 0$ or $u = 0$ on $Z(X_0,2r)\cap\partial\Omega$. Then for each integer k there exists a constant C_k such that*

$$\sup_{X\in D(X_0,r)} |u(X)| \le \frac{C_k}{\{1+rm(V,X_0)\}^k}\Big(\frac{1}{r^n}\int_{D(X_0,2r)} |u|^2\,dY\Big)^{\frac12}.$$

One can deduce the lemma from the maximum principle and Cacciopoli's inequalities of u, see [7] and [10] for detail.

Now let $\Gamma(X,Y)$ denote the fundamental solution for the Schrödinger operator $-\Delta + V$ in $\mathbf{R}^n$ with pole at X, and $\Gamma_0(X,Y)$ be the fundamental solution for the Laplace operator. Clearly, $\Gamma(X,Y) = \Gamma(Y,X)$. Since $V \ge 0$, it is well known that $0 \le \Gamma(X,Y) \le \Gamma_0(X,Y) = \frac{1}{\omega_n(n-2)|X-Y|^{n-2}}$. Moreover, we have the following inter estimations (Theorem 1.14 in [10]), for every $k > 0$,

$$|\Gamma(X,Y)| \le \frac{C_k}{\{1+m(V,X)|X-Y|\}^k}\cdot\frac{1}{|X-Y|^{n-2}} \tag{2.1}$$

$$|\nabla\Gamma(X,Y)| \le \frac{C_k}{\{1+m(V,X)|X-Y|\}^k}\cdot\frac{1}{|X-Y|^{n-1}} \tag{2.2}$$

with the constants $C_k > 0$ independent of X and $Y \in \mathbf{R}^n$.

Lemma 2.5 *Assume $V \in \mathcal{B}_n$ and $|X - Y| \leq 2/m(V, X)$, then*

$$|\nabla_X \Gamma(X, Y) - \nabla_X \Gamma_0(X, Y)| \leq \frac{Cm(V, X)}{|X - Y|^{n-2}}$$

with the constant C independent of X and Y.

Proof Set $r = \frac{1}{2}|X - Y|$, we first note that

$$\Gamma(X, Y) - \Gamma_0(X, Y) = -\int_{\mathbf{R}^n} \Gamma_0(X, Z) V(Z) \Gamma(Z, Y)\, dZ$$

From this and the interior estimates (2.1) and (2.2), we have

$$\begin{aligned}
&|\nabla_X \Gamma(X, Y) - \nabla_X \Gamma_0(X, Y)| \\
&\quad \leq \frac{C}{r^{n-2}} \int_{|Z-X|<r} \frac{V(Z)}{|Z - X|^{n-1}}\, dZ + \frac{C}{r^{n-1}} \int_{|Z-Y|<r} \frac{V(Z)}{|Z - Y|^{n-2}}\, dZ \\
&\qquad + C \int_{\substack{|Z-X|\geq r \\ |Z-Y|\geq r}} \frac{C_k V(Z)\, dZ}{|Z - X|^{n-1}\{1 + m(V, Y)|Z - Y|\}^k |Z - Y|^{n-2}} \\
&\quad = I_1 + I_2 + I_3
\end{aligned}$$

Since $V \in \mathcal{B}_n$, a direct computation shows that $I_1 + I_2 \leq C r^{2-n} m(V, X)$.

To estimate I_3, we set $r_0 = \frac{1}{m(V,X)} \sim \frac{1}{m(V,Y)}$, and use Hölder's inequality, $\mathcal{B}_n$ condition and the doubling property to give

$$\begin{aligned}
I_3 &\leq C \int_{r \leq |Z-Y| \leq r_0} \frac{V(Z)}{|Z - Y|^{2n-3}}\, dZ + C_k \cdot r_0^k \int_{|Z-Y|\geq r_0} \frac{V(Z)}{|Z - Y|^{2n-3+k}}\, dZ \\
&\leq \frac{Cm(V, X)}{r^{n-2}} + C_k r_0^k \sum_{j=1}^{\infty} (2^j r_0)^{3-k-2n} \int_{|Z-Y|\leq 2^{j+1} r_0} V(Z)\, dZ \leq \frac{Cm(V, X)}{r^{n-2}}
\end{aligned}$$

where k sufficiently large. And so we obtain the Lemma 2.4. ■

For $f \in L^p(\partial\Omega)$, $p > 1$, we define the single layer potential

$$\mathcal{S} f(X) = \int_{\partial\Omega} \Gamma(X, Q) f(Q)\, dQ, \quad \text{for } X \in \mathbf{R}^n.$$

From estimate (2.2) and Lemma 2.5, and by using well-known techniques from the theorem of Coifman, McIntosh and Meyer [2], one can show the following lemma (see Theorem 1.18 in [10]).

Lemma 2.6 *Let $f \in L^p(\partial\Omega)$, $1 < p < \infty$ and $u = \mathcal{S}(f)$. Then $\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}$, and for $P \in \partial\Omega$,*

$$\frac{\partial u}{\partial X_i}(P) = \frac{1}{2} f(P) \upsilon_i(P) + \text{p. v.} \int_{\partial\Omega} \nabla_P \Gamma(P, Q) f(Q)\, dQ.$$

The rest of this section is devoted to solving L^2 Neumann problems for bounded connect Lipschitz domains. Firstly, by Lemma 2.6, we can write

$$\frac{\partial}{\partial \upsilon}\mathcal{S}(f)(P) = \Big(\frac{1}{2}I + K\Big)(f)(P) = \Big(\frac{1}{2}I + K_0 + K_1\Big)(f)(P),$$

where $\frac{1}{2}I+K_0$ is the boundary operator related to Laplace's equation, so it is invertible on $L^2(\partial\Omega)$ and is a Fredholm operator with index zero. Since Lemma 2.5 implies that K_1 is compact, $\frac{1}{2}I + K$ is a Fredholm operator with index zero. Moreover, it is not difficult to show that $\frac{1}{2}I+K$ is one-to-one and then invertible on $L^2(\partial\Omega)$. Hence, if Ω is a bounded Lipschitz domain and $p = 2$, the Neumann problem (NLP) is uniquely solvable. Next we denote by $N(X,Y)$ the corresponding Neumann function, and by $G(X,Y)$ the Green function.

Lemma 2.7 *Let Ω be a bounded Lipschitz domain. Assume $k > 0$ be any integer, then*

$$|N(X,Y)| + |G(X,Y)| \le \frac{C_k}{\{1 + m(V,X)|X - Y|\}^k |X - Y|^{n-2}} \tag{2.3}$$

with the constant C_k independent of X, Y and the diameter of domain Ω.

Proof Using Lemma 2.1 and 2.4, this lemma could be proved by an analogous argument as in [10]. ■

Finally in this section we give the following theorem

Theorem 2.8 *Suppose Ω is a bounded Lipschitz domain, $V \in \mathcal{B}_n$ and $g \in L^2(\partial\Omega)$, then there exists a unique solution u of the Schrödinger equation $-\Delta u + Vu = 0$ in Ω such that $\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}$, and $\partial u/\partial \upsilon = g$ a.e. on $\partial\Omega$, in the sense of non-tangential convergence. Moreover, we have the following uniform estimates*

$$\int_\Omega |\nabla u|^2 m(V,X)\,dX + \int_\Omega |u|^2 V^s m(V,X)^{3-2s}\,dX \le C_s \int_{\partial\Omega} |g|^2\,dQ$$

for each $s \in [0,2]$, with the absolute constants C_s independent of u and Ω.

Proof We just need to show the above uniform estimates. Let $g(Q) = (\partial u/\partial \upsilon)(Q)$, it follows from Green's representation formula, Hölder's inequality, the decay estimate (2.3) and Lemma 2.2 that

$$\begin{aligned}|u(X)|^2 &= \Big|\int_{\partial\Omega} N(Q,X)g(Q)\,dQ\Big|^2 \\ &\le Cm(V,X)^{-1}\int_{\partial\Omega} |N(Q,X)|\,|g(Q)|^2\,dQ,\end{aligned}$$

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and

$$\int_{\Omega} |u(X)|^2 V(X)^s m(V,X)^{3-2s}\, dX \tag{2.4}$$
$$\leq C_k \int_{\Omega} \Big(\int_{\partial\Omega} |N(Q,X)|\, |g(Q)|^2\, dQ \Big) V(X)^s m(V,X)^{2-2s}\, dX$$
$$\leq C_k \int_{\partial\Omega} |g(Q)|^2 \Big\{ \int_{\Omega} \frac{V(X)^s m(V,X)^{2-2s}\, dX}{\{1+m(V,Q)|X-Q|\}^k |X-Q|^{n-2}} \Big\}\, dQ$$
$$\leq C \int_{\partial\Omega} |g(Q)|^2\, dQ$$

To estimate the integral of $|\nabla u|^2 m(V,X)$ over the domain Ω, we use the following two inequalities (see Lemma 2.6 and 2.7 in [10])

$$\int_{\Omega} |\nabla u|^2 m(V,X)\, dX \leq C \int_{\partial\Omega} \Big| \frac{\partial u}{\partial v} \Big|\, |u| m(V,Q)\, dQ + C \int_{\Omega} |\nabla u||u| m(V,X)^2\, dX \tag{2.5}$$

and

$$\int_{\partial\Omega} |u|^2 m(V,Q)^2\, dQ \leq C \int_{\Omega} |\nabla u|\, |u| m(V,X)^2\, dX + C \int_{\Omega} |u|^2 m(V,X)^3\, dX. \tag{2.6}$$

Now we have

$$\int_{\Omega} |\nabla u|^2 m(V,X)\, dX \leq C \int_{\partial\Omega} |g|\, |u| m(V,Q)\, dQ + C \int_{\Omega} |\nabla u|\, |u| m(V,X)^2\, dX$$
$$\leq C \int_{\partial\Omega} |g|^2\, dQ + C \int_{\partial\Omega} |u|^2 m(V,Q)^2\, dQ + C \int_{\Omega} |\nabla u|\, |u| m(V,X)^2\, dX$$
$$\leq C \int_{\partial\Omega} |g|^2\, dQ + C \int_{\Omega} |\nabla u|\, |u| m(V,X)^2\, dX + C \int_{\Omega} |u|^2 m(V,X)^3\, dX$$

Thus, by Cauchy inequalities, one can get

$$\int_{\Omega} |\nabla u|^2 m(V,X)\, dX \leq C \int_{\partial\Omega} |g|^2\, dQ + C \int_{\Omega} |u|^2 m(V,X)^3\, dX.$$

This and (2.4) imply the theorem. ∎

3 L^2 Data and Rellich Estimates

From now on, we assume that Ω is an unbounded region above a Lipschitz graph, and write $\Omega = \{(X', X_n) \in R^n : X_n > \varphi(X')\}$, where φ is a Lipschitz function. We will use the following notation:

$$\Omega_R = \{(X', X_n) \in \mathbf{R}^n : |X'| < R, \varphi(X') < X_n < \varphi(X') + R\}.$$

The main result in this section is the following Rellich estimate.

Theorem 3.1 *Let $V \in \mathcal{B}_n$, and $-\Delta u + Vu = 0$ in Ω, the region above a Lipschitz graph. Also assume that $(\nabla u)^* \in L^2(\partial\Omega)$, ∇u has non-tangential limits almost everywhere on $\partial\Omega$ and $|u(X)| + |X|\,|\nabla u(X)| = O(|X|^{2-n})$ as $|X| \to \infty$. Then*

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial \upsilon}\right|^2 dQ \sim \int_{\partial\Omega} |\nabla_t u|^2\, dQ + \int_{\partial\Omega} |u|^2 m(V,Q)^2\, dQ \tag{3.1}$$

where $\nabla_t u = \nabla u - (\partial u/\partial \upsilon)\vec{\upsilon}$ denotes the tangential derivative of u on $\partial\Omega$.

Before carrying out the proof of Theorem 3.1, we give several lemmas:

Lemma 3.2 *Suppose the same conditions as in Theorem 3.1, then*

$$\int_{\Omega} |\nabla u|^2 m(V,X)\, dX + \int_{\Omega} |u|^2 V^s m(V,X)^{3-2s}\, dX \le C_s \int_{\partial\Omega} |g|^2\, dQ \tag{3.2}$$

for each $s \in [0,2]$, with the absolute constants C_s independent of u, V and Ω.

Proof Let $R > 0$ be sufficiently large, it is suffices to show the above inequality with Ω replaced by Ω_R. Noting that the Neumann function in Ω_R has the same estimates (2.3), and the constants C_k are independent of R. Therefore, we can show the above inequality along the same lines of the proof of Theorem 2.8. ■

Lemma 3.3 *Suppose the same conditions as in Theorem 3.1, then*

$$\int_{\Omega} |\nabla u|^2 m(V,X)\, dX \le C \int_{\partial\Omega} \left|\frac{\partial u}{\partial \upsilon}\right| |u| m(V,Q)\, dQ + C \int_{\Omega} |u|^2 m(V,X)^3\, dX \tag{3.3}$$

and

$$\int_{\partial\Omega} |u(Q)|^2 m(V,Q)^2\, dQ \le C \int_{\partial\Omega} \left|\frac{\partial u}{\partial \upsilon}\right|^2 dQ \tag{3.4}$$

Proof We recall the inequalities (2.5) and (2.6) in the proof of Theorem 2.8, we can see that (3.3) could be obtained from (2.5) and Cauchy's inequality, and (3.4) from (2.6) and Cauchy's inequality. ■

Lemma 3.4 *Suppose the same conditions as in Theorem 3.1, then*

$$\int_{\Omega} |u(X)|^2 \frac{V^2(X)}{m(V,X)}\, dX \le C \int_{\Omega} |\nabla u(X)|^2 m(V,X)\, dX + C \int_{\partial\Omega} |u(Q)|^2 m(V,Q)^2\, dQ + C \int_{\Omega} |u(X)|^2 m(V,X)^3\, dX.$$

Proof Let $X_0 \in \Omega$ and $r_0 = m(V, X_0)^{-1}$, and let $r = tr_0$, $1 \le t \le 2$. Using Lemma 2.3 and integrating in t, we get

$$\int_{\substack{|X-X_0|\le r_0\\ X\in\Omega}} |u|^2 V^2\, dX \le C_n \eta(X_0, 2r_0)\Big\{ \int_{\substack{|X-X_0|\le 2r_0\\ X\in\Omega}} |\nabla u|^2\, dX + \frac{1}{r_0}\int_{\substack{|Q-X_0|\le 2r_0\\ Q\in\partial\Omega}} |u|^2\, dQ + \frac{1}{r_0^2}\int_{\substack{|X-X_0|\le 2r_0\\ X\in\Omega}} |u|^2\, dX \Big\} \tag{3.5}$$

Recalling that $\eta(X_0, 2r_0) \le C r_0^{-2}$ with an absolute constant C. From this and Lemma 2.1, the inequality (3.5) may be rewritten as

$$\int_{|X-X_0|\le \frac{1}{m(V,X_0)}\, X\in\Omega} |u|^2 V^2 m^{n-1}\, dX \le C \int_{\substack{|X-X_0|\le \frac{1}{m(V,X_0)}\\ X\in\Omega}} |\nabla u|^2 m^{n+1}\, dX + C\int_{\substack{|Q-X_0|\le \frac{1}{m(V,X_0)}\\ Q\in\partial\Omega}} |u|^2 m^{n+2}\, dQ + C\int_{\substack{|X-X_0|\le \frac{1}{m(V,X_0)}\\ X\in\Omega}} |u|^2 m^{n+3}\, dX$$

with the constant C independent of X_0. Integrating both sides of the above inequality in X_0 over Ω, one can then prove the lemma by the property of $m(V, X)$ in Lemma 2.1. ∎

We are now in the position to give

The Proof of Theorem 3.1 Let $\vec{\upsilon} = (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$ be the unit normal to the boundary $\partial\Omega$, then $\upsilon_n = -1/\sqrt{1+|\nabla\varphi|^2}$. We choose $\vec{h} = (0, \dots, 0, 1)$. Then a simple computation shows that $\operatorname{div}\big(\vec{h}|\nabla u|^2 - 2(\vec{h}\nabla u)\nabla u\big) = -2(\vec{h}\nabla u)\Delta u$, and so we have the following Rellich identity

$$\int_{\partial\Omega} |\nabla u|^2 \upsilon_n\, dQ = 2\int_{\partial\Omega} \frac{\partial u}{\partial X_n}\frac{\partial u}{\partial \upsilon}\, dQ - 2\int_{\Omega} \Delta u \frac{\partial u}{\partial X_n}\, dX \tag{3.6}$$

From this identity and the Cauchy inequality follows

$$\begin{aligned}\int_{\partial\Omega} |\nabla u|^2\, dQ &\le C\int_{\partial\Omega} \Big|\frac{\partial u}{\partial \upsilon}\Big|^2\, dQ + C\int_{\Omega} |\nabla u|\,|u| V\, dX \\ &\le C\int_{\partial\Omega} \Big|\frac{\partial u}{\partial \upsilon}\Big|^2\, dQ + C\int_{\Omega} |\nabla u|^2 m(V,X)\, dX + C\int_{\Omega} |u|^2 \frac{V^2\, dX}{m(V,X)} \\ &\le C\int_{\partial\Omega} \Big|\frac{\partial u}{\partial \upsilon}\Big|^2\, dQ,\end{aligned}$$

where the last inequality is because of Lemma 3.2. This, together with (3.4) in Lemma 3.3, yields

$$\int_{\partial\Omega} |\nabla_t u|^2\, dQ + \int_{\partial\Omega} |u|^2 m(V,Q)^2\, dQ \le C\int_{\partial\Omega} \Big|\frac{\partial u}{\partial \upsilon}\Big|^2\, dQ. \tag{3.7}$$

On the other hand, noting $\nabla u = \nabla_t u + \frac{\partial u}{\partial \vec{v}} \cdot \vec{v}$, one thus see from the Rellich identity (3.6) that

$$\begin{aligned}\int_{\partial\Omega}\Big|\frac{\partial u}{\partial v}\Big|^2 dQ &\le C\int_{\partial\Omega}|\nabla_t u|^2\,dQ + C\int_\Omega |\nabla u|\,|u|V\,dX\\ &\le C\int_{\partial\Omega}|\nabla_t u|^2\,dQ + C\int_\Omega|\nabla u|^2 m(V,X)\,dX + C\int_\Omega |u|^2\frac{V^2\,dX}{m(V,X)}\\ &\le C\int_{\partial\Omega}|\nabla_t u|^2\,dQ + C\int_{\partial\Omega}\Big|\frac{\partial u}{\partial v}\Big|\,|u|m(V,Q)\,dQ\\ &\quad + C\int_\Omega |u|^2 m(V,X)^3\,dX + C\int_{\partial\Omega}|u|^2 m(V,Q)^2\,dQ\end{aligned}$$

where we have used Lemma 3.4 and the inequality (3.3) of Lemma 3.3. Thus we use Cauchy's inequality again to obtain

$$\begin{aligned}\int_{\partial\Omega}\Big|\frac{\partial u}{\partial v}\Big|^2 dQ &\le C\int_{\partial\Omega}|\nabla_t u|^2\,dQ + C\int_{\partial\Omega}|u|^2 m(V,Q)^2\,dQ\\ &\quad + C\int_\Omega |u|^2 m(V,X)^3\,dX\end{aligned} \tag{3.8}$$

Now we can deduce Theorem 3.1 from (3.7), (3.8) and the following Lemma 3.5. ∎

Lemma 3.5 *Suppose the same conditions as in Theorem 3.1. Then*

$$\int_\Omega |u(X)|^2 m(V,X)^3\,dX \le C\int_{\partial\Omega}|u(Q)|^2 m(V,Q)^2\,dQ.$$

Proof Let $f \in C_0^\infty(\Omega)$ and v be the solution to $-\nabla v + Vv = f$ in Ω and $v = 0$ on $\partial\Omega$, then

$$\Big|\int_\Omega uf\Big| = \Big|\int_{\partial\Omega} u\cdot\frac{\partial v}{\partial v}\Big| \le \Big(\int_{\partial\Omega}|u|^2 m(V,Q)^2\,dQ\Big)^{\frac12}\Big(\int_{\partial\Omega}\Big|\frac{\partial v}{\partial v}\Big|^2\frac{dQ}{m(V,Q)^2}\Big)^{\frac12}.$$

Then, by duality, it suffices to show that

$$\int_{\partial\Omega}\Big|\frac{\partial v}{\partial v}\Big|^2\frac{dQ}{m(V,Q)^2} \le C\int_\Omega \frac{|f(X)|^2}{m(V,X)^3}\,dX. \tag{3.9}$$

To show (3.9), let $Q \in \partial\Omega$ and $r = 1/m(V,Q)$, using the Rellich identity (3.6) for the function $v\eta$ on $D(Q,2r)$, where $\eta \in C_0^\infty\big(B(Q,2r)\big)$, $\eta \equiv 1$ on $B(Q,r)$ and $r|\nabla\eta| + r^2|\triangle\eta| \le C$, we obtain

$$\int_{B(Q,2r)\cap\partial\Omega}\eta^2 v_n\Big|\frac{\partial v}{\partial v}\Big|^2 dP = 2\int_{D(Q,2r)}\triangle(v\eta)\frac{\partial(v\eta)}{\partial X_n}\,dX.$$

Therefore, by Cauchy's inequality,

$$\int_{|P-Q|\le r, P\in\partial\Omega}\left|\frac{\partial v}{\partial \upsilon}\right|^2 dP \le Cr\int_{D(Q,2r)}|f(X)|^2\,dX + C\int_{D(Q,2r)}\Big(rV^2+\frac{V}{r}+\frac{1}{r^3}\Big)|v|^2\,dX + \frac{C}{r}\int_{D(Q,2r)}|\nabla v|^2\,dX.$$

By the usual trick, this gives

$$(3.10)\qquad \int_{\partial\Omega}\left|\frac{\partial v}{\partial \upsilon}\right|^2\frac{dQ}{m(V,Q)^2}\le C\int_\Omega\frac{|f(X)|^2}{m(V,X)^3}\,dX + C\int_\Omega\frac{|\nabla v|^2\,dX}{m(V,X)} + C\int_\Omega\Big(\frac{V^2}{m(V,X)^3}+\frac{V}{m(V,X)}+m(V,X)\Big)|v|^2\,dX$$

Note that $v(X)=\int_\Omega G(X,Y)f(Y)\,dY$, where $G(X,Y)$ denotes the Green function on the domain Ω, using Hölder's inequality, Fubini's theorem, decay estimates of G and Lemma 2.2, and using arguments similar to the proof of Theorem 2.8, we obtain

$$(3.11)\qquad \int_\Omega V(X)^s m(V,X)^{1-2s}|v(X)|^2\,dX\le C\int_\Omega\frac{|f(X)|^2}{m(V,X)^3}\,dX,$$

where $s\in[0,2]$. We also get, from integration by parts,

$$\int_{D(X_0,r)}|\nabla v|\,dX\le\frac{C}{r}\int_{D(X_0,2r)}|\nabla v|\,|v|\,dX + C\int_{D(X_0,2r)}|v|\,|f|\,dX,$$

where $r=m(V,X_0)$ and $X_0\in\bar\Omega$. Then

$$(3.12)\qquad \int_\Omega\frac{|\nabla v|^2}{m(V,X)}\,dX\le C\int_\Omega|v(X)|^2m(V,X)\,dX + C\int_\Omega\frac{|f(X)|^2}{m(V,X)^3}\,dX \le C\int_\Omega\frac{|f(X)|^2}{m(V,X)^3}\,dX.$$

Clearly, (3.10), (3.11) and (3.12) imply (3.9). Thus we complete the proof of the lemma. ∎

Now from Theorem 3.1 and the standard arguments (see [3] or [10]), we can obtain the following theorem.

Theorem 3.6 *Suppose Ω is a region above a Lipschitz graph, $V\in\mathcal{B}_n$ and $g\in L^2(\partial\Omega)$, then the same results of Theorem 2.8 are valid.*

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4 The Uniqueness

We always denote by $N(X,Y)$ the Neumann function related to the L^2-Neumann problem for Schrödinger equation $-\triangle u + Vu = 0$ in Ω, the region above a Lipschitz graph. It is easy to see that $N(X,Y)$ also satisfies the decay estimates, Lemma 2.7.

To prove the uniqueness, it is important to raise the integrability of the supposed solution u especially for H^p boundary data. The following lemma is crucial.

Lemma 4.1 *Suppose $-\triangle u + Vu = 0$ in Ω, $V \in \mathcal{B}_n$, and let $0 < p < n-1$ and $p^* = (n-1)p/(n-1-p)$. Then there exists a constant $C > 0$ depending only on p and the Lipschitz character of Ω such that, for any $0 < \alpha < \min\{1, p\}$,*

$$\int_{\partial\Omega\cap\{w^*(Q)\le|C_0|\}} w^*(Q)^{p^*/\alpha}\,dQ + \int_{\partial\Omega\cap\{w^*(Q)\ge|C_0|\}} w^*(Q)^{p^*}\,dQ \le C\|(\nabla w)^*\|_{L^p(\partial\Omega)}^{p^*}$$

where $w = u - C_0$, and C_0 is a constant.

Proof Without loss of generality we may assume that $(\nabla u)^* \in L^p(\partial\Omega)$. Fix $Q \in \partial\Omega$ and $X = (X', X_n) \in \Gamma(Q)$. It is clear that $(X', s) \in \Gamma(Q)$ for $s > X_n$, and that $(X', s) \in \Gamma(P)$ if $|P - Q| \le C(s - X_n)$, where C is a constant depending only on the Lipschitz character m. Thus

$$|\nabla u(X', s)| \le C(s - X_n)^{-(n-1)/\alpha}\Big(\int_{\Lambda(Q,C(s-X_n))} |(\nabla u)^*(P)|^\alpha\,dP\Big)^{1/\alpha} \tag{4.1}$$

for any $\alpha > 0$. This implies that $\lim_{s\to\infty} \nabla u(X', s) = 0$, hence $\lim_{s\to\infty} u(X', s)$ exists and is independent of X'. Then we may take a constant C_0 such that $w = u - C_0$ vanishes at infinity. Also, after replacing u by $u_\tau(X) = u\big(X + \tau(0,1)\big)$, we may assume that $w^*(Q) < \infty$ for almost every $Q \in \partial\Omega$.

On the other hand, since $B\big((X', s), C(s - X_n)\big) \subset \Omega$ for proper small constant C, we can see from the inner estimates that

$$\begin{aligned} |\nabla w(X', s)| &\le \frac{C}{s - X_n}\Big\{\frac{1}{(s - X_n)^n}\int_{B((X',s),C(s-X_n))} |u(Y)|^2\,dY\Big\}^{\frac{1}{2}} \\ &\le C\frac{w^*(Q) + |C_0|}{s - X_n} \end{aligned} \tag{4.2}$$

for any $s > X_n$. Now we can see from (4.1) and (4.2) that

$$\begin{aligned} |w(X', X_n)| &\le \int_{X_n}^\infty \Big|\frac{\partial w(X', s)}{\partial s}\Big|\,ds \\ &\le C\int_{X_n}^\infty \Big|\frac{w^*(Q) + |C_0|}{s - X_n}\Big|^{1-\alpha} \int_{\Lambda(Q,C(s-X_n))} \frac{(\nabla u)^*(P)^\alpha}{(s - X_n)^{n-1}}\,dP\,ds \\ &\le C[w^*(Q) + |C_0|]^{1-\alpha}\int_{\partial\Omega} \frac{(\nabla w)^*(P)^\alpha}{|Q - P|^{n-1-\alpha}}\,dP. \end{aligned}$$

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This follows

$$w^*(Q) \le C[w^*(Q) + |C_0|]^{1-\alpha} \int_{\partial\Omega} \frac{(\nabla w)^*(P)^\alpha}{|Q-P|^{n-1-\alpha}}\, dP \tag{4.3}$$

Noting $(\alpha/p^*) = \big(\alpha/p) - (\alpha/(n-1)\big)$ and $p/\alpha > 1$, we may obtain this lemma from (4.3) by using the fraction integral theorem (see [12]), which completes the proof. ∎

Theorem 4.2 *Let $V \in \mathcal{B}_n$, $-\triangle u + Vu = 0$ in Ω and $(\nabla u)^* \in L^p(\partial\Omega)$ for some $(n-1)/n < p < 2$. Also assume $\partial u/\partial \upsilon = 0$ non-tangentially almost everywhere on $\partial\Omega$ if $p \ge 1$, or $\partial u/\partial\upsilon$ vanishes on $\partial\Omega$ in the H^p sense if $p < 1$. Then $u \equiv 0$ in Ω.*

Proof We prove the corresponding result in the case $(n-1)/n < p < 1$. Fix any point $Y \in \Omega$, and we may assume $|Y| < r$. We let $\psi \in C_0^\infty(\mathbf{R}^n)$ be a cutoff function satisfying $\psi(X) = 1$ for $|X - Y| < r$, $\psi(X) = 0$ for $|X - Y| > 2r$, and $r|\nabla\psi| + r^2|\triangle\psi| \le C$ with the constant C independent of r. Let $w_\tau = u_\tau - C_0$, we have

$$\begin{aligned} w_\tau(Y) &= -2\int_\Omega \nabla w_\tau \nabla\psi N\, dX - \int_\Omega w_\tau \triangle\psi N\, dX \\ &\quad + \int_\Omega (V - V_\tau) w_\tau \psi N\, dX - C_0 \int_\Omega V_\tau \psi N\, dX \\ &\quad + \int_{\partial\Omega} N w_\tau \frac{\partial\psi}{\partial\upsilon}\, dQ + \int_{\partial\Omega} N\psi \frac{\partial w_\tau}{\partial\upsilon}\, dQ \\ &= K_1(Y) + K_2(Y) + K_3(Y) + K_4(Y) + K_5(Y) + K_6(Y) \end{aligned} \tag{4.4}$$

Recalling that $(\nabla u)^* \in L^p(\partial\Omega)$ and $p^* = (n-1)p/(n-1-p) > 1$, let $E(r) = \{X : r < |X - Y| < 2r\}$. In view of Lemma 4.1, we have

$$|K_5(Y)| \le \frac{C}{r^{n-1}} \int_{\partial\Omega\cap E(r)} w^*(Q)\, dQ \le \Big(\frac{C}{r^{n-1}}\Big)^{\frac{1}{p^*}} + \Big(\frac{C}{r^{n-1}}\Big)^{\frac{\alpha}{p^*}}$$

and so $K_5(Y) \to 0$ if $r \to \infty$. An analogous estimate yields $K_2(Y) \to 0$ if $r \to \infty$.

In order to estimate the term $K_1(Y)$, we note that (4.1) implies

$$|\nabla u_\tau(X)| \le C\delta(X)^{-(n-1)/p} \|(\nabla u)^*\|_{L^p(\partial\Omega)},$$

where $\delta(X)$ denotes the distance from X to $\partial\Omega$. Thus

$$\begin{aligned} |K_1(Y)| &\le \frac{C}{r^{n-1}} \int_{\Omega\cap E(r)} |\nabla u_\tau(X)|\, dX \\ &\le \frac{C}{r^{n-1}} \|(\nabla u)^*\|_{L^p(\partial\Omega)}^{1-p} \int_0^{Cr}\int_{\partial\Omega} (\nabla u)^*(Q)^p s^{-(n-1)(1-p)/p}\, dQ\, ds \\ &\le \frac{C}{r^{-1+(n-1)/p}} \|(\nabla u)^*\|_{L^p(\partial\Omega)} \to 0, \quad r \to \infty. \end{aligned}$$

We also note that $|w_\tau(X)| \le C\delta^{-(n-1)/\alpha}\|w^*\|_{L^\alpha(\partial\Omega)}$ for any $\alpha > 0$; thus we can see from Lemma 4.1 that

(4.5)
$$\int_{D(0,3r)} |w_\tau(X)|^{n/(n-1)}\,dX \le C\|w^*\|^{1/(n-1)}_{L^{p^*}(\partial\Omega)} \int_{D(0,3r)} \delta(X)^{-1/p^*}|w_\tau(X)|\,dX \le C(r),$$

with $C(r)$ independent of τ. Since $V \in \mathcal{B}_{n+\epsilon}$ for some $\epsilon > 0$, one can set $(1/\beta) = [(n-1)/n] + [1/(n+\epsilon)]$ and $(1/t) = (1/\beta) - (2/n)$, and then $t > \beta > 1$. Using fraction integral theorem and Cauchy's inequality, we get

$$\begin{aligned}\|K_3\|_{L^t(D(0,r))} &\le C\|(V - V_\tau)w_\tau\|_{L^\beta(D(0,3r))}\\ &\le C\|w_\tau\|_{L^{n/(n-1)}(D(0,3r))}\|V - V_\tau\|_{L^{n+\epsilon}(D(0,3r))}.\end{aligned}$$

From this and (4.5), we get

$$\|K_3\|_{L^t(D(0,r))} \le C\|(V - V_\tau)\|_{L^{n+\epsilon}(D(0,3r))} \to 0, \quad \tau \to 0.$$

The term K_6 vanishes as $\tau \to 0$ because the normal derivative vanishes in the H^p-sense and the fact $N(\cdot, Y) \in C^{\alpha(p)}$ for $\alpha(p) = (n-1)(1-p)/p$. Finally, by Lemma 2.2,

$$|K_4(Y)| \le C \int_{|X-Y|\le 2r} \frac{V(X)\,dX}{\{1 + |X-Y|m(V,Y)\}^k|X-Y|^{n-2}} \le C.$$

Combining all the estimates above, we obtain

$$|u(Y)| \le C_R, \quad \text{for any } r > R, \text{ and } |Y| < r,$$

with the constant C_R only dependent or R, where R is a constant sufficient large if needed. Thus, using the decayed estimates in Lemma 2.4, we get $u(Y) = 0$. The proof is complete. ■

Remark 4.3 We can use similar arguments to prove Theorem 4.2 in the case $1 \le p \le 2$.

5 The Existence and Regularity

In this section, we will give the proof of Theorem 1.1 and 1.2. We first recall the following decay Hölder estimates for Neumann function related to the L^2 Neumann problem, see [8] or [10, p. 171].

Lemma 5.1 *Let $V \in \mathcal{B}_n$, and $k > 0$ be any integer. Then there exist $0 < \alpha < 1$, and a positive constant C_k such that, for $X, Y, Z \in \bar{\Omega}$ with $|Z - X| \le \frac{1}{8+8m}|X - Y|$,*

$$|N(X,Y) - N(Z,Y)| \le \frac{C_k|Z-X|^\alpha}{\{1 + m(V,X)\}^k|X-Y|^{n-2+\alpha}}, \tag{5.1}$$

Lemma 5.2 *Given an atom a for H^p on $\partial\Omega$, and $1-\epsilon < p \le 1$ for some small $\epsilon > 0$, let $V \in \mathcal{B}_n$ and let u be the solution to $-\triangle u + Vu = 0$ in Ω such that $(\nabla u)^* \in L^2(\partial\Omega)$ and $\partial u/\partial \upsilon = a$ a.e. on $\partial\Omega$ in the sense of nontangential convergence. Then $\partial u/\partial\upsilon = a$ in the H^p-sense. Moreover*

$$\int_{\partial\Omega} |(\nabla u)^*|^p \, dQ \le C \tag{5.2}$$

with the constant C independent of the atom a.

Proof Suppose supp $a \subset \Lambda(Q_0, r_0)$ for some $Q_0 \in \partial\Omega$ and $r_0 > 0$, $\|a\|_{L^2(\partial\Omega)} \le r_0^{-(n-1)(1/p-1/2)}$. Since $\int_{\partial\Omega} a(Q)\, dQ = 0$, we can write

$$u(X) = \int_{\partial\Omega} \big(N(X,Q) - N(X,Q_0)\big)\, a(Q)\, dQ.$$

Put $r_1 = (8+8m)r_0$, then for $X \in \Omega$, $|X - Q_0| \ge r_1$, we obtain from Lemma 5.1 that

$$|u(X)| \le \frac{Cr_0^{\alpha+(n-1)/2}}{|X-Q_0|^{n-2+\alpha}} \|a\|_{L^2} \le \frac{Cr_0^{\alpha-(n-1)(1-p)/p}}{|X-Q_0|^{n-2+\alpha}} \tag{5.3}$$

Let $r \ge 8r_1$ and $\Omega_t = \Omega - Z(Q_0, tr)$ for $\frac{1}{4} \le t \le \frac{1}{2}$. Using Cauchy's inequality and the L^2-estimate in Ω_t, we get

$$\begin{aligned} I(r) &= \int_{\substack{2r \le |Q-Q_0| \le r \\ Q \in \partial\Omega}} |(\nabla u)^*(Q)|^p \, dQ \\ &\le Cr^{(n-1)(2-p)/2} \Big(\int_{\partial\Omega_t} |(\nabla u)^*|^2 \, dQ \Big)^{p/2} \\ &\le Cr^{(n-1)(2-p)/2} \Big(\int_{\Omega \cap \partial Z(Q_0, tr)} \Big| \frac{\partial u}{\partial \upsilon} \Big|^2 \, dQ \Big)^{p/2}. \end{aligned}$$

Integrating in t, from Cacciopoli's inequality and (5.3) we obtain that

$$\begin{aligned} I(r) &\le Cr^{n-1-np/2} \Big\{ \int_{\substack{r/4 \le |X-Q_0| \le r/2 \\ X \in \Omega}} |\nabla u(X)|^2 \, dX \Big\}^{p/2} \\ &\le Cr^{n-1-p-np/2} \Big\{ \int_{\substack{r/8 \le |X-Q_0| \le r \\ X \in \Omega}} |u(X)|^2 \, dX \Big\}^{p/2} \le C \Big[\frac{r_0}{r} \Big]^{\alpha p - (n-1)(1-p)}. \end{aligned}$$

Also, the L^2-estimate gives

$$\begin{aligned} I_1 &= \int_{\Lambda(Q_0, 8r_1)} |(\nabla u)^*|^p \, dQ \le Cr_0^{(n-1)(2-p)/2} \Big(\int_{\partial\Omega} |(\nabla u)^*|^2 \, dQ \Big)^{p/2} \\ &\le Cr_0^{(n-1)(2-p)/2} \Big(\int_{\partial\Omega} |a|^2 \, dQ \Big)^{p/2} \le C. \end{aligned}$$

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Since $\alpha p - (n-1)(1-p) > 0$ when we take $p > 1 - \frac{\alpha}{\alpha+n-1}$, we obtain

$$\int_{\partial\Omega} |(\nabla u)^*|^p \, dQ \le I_1 + \sum_{j=3}^{+\infty} I(2^j r_1) \le C,$$

with the constant C independent of the atom a. The lemma is proved. ∎

Finally, we turn to the proof of Theorem 1.1 and 1.2.

Proof of Theorem 1.1 The case $p = 2$ of Theorem 1.1 is treated in Theorem 3.6.

For $1 < p < 2$, the uniqueness is contained in Theorem 4.2, while the existence follows by interpolation between the L^2-case and the H^1-case which is contained in Lemma 5.2. So it is sufficient to find the uniform estimates of Theorem 1.1.

We first consider the case $p = 1$ and use the same notation as in the proof of Lemma 5.2. Analogously, for each $s \in [0,1]$, we can estimate, by Lemma 2.2,

$$\begin{aligned}
&\int_{D(Q_0,r_1)} |u(X)| V(X)^s m(V,X)^{2-2s} \, dX \\
&\qquad \le \int_{D(Q_0,r_1)} \int_{\partial\Omega} |N(Q,X)| \, |a(Q)| V(X)^s m(V,X)^{2-2s} \, dQ \, dX \\
&\qquad \le \int_{\partial\Omega} |a(Q)| \, dQ \le C
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|X-Q_0|\sim 2^j r_1} |u(X)| V(X)^s m(V,X)^{2-2s} \, dX \\
&\quad \le \int_{|X-Q_0|\sim 2^j r_1} \int_{\Lambda(Q_0,r_0)} |N(Q,X) - N(Q_0,X)| \, |a(Q)| V(X)^s m(V,X)^{2-2s} \, dQ \, dX \\
&\quad \le C_k r_0^\alpha \int_{|X-Q_0|\sim 2^j r_1} \frac{V(X)^s m(V,X)^{2-2s} \, dX}{|X-Q_0|^{n-2+\alpha} \{1 + |X-Q_0| m(V,Q_0)\}^k} \\
&\quad \le C 2^{-j\alpha}.
\end{aligned}$$

Therefore there exists an absolute constant C such that

$$\int_\Omega |u(X)| V(X)^s m(V,X)^{2-2s} \, dX \le C_s, \tag{5.4}$$

for the solution function u related to the boundary data a, any atom for H^1. On the other hand, recalling Theorem 3.6, we have

$$\int_\Omega |u(X)|^2 V(X)^s m(V,X)^{3-2s} \, dX \le C_s \int_{\partial\Omega} |g(Q)|^2 \, dQ, \tag{5.5}$$

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for any boundary data $g \in L^2(\partial\Omega)$, where $s \in [0,2]$. Now combining (5.4) with (5.5), we can get the desired results by interpolation theorem. Theorem 1.1 is obtained. ■

The Proof of Theorem 1.2 Theorem 1.2 can be regarded as a corollary of Theorem 4.2 and Lemma 5.2. ■

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