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Effective equidistribution of horospherical flows in infinite volume rank-one homogeneous spaces

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Abstract. We prove effective equidistribution of horospherical flows in $SO(n, 1)^{\circ}/\Gamma$ when Γ is geometrically finite and the frame flow is exponentially mixing for the Bowen–Margulis–Sullivan measure. We also discuss settings in which such an exponential mixing result is known to hold. As part of the proof, we show that the Patterson–Sullivan measure satisfies some friendly like properties when Γ is geometrically finite.

Key words: hyperbolic surface, orbit closure, unipotent flows
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Contents

1. *Introduction*

The group $G = \mathrm{SO}(n, 1)^\circ$ with $n \geq 2$ can be considered as the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^n$. Let $\Gamma \subseteq G$ be a geometrically finite and Zariski dense subgroup of G with infinite covolume. In this paper, we establish an effective rate of equidistribution of orbits under the action of a horospherical subgroup $U \subseteq G$ under a certain exponential mixing assumption (Assumption 1.1).

An early result on the equidistribution of horocyclic flows in G/Γ for $G = \mathrm{SL}_2(\mathbb{R})$ and Γ a lattice was obtained by Dani and Smillie in [4]. They proved that if $U = \{(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) : t \in \mathbb{R}\}$ and if x does not have a closed U-orbit in G/Γ, then for every $f \in C_c(X)$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_t x)\, dt = m(f), \tag{1}$$

where m denotes the normalized Haar probability measure on X. The lattice case is well understood in general, thanks to Ratner's celebrated theorems on unipotent flows [28].

Results such as these are not considered to be *effective*, because they do not address the rate of convergence, and this is important in many applications. Burger proved effective equidistribution of horocyclic flows for $\mathrm{SL}_2(\mathbb{R})/\Gamma$ when Γ is a uniform lattice or convex cocompact with critical exponent at least $1/2$ in [3]. Sarnak proved effective equidistribution of translates of closed horocycles when Γ is a non-uniform lattice in [31]. More general results were obtained for non-uniform lattices using representation theoretic methods by Flaminio and Forni in [9], and also by Strömbergsson in [36]. The case where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ was also obtained independently by Sarnak and Ubis in [32]. The higher dimensional setting has recently been considered by Katz [15] and McAdam [21]. McAdam proved equidistribution of abelian horospherical flows in $\mathrm{SL}_n(\mathbb{R})/\Gamma$ for $n \geq 3$ when Γ is a cocompact lattice or $\mathrm{SL}_n(\mathbb{Z})$, and Katz proved equidistribution in greater generality when Γ is a lattice in a semi-simple linear group without compact factors.

In infinite volume, we cannot hope for a result such as (1) for the Haar measure: by the Hopf ratio ergodic theorem, for almost every point,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_t x)\, dt = 0.$$

This tells us that this is not the correct measure to consider. A key characteristic of the Haar probability measure in the lattice case is that it is the unique U-invariant ergodic Radon measure that is not supported on a closed U orbit, [4, 10]. By [3, 29, 40], the measure with this property in the infinite volume setting is the *Burger–Roblin* (BR) measure, which is defined fully in §3. The correct normalization will be given by the *Patterson–Sullivan* (PS) measure, which is a geometrically defined measure on U orbits. This is also defined in §3.

Maucourant and Schapira proved equidistribution of horocycle flows on geometrically finite quotients of $\mathrm{SL}_2(\mathbb{R})$ in [20], and in [24], Mohammadi and Oh generalize these results to geometrically finite quotients of $\mathrm{SO}(n, 1)^\circ$ for $n \geq 2$, but these results are not effective. Oh and Shah also proved equidistribution on the unit tangent bundle of geometrically finite

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hyperbolic manifolds in [26]. In [8], Edwards proves effective results for geometrically finite quotients of $\mathrm{SL}_2(\mathbb{R})$.

In this paper, we extend these results to geometrically finite quotients of $\mathrm{SO}(n,1)^\circ$, under the assumption of exponential mixing of the frame flow for the *Bowen–Margulis–Sullivan* (BMS) measure, which is defined in §3. More explicitly, in §§6–8 (but not §4 or §5), we will assume the following holds, where $\{a_s : s \in \mathbb{R}\}$ denotes the frame flow on G/Γ.

Assumption 1.1. (Exponential mixing) There exist $c, \kappa > 0$ and $\ell \in \mathbb{N}$ which depend only on Γ, such that for $\psi, \varphi \in C_c^\infty(G/\Gamma)$ and $s > 0$,

$$\left| \int_X \psi(a_s x)\varphi(x)\, dm^{\mathrm{BMS}}(x) - m^{\mathrm{BMS}}(\psi)m^{\mathrm{BMS}}(\varphi) \right| < c S_\ell(\psi) S_\ell(\varphi) e^{-\kappa s}.$$

Assumption 1.1 is known to hold when Γ is convex cocompact by [30]. In [23], Mohammadi and Oh prove such a result for geometrically finite Γ under a spectral gap assumption (see Definition 2.1), using decay of matrix coefficients. Edwards and Oh recently proved effective mixing for the geodesic flow on the unit tangent bundle of a geometrically finite hyperbolic manifold when the critical exponent is larger than $(n-1)/2$ in [7]. Further details about this assumption are discussed in §2.

We will need to restrict consideration to points satisfying the following geometric property, which means that the point does not travel into a cusp ‘too fast’. Here, d is a left-invariant Riemannian metric on G/Γ that projects to the hyperbolic distance on $\mathbb{H}^n$.

Definition 1.2. For $0 < \varepsilon < 1$ and $s_0 \geq 1$, we say that $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$ is (ε, s_0)-*Diophantine* if for all $s \geq s_0$,

$$d(\mathcal{C}_0, a_{-s}x) < (1-\varepsilon)s,$$

where $\mathcal{C}_0$ is a compact set arising from the thick-thin decomposition, and is fully defined in §3.2. We say that $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$ is *Diophantine* if x is (ε, s_0)-Diophantine for some ε and s_0.

Here, $\Lambda(\Gamma)$ denotes the set of limit points of Γ, and is defined fully in §3, as is the notation $x^\pm$. In the case that Γ is a lattice, the condition $x^- \in \Lambda(\Gamma)$ is always satisfied. Also, if Γ is convex cocompact, every point $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$ will be Diophantine, because all limit points are *radial* in this case (see §3).

Note that x is (ε, s_0)-Diophantine if $(1-\varepsilon)s$ is a bound on the asymptotic excursion rate of the geodesic $\{a_{-s}x\}$, that is,

$$\limsup_{s\to\infty} \frac{d(\mathcal{C}_0, a_{-s}x)}{s} \leq 1 - \varepsilon. \tag{2}$$

Sullivan’s logarithm law for geodesics when Γ is geometrically finite with $\delta_\Gamma > (n-1)/2$ was shown in [16, 35] (and is a strengthening of Sullivan’s logarithm law for

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non-compact lattices [38, §9]), and implies that for almost all $x \in G/\Gamma$,

$$\limsup_{s\to\infty} \frac{d(\mathcal{C}_0, a_{-s}x)}{\log s} = \frac{1}{2\delta_\Gamma - k}, \tag{3}$$

where k is the maximal cusp rank. In [16], Kelmer and Oh showed a strengthening of the above, considering excursion to individual cusps and obtaining a limit for the shrinking target problem of the geodesic flow. Note also that the result stated in [16] is for $x \in \mathrm{T}^1(G/\Gamma)$, but since the distance function there is assumed to be K-invariant, where $\mathbb{H}^n = K\backslash G$, and the set $\mathcal{C}_0$ is K-invariant as well (see §3.2), we can deduce the form above.

It follows from (3) that the limit on the left-hand side of (2) is zero for almost every point $x \in G/\Gamma$ (with respect to the invariant volume measure) in this case. Moreover, for any ε, the Hausdorff dimension of the set of directions in $\mathrm{T}^1(\mathbb{H}^n/\Gamma)$ around a fixed point in $\mathbb{H}^n/\Gamma$ that do not satisfy (2) is computed in [22, Theorem 1]. For geometrically finite Γ, the Hausdorff dimension of the set of directions around a fixed point that do not satisfy (2) can be found in [11, 35].

The main goal of this paper is to establish the following two theorems. Here, m^{BR} denotes the BR measure, m^{BMS} denotes the BMS measure, and μ^{PS} denotes the PS measure. These measures are defined in §3. Throughout the paper, the notation

$$x \ll y$$

means there exists a constant c such that

$$x \le cy.$$

If a subscript is denoted, e.g. $\ll_\Gamma$, this explicitly indicates that this constant depends on Γ.

Let $U = \{u_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^{n-1}\}$ denote the expanding horospherical flow. Let $B_U(r)$ denote the ball in U of radius r with the max norm on $\mathbb{R}^{n-1}$. See §3 for more details on notation.

THEOREM 1.3. *Assume that Γ satisfies Assumption 1.1. For any $0 < \varepsilon < 1$ and $s_0 \ge 1$, there exist constants $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying for every $\psi \in C_c^\infty(G/\Gamma)$, there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that for every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine, and for every $r \gg_{\Gamma,\varepsilon} s_0$,*

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(r))} \int_{B_U(r)} \psi(u_{\boldsymbol{t}}x)\, d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - m^{\mathrm{BMS}}(\psi) \right| \le c S_\ell(\psi) r^{-\kappa},$$

where $S_\ell(\psi)$ is the ℓ-Sobolev norm.

For the Haar measure, we will prove the following equidistribution result.

THEOREM 1.4. *Assume that Γ satisfies Assumption 1.1. For any $0 < \varepsilon < 1$ and $s_0 \ge 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying for every $\psi \in C_c^\infty(G/\Gamma)$, there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that for every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine, and for all $r \gg_{\Gamma,\operatorname{supp}\psi,\varepsilon} s_0$,*

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(r))} \int_{B_U(r)} \psi(u_{\boldsymbol{t}}x)\, d\boldsymbol{t} - m^{\mathrm{BR}}(\psi) \right| \le c S_\ell(\psi) r^{-\kappa},$$

where $S_\ell(\psi)$ is the ℓ-Sobolev norm.

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Note that the assumption that x is Diophantine is required to obtain quantitative non-divergence results in §4, which is key in proving the above theorems. The dependence on a Diophantine condition is necessary, and is analogous to known effective equidistribution results for when Γ is a non-cocompact lattice (see [**21**, **36**]).

In [**39**], we apply the above result to obtain a quantitative ratio theorem for the distribution of orbits for Γ acting on $U\backslash G$. This improves upon the work of Maucourant and Schapira in [**20**].

A key step toward proving Theorem 1.3 is the following, which is proved in §6.

THEOREM 1.5. *Assume that* Γ *satisfies Assumption 1.1. There exist* $\kappa = \kappa(\Gamma)$ *and* $\ell = \ell(\Gamma)$ *which satisfy the following: for any* $\psi \in C_c^\infty(X)$*, there exists* $c = c(\Gamma, \operatorname{supp}\psi) > 0$ *such that for any* $f \in C_c^\infty(B_U(r))$*,* $0 < r < 1$*,* $x \in \operatorname{supp} m^{\mathrm{BMS}}$*, and* $s \gg_\Gamma d(\mathcal{C}_0, x)$*, we have*

$$\left| \int_U \psi(a_s u_{\boldsymbol{t}} x) f(\boldsymbol{t})\, d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BMS}}(\psi) \right| < c S_\ell(\psi) S_\ell(f) e^{-\kappa s}.$$

In §6, we also prove an analogous statement for the Haar measure. Such a result is proven in [**23**] under a spectral gap assumption on Γ, but we show in this paper how to prove it whenever the frame flow is exponentially mixing.

The proof will use similar techniques as in [**24**, **26**]; in particular, we will rely on Margulis' 'thickening trick' from his thesis [**19**].

In the proofs of our main theorems (Theorems 1.3 and 1.4), we use partition of unity arguments. In particular, the bounds we get are on slightly bigger sets. As a result, we need an effective bound on the PS measure of a small neighborhood of a boundary of a ball relative to the PS measure of that ball. The following theorem achieves this. It is shown using [**5**, Lemma 3.8] and [**35**, Theorem 2].

THEOREM 1.6. *There exists a constant* $\alpha = \alpha(\Gamma) > 0$*, such that for every* $x \in G/\Gamma$ *that is* (ε, s_0)*-Diophantine, for every* $0 < s \leq T^{\varepsilon/(1-\varepsilon)}$*, every* $0 < \xi \ll_\Gamma 1$*, and every* $T \gg_{\Gamma,\varepsilon} s_0$*,*

$$\frac{\mu_{a_{-s}x}^{\mathrm{PS}}(B_U(\xi + T))}{\mu_{a_{-s}x}^{\mathrm{PS}}(B_U(T))} - 1 \ll_\Gamma \xi^\alpha.$$

In the appendix, a stronger version is obtained under the assumption that all cusps of G/Γ have maximal rank.

This paper is organized as follows. In §2, we discuss under what conditions Assumption 1.1 is known to hold. In §3, we set out notation used in the article, and define the measures we will be using, along with proving some important facts about them. In §4, we prove quantitative non-divergence of horospherical orbits of Diophantine points, which is needed in the following sections. In §5, we control the PS measure of the boundary of a set by proving Theorem 1.6. In §6, we use Margulis' 'thickening trick' to prove Theorem 1.5 and an analogous result for the Haar measure, which are key in the proofs of Theorems 1.3 and 1.4. In §7, we use quantitative non-divergence and Theorem 1.5 to prove Theorem 1.3. In §8, we use Theorem 1.6 and the Haar measure analogue of Theorem 1.5 to prove Theorem 1.4. Several technical details of the proof of Theorem 1.6 are in the appendix, §A, and in §A.3, we prove stronger statements hold in the setting

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that all cusps have maximal rank, because the PS measure is absolutely friendly (see [5, Theorem 1.9]).

2. *Known exponential mixing results*

Throughout the paper, we assume the existence of an exponential mixing result (see Assumption 1.1). In this section, we elaborate on the conditions under which such a result is known. Here we assume that Γ is a Zariski dense discrete subgroup of G.

There is a natural action of G on $\mathbb{H}^n$ and $\partial\mathbb{H}^n$, the hyperbolic n-space and its boundary, respectively. Let $\Lambda(\Gamma) \subseteq \partial(\mathbb{H}^n)$ denote the limit set of X, that is, the set of all accumulation points of Γz for some $z \in \mathbb{H}^n \cup \partial(\mathbb{H}^n)$. The *convex core* of X is the image in X of the minimal convex subset of $\mathbb{H}^n$ which contains all geodesics connecting any two points in $\Lambda(\Gamma)$. We say that Γ is *convex cocompact* if the convex core of $\mathbb{H}^n/\Gamma$ is compact, and *geometrically finite* if a unit neighborhood of the convex core of Γ has finite volume.

For Γ convex cocompact, Assumption 1.1 was proved by Sarkar and Winter in [30, Theorem 1.1].

Fix a point $w_o \in \mathrm{T}^1(\mathbb{H}^n)$ and denote $M = \mathrm{Stab}_G(w_o)$. Denote by $\hat{G}$ and $\hat{M}$ the unitary dual of G and M respectively. A representation $(\pi, \mathcal{H}) \in \hat{G}$ is called *tempered* if for any K-finite $v \in \mathcal{H}$, the associated matrix coefficient function $g \mapsto \langle \pi(g)v, v\rangle$ belongs to $L^{2+\varepsilon}(G)$ for any $\varepsilon > 0$, and *non-tempered* otherwise. The non-tempered part of $\hat{G}$ consists of the trivial representation, and complementary series representations $\mathcal{U}(v, s - n + 1)$ parameterized by $v \in \hat{M}$ and $s \in I_v$, where $I_v \subseteq ((n-1)/2, n-1)$ is an interval depending on v (see [12]).

Definition 2.1. The space $L^2(X)$ has a *spectral gap* if there exist $(n-1)/2 < s_0 = s_0(\Gamma) < \delta$ and $n_0 = n_0(\Gamma) \in \mathbb{N}$ such that:

(1) the multiplicity of $\mathcal{U}(v, \delta_\Gamma - n + 1)$ contained in $L^2(X)$ is at most $dim(v)^{n_0}$ for any $v \in \hat{M}$;

(2) $L^2(X)$ does not weakly contain any $\mathcal{U}(v, s - n + 1)$ with $s \in (s_0, \delta)$ and $v \in \hat{M}$.

According to [23, Theorem 3.27], if $\delta_\Gamma > (n-1)/2$ for $n = 2, 3$, or if $\delta_\Gamma > n - 2$ for $n \geq 4$, then $L^2(X)$ has a spectral gap. If $\delta_\Gamma \leq (n-1)/2$, then there is no spectral gap, but it was conjectured that whenever $\delta_\Gamma > (n-1)/2$, $L^2(X)$ has a spectral gap (see [23]). Note that if there are cusps of maximal rank $n-1$, it follows that $\delta_\Gamma > (n-1)/2$.

For Γ geometrically finite such that $L^2(X)$ has a spectral gap and $\delta_\Gamma > (n-1)/2$, Mohammadi and Oh stated in [23, Theorem 1.6] an exponential mixing result similar to Assumption 1.1. In their statement, the constant c depends on Γ and the support of the functions. The dependence on the support of the functions arises in the last part of the proof (see [23, §6.3]) and can be omitted by using the following lemma (the BR-measure is defined in §3.3), hence obtaining a result of the form needed in Assumption 1.1.

LEMMA 2.2. *If $\delta > (n-1)/2$, then there exists $c = c(\Gamma) > 0$ such that any $B \subset X$ of diameter smaller than 1 satisfies*

$$m^{\mathrm{BR}}(B) \leq c.$$

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Proof. For any $g \in G$ denote

$$\Phi_0(g) = |\nu_{g(o)}|,$$

where o is the projection of w_o onto $\mathbb{H}^n$ and for any $x \in \mathbb{H}^n$, ν_x is the Patterson–Sullivan density defined in §3.1. Since Φ_0 is Γ-invariant, it can be considered as a smooth function on X. Moreover, by assuming B contains $K = \mathrm{Stab}_G(o)$ and using the Cauchy–Schwartz inequality, we get

$$\begin{aligned} m^{\mathrm{BR}}(B) &= \int_B \Phi_0(g)\, dm^{\mathrm{Haar}}(g) \\ &\le \sqrt{dm^{\mathrm{Haar}}(B)}\, \|\Phi_0\|_2 \\ &\ll \|\Phi_0\|_2. \end{aligned}$$

According to [37, §7] and by the assumption $\delta > (n-1)/2$, we have that $\phi_0 \in L^2(X)$. □

3. *Notation and preliminaries*

Recall from §1 that $G = \mathrm{SO}(n,1)^\circ$ and $\Gamma \subseteq G$ is a geometrically finite Kleinian subgroup of G. Denote

$$X := G/\Gamma.$$

Here, G acts transitively on $\mathbb{H}^n$, the hyperbolic n-space. Fix a reference point $o \in \mathbb{H}^n$ and let $K = \mathrm{Stab}_G(o)$, then $K\backslash G = \mathbb{H}^n$. Let $\pi : G \to \mathbb{H}^n$ be the projection

$$\pi(g) = g(o). \tag{4}$$

We will abuse notation and also write π for the induced map from G/Γ to $\mathbb{H}^n/\Gamma$. For convenience, we will assume throughout the paper that we have chosen o so that $o\Gamma \in \pi(\mathcal{C}_0)$, where $\mathcal{C}_0$ is defined in §3.2. This says that $o\Gamma$ is in the convex core of $\mathbb{H}^n/\Gamma$.

Let d denote the left G-invariant metric on G which induces the hyperbolic metric on $K\backslash G = \mathbb{H}^n$.

Recall from §2 that $\Lambda(\Gamma) \subseteq \partial(\mathbb{H}^n)$ denotes the limit set of X. We denote the Hausdorff dimension of $\Lambda(\Gamma)$ by δ_Γ. It is equal to the critical exponent of Γ (see [27]).

We say that a limit point $\xi \in \Lambda(\Gamma)$ is *radial* if there exists a compact subset of X so that some (and hence every) geodesic ray toward ξ has accumulation points in that set. An element $g \in G$ is called *parabolic* if the set of fixed points of g in $\partial(\mathbb{H}^n)$ is a singleton. We say that a limit point is *parabolic* if it is fixed by a parabolic element of Γ. A parabolic limit point $\xi \in \Lambda(\Gamma)$ is called *bounded* if the stabilizer Γ_ξ acts cocompactly on $\Lambda(\Gamma) - \{\xi\}$.

We denote by $\Lambda_r(\Gamma)$ and $\Lambda_{bp}(\Gamma)$ the set of all radial limit points and the set of all bounded parabolic limit points respectively. Since Γ is geometrically finite (see [2]),

$$\Lambda(\Gamma) = \Lambda_r(\Gamma) \cup \Lambda_{bp}(\Gamma).$$

Fix $w_o \in \mathrm{T}^1(\mathbb{H}^n)$ and let $M = \mathrm{Stab}_G(w_o)$ so that $\mathrm{T}^1(\mathbb{H}^n)$ may be identified with $M\backslash G$. For $w \in \mathrm{T}^1(\mathbb{H}^n)$,

$$w^\pm \in \partial\mathbb{H}^n$$

denotes the forward and backward endpoints of the geodesic w determined. For $g \in G$, we define

$$g^{\pm} := w_o^{\pm} g.$$

Without loss of generality, we may assume that $w_o^{\pm} \in \Lambda(\Gamma)$, and hence every $\gamma \in \Gamma$ will satisfy $\gamma^{\pm} \in \Lambda(\Gamma)$.

Let $A = \{a_s : s \in \mathbb{R}\}$ be a one parameter diagonalizable subgroup such that M and A commute, and such that the right action on $M\backslash G = \mathrm{T}^1(\mathbb{H}^n)$ corresponds to unit speed geodesic flow. We parameterize A by $A = \{a_s : s \in \mathbb{R}\}$, where

$$a_s = \begin{pmatrix} e^s & & \\ & I & \\ & & e^{-s} \end{pmatrix} \tag{5}$$

and I denotes the $(n-1) \times (n-1)$ identity matrix.

Let U denote the expanding horospherical subgroup

$$U = \{g \in G : a_{-s} g a_s \to e \text{ as } s \to +\infty\},$$

let $\tilde{U}$ be the contracting horospherical subgroup

$$\tilde{U} = \{g \in G : a_s g a_{-s} \to e \text{ as } s \to +\infty\},$$

and let $P = MA\tilde{U}$ be the parabolic subgroup.

The group U is a connected abelian group, isomorphic to $\mathbb{R}^{n-1}$. We may use the parameterization $\mathbf{t} \mapsto u_{\mathbf{t}}$ so that for any $s \in \mathbb{R}$,

$$a_s u_{\mathbf{t}} a_{-s} = u_{e^s \mathbf{t}}. \tag{6}$$

Similarly, we parameterize $\tilde{U}$ by $\mathbf{t} \mapsto v_{\mathbf{t}} \in \tilde{U}$ so that for $s \in \mathbb{R}$,

$$a_s v_{\mathbf{t}} a_{-s} = v_{e^{-s} \mathbf{t}}. \tag{7}$$

More explicitly, if $\mathbf{t} \in \mathbb{R}^{n-1}$ is viewed as a row vector,

$$u_{\mathbf{t}} = \begin{pmatrix} 1 & \mathbf{t} & \frac{1}{2}\|\mathbf{t}\|^2 \\ & I & \mathbf{t}^T \\ & & 1 \end{pmatrix} \tag{8}$$

and

$$v_{\mathbf{t}} = \begin{pmatrix} 1 & & \\ \mathbf{t}^T & I & \\ \frac{1}{2}\|\mathbf{t}\|^2 & \mathbf{t} & 1 \end{pmatrix}.$$

For a subset H of G and $\eta > 0$, H_η denotes the closed η-neighborhood of e in H, that is,

$$H_\eta = \{h \in H : d(h, e) \le \eta\}.$$

For any $r > 0$, let

$$B_U(r) = \{u_{\mathbf{t}} : \|\mathbf{t}\| \le r\} \quad \text{and} \quad B_{\tilde{U}}(r) = \{v_{\mathbf{t}} : \|\mathbf{t}\| \le r\},$$

where $\|\mathbf{t}\|$ is the sup-norm of $\mathbf{t} \in \mathbb{R}^{n-1}$.

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LEMMA 3.1. *For $0 < \eta < 1/4$ and $p \in P_\eta$, there exists $\rho_p : B_U(1) \to B_U(1 + O(\eta))$ that is a diffeomorphism onto its image and a constant $D = D(\eta) < 3\eta$ such that*

$$u_{\boldsymbol{t}} p^{-1} \in P_D u_{\rho_p(\boldsymbol{t})}.$$

Explicitly, if $p = a_s v_{\boldsymbol{r}}$, then $\rho_p(\boldsymbol{t}) = (\boldsymbol{t} - \frac{1}{2}\|\boldsymbol{t}\|^2 \boldsymbol{r})/e^s(1 - (\boldsymbol{t} \cdot \boldsymbol{r}) + \frac{1}{4}\|\boldsymbol{r}\|^2\|\boldsymbol{t}\|^2)$.

Proof. For $s \in \mathbb{R}$ and $\mathbf{r} \in \mathbb{R}^{n-1}$, let $p = a_s v_{\mathbf{r}}$. Then

$$p^{-1} = \begin{pmatrix} e^{-s} & & \\ -e^{-s}\mathbf{r}^T & I & \\ \frac{1}{2}e^{-s}\|\mathbf{r}\|^2 & -\mathbf{r} & e^s \end{pmatrix},$$

so

$$u_{\mathbf{t}} p^{-1} = \begin{pmatrix} e^{-s}(1 - (\mathbf{t}\cdot\mathbf{r}) + \frac{1}{4}\|\mathbf{r}\|^2\|\mathbf{t}\|^2) & \mathbf{t} - \frac{1}{2}\|\mathbf{t}\|^2\mathbf{r} & \frac{1}{2}e^s\|\mathbf{t}\|^2 \\ -e^{-s}\mathbf{r}^T + \frac{1}{2}e^{-s}\|\mathbf{r}\|^2\mathbf{t}^T & I - \mathbf{t}^T\mathbf{r} & e^s\mathbf{t}^T \\ \frac{1}{2}e^{-s}\|\mathbf{r}\|^2 & -\mathbf{r} & e^s \end{pmatrix}.$$

Now, if $p' = a_{s'} v_{\mathbf{r}'}$, we obtain that

$$p' u_{\mathbf{t}'} = \begin{pmatrix} e^{s'} & e^{s'}\mathbf{t}' & \frac{1}{2}e^{s'}\|\mathbf{t}'\|^2 \\ \mathbf{r}'^T & \mathbf{r}'^T\mathbf{t}' + I & \frac{1}{2}\|\mathbf{t}'\|^2\mathbf{r}'^T + \mathbf{t}'^T \\ \frac{1}{2}e^{-s'}\|\mathbf{r}'\|^2 & \frac{1}{2}e^{-s'}\|\mathbf{r}'\|^2\mathbf{t}' + e^{-s'}\mathbf{r}' & e^{-s'}(\frac{1}{4}\|\mathbf{r}'\|^2\|\mathbf{t}'\|^2 + (\mathbf{r}'\cdot\mathbf{t}') + 1) \end{pmatrix}.$$

We wish to solve for $\mathbf{t}'$.

Setting entries equal yields

$$\mathbf{t} - \tfrac{1}{2}\|\mathbf{t}\|^2\mathbf{r} = e^{s'}\mathbf{t}'$$

and

$$e^{s'} = e^{-s}\big(1 - (\mathbf{t}\cdot\mathbf{r}) + \tfrac{1}{4}\|\mathbf{r}\|^2\|\mathbf{t}\|^2\big). \tag{9}$$

Combining these implies that

$$\mathbf{t}' = \frac{\mathbf{t} - 1/2\|\mathbf{t}\|^2\mathbf{r}}{e^s(1 - (\mathbf{t}\cdot\mathbf{r}) + 1/4\|\mathbf{r}\|^2\|\mathbf{t}\|^2)}.$$

We define $\rho_p(\mathbf{t})$ to be this quantity. One can directly check that it satisfies the claim. □

3.1. *Patterson–Sullivan and Lebesgue measures.* For $x, y \in \mathbb{H}^n$ and $\xi \in \partial(\mathbb{H}^n)$, the *Busemann function* is given by

$$\beta_\xi(x, y) := \lim_{t\to\infty} d(x, \xi_t) - d(y, \xi_t),$$

where ξ_t is a geodesic ray towards ξ.

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A family of finite measures $\{\mu_x : x \in \mathbb{H}^n\}$ on $\partial(\mathbb{H}^n)$ is called a Γ*-invariant conformal density of dimension* $\delta_\mu > 0$ if for every $x, y \in \mathbb{H}^n$, $\xi \in \partial(\mathbb{H}^n)$, and $\gamma \in \Gamma$,

$$\gamma_* \mu_x = \mu_{x\gamma} \quad \text{and} \quad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_\mu \beta_\xi(y,x)}, \tag{10}$$

where $\gamma_*\mu_x(F) = \mu_x(F\gamma)$ for any Borel subset F of $\partial(\mathbb{H}^n)$.

We let $\{\nu_x\}_{x\in\mathbb{H}^n}$ denote the Patterson–Sullivan density on $\partial\mathbb{H}^n$, that is, the unique (up to scalar multiplication) conformal density of dimension δ_Γ.

For each $x \in \mathbb{H}^n$, we denote by m_x the unique probability measure on $\partial(\mathbb{H}^n)$ which is invariant under the compact subgroup $\mathrm{Stab}_G(x)$. Then $\{m_x : x \in \mathbb{H}^n\}$ forms a G-invariant conformal density of dimension $n-1$, called the Lebesgue density. Fix $o \in \mathbb{H}^n$.

For $g \in G$, we can define measures on Ug using the conformal densities defined previously. The Patterson–Sullivan measure (abbreviated as the PS measure)

$$d\mu^{\mathrm{PS}}_{Ug}(u_{\mathbf{t}}g) := e^{\delta_\Gamma \beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))} d\nu_o((u_{\mathbf{t}}g)^+), \tag{11}$$

and the Lebesgue measure

$$\mu^{\mathrm{Leb}}_{Ug}(u_{\mathbf{t}}g) := e^{(n-1)\beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))} dm_o((u_{\mathbf{t}}g)^+).$$

We similarly define the opposite PS measure on $\tilde{U}g$:

$$d\mu^{\mathrm{PS}\,-}_{\tilde{U}g}(v_{\mathbf{t}}g) := e^{\delta_\Gamma \beta_{(v_{\mathbf{t}}g)^-}(o, v_{\mathbf{t}}g(o))} d\nu_o((v_{\mathbf{t}}g)^-). \tag{12}$$

The conformal properties of m_x and ν_x imply that these definitions are independent of the choice of $o \in \mathbb{H}^n$.

We often view μ^{PS}_{Ug} as a measure on U via

$$d\mu^{\mathrm{PS}}_{g}(\mathbf{t}) := d\mu^{\mathrm{PS}}_{Ug}(u_{\mathbf{t}}g),$$

and similarly for $\mu^{\mathrm{PS}\,-}_{\tilde{U}g}$ on $\tilde{U}$. For $g \in G$, $s \in \mathbb{R}$ and $E \subseteq U$ a Borel subset (or $E \subseteq \tilde{U}$ for $\mu^{\mathrm{PS}\,-}$), these measures satisfy

$$\mu^{\mathrm{Leb}}_{g}(E) = e^{(n-1)s} \mu^{\mathrm{Leb}}_{a_{-s}g}(a_{-s}Ea_s), \tag{13}$$

$$\mu^{\mathrm{PS}}_{g}(E) = e^{\delta_\Gamma s} \mu^{\mathrm{PS}}_{a_{-s}g}(a_{-s}Ea_s), \tag{14}$$

$$\mu^{\mathrm{PS}\,-}_{g}(E) = e^{\delta_\Gamma s} \mu^{\mathrm{PS}\,-}_{a_{s}g}(a_{s}Ea_{-s}). \tag{15}$$

In particular,

$$\mu^{\mathrm{PS}}_{g}(B_U(e^s)) = e^{\delta_\Gamma s} \mu^{\mathrm{PS}}_{a_{-s}g}(B_U(1)) \quad \text{and} \quad \mu^{\mathrm{PS}\,-}_{g}(B_{U^-}(e^s)) = e^{\delta_\Gamma s} \mu^{\mathrm{PS}\,-}_{a_{s}g}(B_U(1)).$$

The measure

$$d\mu^{\mathrm{Leb}}_{Ug}(u_{\mathbf{t}}g) = d\mu^{\mathrm{Leb}}_{U}(u_{\mathbf{t}}) = d\mathbf{t}$$

is independent of the orbit Ug and is simply the Lebesgue measure on $U \equiv \mathbb{R}^{n-1}$ up to a scalar multiple.

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We will need the following fundamental results, which are stated for μ^{PS} and U, but also hold if we replace them with $\mu^{\mathrm{PS}\,-}$ and $\tilde{U}$.

LEMMA 3.2. *The map $g \mapsto \mu_g^{\mathrm{PS}}$ is continuous, where the topology on the space of regular Borel measures on U is given by $\mu_n \to \mu \iff \mu_n(f) \to \mu(f)$ for all $f \in C_c(U)$.*

Proof. This is clear from the definition of the PS measure, since it is defined using the Busemann function and stereographic projection. □

COROLLARY 3.3. *For any compact set $\Omega \subseteq G$ and any $r > 0$,*

$$0 < \inf_{g\in\Omega, g^+\in\Lambda(\Gamma)} \mu_g^{\mathrm{PS}}(B_U(r)g) \le \sup_{g\in\Omega, g^+\in\Lambda(\Gamma)} \mu_g^{\mathrm{PS}}(B_U(r)g) < \infty.$$

To define the PS measure on Ux for $x \in X$, note that

$$\text{if } x^- \in \Lambda_r(\Gamma), \quad \text{then } u \mapsto ux \text{ is injective,} \tag{16}$$

and we can define the PS measure on $Ux \subseteq X$, denoted μ_x^{PS}, simply by pushforward of μ_g^{PS}, where $x = g\Gamma$. In general, defining μ_x^{PS} requires more care, see e.g. [24, §2.3] for more details. As before, we can view μ_x^{PS} as a measure on U via

$$d\mu_x^{\mathrm{PS}}(\mathbf{t}) = d\mu_x^{\mathrm{PS}}(u_{\mathbf{t}}x).$$

3.2. *Thick–thin decomposition and the height function.* There exists a finite set of Γ-representatives $\xi_1, \ldots, \xi_q \in \Lambda_{bp}(\Gamma)$. For $i = 1, \ldots, q$, fix $g_i \in G$ such that $g_i^- = \xi_i$, and for any $R > 0$, set

$$\mathcal{H}_i(R) := \bigcup_{s>R} Ka_{-s}Ug_i \quad \text{and} \quad \mathcal{X}_i(R) := \mathcal{H}_i(R)\Gamma \tag{17}$$

(recall, $K = \mathrm{Stab}_G(o)$). Each $\mathcal{H}_i(R)$ is a horoball of depth R.

The *rank* of $\mathcal{H}_i(R)$ is the rank of the finitely generated abelian subgroup $\Gamma_{\xi_i} = \mathrm{Stab}_\Gamma(\xi_i)$. We say that the cusp has maximal rank if rank $\Gamma_\xi = n - 1$. It is known that each rank is strictly smaller than $2\delta_\Gamma$.

We denote

$$\operatorname{supp} m^{\mathrm{BMS}} := \{g\Gamma \in X : g^\pm \in \Lambda(\Gamma)\}.$$

(For now, this is simply notation. The measure m^{BMS} will be defined in the next section, and this set is its support. It projects onto the convex core of $\mathbb{H}^n/\Gamma$.) Note that the condition $g^\pm \in \Lambda(\Gamma)$ is independent of the choice of representative of $x = g\Gamma$ in the above definition, because $\Lambda(\Gamma)$ is Γ-invariant. Thus, the notation $x^\pm \in \Lambda(\Gamma)$ is well defined, even though $x^\pm$ itself is not.

According to [2], there exists $R_0 \ge 1$ such that $\mathcal{X}_1(R_0), \ldots, \mathcal{X}_q(R_0)$ are disjoint, and for some compact set $\mathcal{C}_0 \subset G/\Gamma$,

$$\operatorname{supp} m^{\mathrm{BMS}} \subseteq \mathcal{C}_0 \sqcup \mathcal{X}_1(R_0) \sqcup \cdots \sqcup \mathcal{X}_q(R_0).$$

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For $1 \leq i \leq q$ and $R \geq R_0$, denote

$$\mathcal{X}(R) := \mathcal{X}_1(R) \sqcup \cdots \sqcup \mathcal{X}_q(R), \quad \mathcal{C}(R) := \operatorname{supp} m^{\mathrm{BMS}} - \mathcal{X}(R).$$

We will need a version of Sullivan's shadow lemma, obtained by Schapira–Maucourant (see Proposition 5.1 and Remark 5.2 in [20]).

PROPOSITION 3.4. *There exists a constant* $\lambda = \lambda(\Gamma) \geq 1$ *such that for all* $x \in \operatorname{supp} m^{\mathrm{BMS}}$ *and all* $T > 0$*, we have*

$$\lambda^{-1} T^{\delta_\Gamma} e^{(k_1(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{-\log T} x))} \leq \mu_x^{\mathrm{PS}}(B_U(T)) \tag{18}$$

$$\leq \lambda T^{\delta_\Gamma} e^{(k_1(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{-\log T} x))} \tag{19}$$

and

$$\lambda^{-1} T^{\delta_\Gamma} e^{(k_2(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log T} x))} \leq \mu_x^{\mathrm{PS}\,-}(B_{\tilde{U}}(T))$$
$$\leq \lambda T^{\delta_\Gamma} e^{(k_2(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log T} x))}, \tag{20}$$

where $k_1(x, T)$ *is the rank of* $\mathcal{X}_i(R_0)$ *if* $a_{-\log T} x \in \mathcal{X}_i(R_0)$ *for some* $1 \leq i \leq \ell$ *and equals 0 if* $a_{-\log T} x \in \mathcal{C}_0$*, and* $k_2(x, T)$ *is defined analogously for* $a_{\log T} x$*. Recall the definition of* π *from (4) as the projection from G to* $\mathbb{H}^n$*.*

Definition 3.5. For $x \in G/\Gamma$, we define the *height* of x by

$$\operatorname{height}(x) = d(\pi(\mathcal{C}_0), \pi(x)), \tag{21}$$

where $\pi : G/\Gamma \to \mathbb{H}^n/\Gamma$ is the projection map as in (4), recalling that $\mathbb{H}^n/\Gamma \cong K\backslash G/\Gamma$.

LEMMA 3.6. *For any* $x \in \operatorname{supp} m^{\mathrm{BMS}}$ *and* $R \geq R_0$*, we have that*

$$x \in \mathcal{C}(R) \iff \operatorname{height}(x) \leq R - R_0.$$

Proof. The claim follows from the disjointness of $\mathcal{X}_i(R_0)$, $1 \leq i \leq q$ from $\mathcal{C}_0$, and the fact that $\mathcal{X}_i(R) \subseteq \mathcal{X}_i(R_0)$.

If $x \in \mathcal{C}(R)$, then either $x \in \mathcal{C}_0$, in which case $\operatorname{height}(x) = 0$ and we are done, or $x \in \mathcal{X}_i(R_0)$. Assume the latter, then the Busemann function between x and the boundary of $\mathcal{X}_i(R_0)$ (which intersects $\mathcal{C}_0$) is at most $R - R_0$. Thus, we may deduce the claim in this case.

Next, assume $x \in \mathcal{X}_i(R)$ for some i. The Busemann function between two points in different horoballs is at least $R - R_0$. Since a point from $\mathcal{X}_i(R)$ cannot go into $\mathcal{C}_0$ without passing through $\mathcal{X}_i(R_0)$, this is a lower bound for the distance between the base points, that is, the height. □

COROLLARY 3.7. *Let* $x \in G/\Gamma$ *be* (ε, s_0)*-Diophantine. Then*

$$\operatorname{height}(x) < (2 - \varepsilon) s_0.$$

Proof. By Definition 1.2,

$$d(\mathcal{C}_0, a_{-s_0} x) < (1 - \varepsilon) s_0.$$

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Hence, we have that

$$\begin{aligned}\text{height}(x) &\leq d(\mathcal{C}_0, x)\\ &< d(\mathcal{C}_0, a_{-s_0}x) + d(a_{-s_0}x, x)\\ &< (1-\varepsilon)s_0 + s_0.\end{aligned}$$ □

The *injectivity radius* at $x \in X$ is defined to be the supremum over all $\varepsilon > 0$ such that the map

$$h \mapsto hx \text{ is injective on } G_\varepsilon.$$

We denote the injectivity radius at x by

$$\text{inj}(x).$$

The injectivity radius of a set Ω is defined to be

$$\inf_{x\in\Omega} \text{inj}(x).$$

By the proof of [25, Proposition 6.7], there exists a constant $\sigma = \sigma(\Gamma) > 0$ such that for all $x \in \text{supp}\, m^{\text{BMS}}$,

$$\sigma^{-1}\, \text{inj}(x) \leq e^{-\text{height}(x)} \leq \sigma\, \text{inj}(x). \tag{22}$$

The following fact is well known, but we include a proof for completion.

LEMMA 3.8. *There exists $T_0 = T_0(\Gamma) > 0$ which satisfies the following. Let $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$, and let $R > 0$ be such that $d(\mathcal{C}_0, x) < R$. Then there exists $\boldsymbol{t} \in B_U(2(R+T))$ such that*

$$(u_{\boldsymbol{t}}x)^\pm \in \Lambda(\Gamma).$$

In particular, for every $0 < \varepsilon < 1$, $s_0 \geq 1$, and (ε, s_0)-Diophantine point x, there exists $|\boldsymbol{t}| \ll_\Gamma s_0$ such that

$$(u_{\boldsymbol{t}}x)^\pm \in \Lambda(\Gamma).$$

Proof. Let $g, h' \in G$ be such that $x = g\Gamma$, $h'^- = g^-$, $h'\Gamma \in K\mathcal{C}_0$, and

$$d(g, h') \leq \text{height}(x) < R.$$

Since $K\mathcal{C}_0$ is a compact set, by [24, Lemma 3.3], there exists a constant T_0, which only depends on $\mathcal{C}_0$ (that is, on Γ) such that for some $\mathbf{t} \in B_U(T_0)$,

$$(u_{\mathbf{t}}h')^\pm \in \Lambda(\Gamma).$$

Fix $h := u_{\mathbf{t}}h'$ and observe that

$$d(g, h) < R + T_0. \tag{23}$$

We must flow $h\Gamma$ with an element of A so that it lies on Ux.

Because $h^- = g^-$, if $s = \beta_{g^-}(h, g)$, then

$$a_s h \in Ug.$$

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Since $\beta_{g^-}(h, g) \le d(h, g)$, we arrive at

$$\begin{aligned} d(g, a_s h) &\le d(g, h) + d(h, a_s h) \\ &\le 2d(g, h) \\ &\le 2(R + T). \end{aligned}$$

For (ε, s_0)-Diophantine x, observe that

$$\begin{aligned} d(\mathcal{C}_0, x) &\le d(\mathcal{C}_0, a_{-s_0} x) + d(a_{-s_0}, x) \\ &< (1 - \varepsilon)s_0 + s_0 \\ &< 2s_0, \end{aligned}$$

so we see that $R = 2s_0$ works for all such points. □

3.3. *Bowen–Margulis–Sullivan and Burger–Roblin measures.* Recall $\pi : G \to \mathbb{H}^n$ from (4). In this section, we will abuse notation and write π for the restriction of π to $\mathrm{T}^1(\mathbb{H}^n) \cong M\backslash G$. Recalling the fixed reference point $o \in \mathbb{H}^n$ as before, the map

$$w \mapsto (w^+, w^-, s := \beta_{w^-}(o, \pi(w)))$$

is a homeomorphism between $\mathrm{T}^1(\mathbb{H}^n)$ and

$$(\partial(\mathbb{H}^n) \times \partial(\mathbb{H}^n) - \{(\xi, \xi) : \xi \in \partial(\mathbb{H}^n)\}) \times \mathbb{R}.$$

This homeomorphism allows us to define the Bowen–Margulis–Sullivan (BMS) and Burger–Roblin (BR) measures on $\mathrm{T}^1(\mathbb{H}^n)$, denoted by $\tilde{m}^{\mathrm{BMS}}$ and $\tilde{m}^{\mathrm{BR}}$ respectively:

$$d\tilde{m}^{\mathrm{BMS}}(w) := e^{\delta_\Gamma \beta_{w^+}(o,\pi(w))} e^{\delta_\Gamma \beta_{w^-}(o,\pi(w))} d\nu_o(w^+) d\nu_o(w^-) ds,$$

$$d\tilde{m}^{\mathrm{BR}}(w) := e^{(n-1)\beta_{w^+}(o,\pi(w))} e^{\delta_\Gamma \beta_{w^-}(o,\pi(w))} dm_o(w^+) d\nu_o(w^-) ds.$$

The conformal properties of $\{\nu_x\}$ and $\{m_x\}$ imply that these definitions are independent of the choice of $o \in \mathbb{H}^n$. Using the identification of $\mathrm{T}^1(\mathbb{H}^n)$ with $M\backslash G$, we lift the above measures to G so that they are all invariant under M from the left. By abuse of notation, we use the same notation ($\tilde{m}^{\mathrm{BMS}}$ and $\tilde{m}^{\mathrm{BR}}$). These measures are right Γ-invariant, and hence induce locally finite Borel measures on X, which are the Bowen–Margulis–Sullivan measure m^{BMS} and the Burger–Roblin measure m^{BR} respectively.

Note that

$$\operatorname{supp} m^{\mathrm{BMS}} := \{x \in X : x^\pm \in \Lambda(\Gamma)\}$$

and

$$\operatorname{supp} m^{\mathrm{BR}} = \{x \in X : x^- \in \Lambda(\Gamma)\}.$$

Recall $P = MA\tilde{U}$, which is exactly the stabilizer of w_o^+ in G. We can define another measure ν on Pg for $g \in G$, which will give us a product structure for $\tilde{m}^{\mathrm{BMS}}$ and $\tilde{m}^{\mathrm{BR}}$ that will be useful in our approach. For any $g \in G$, define

$$d\nu(pg) := e^{\delta_\Gamma \beta_{(pg)^-}(o, pg(o))} d\nu_o(w_o^- pg) dm ds \tag{24}$$

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on Pg, where $s = \beta_{(pg)^-}(o, pg(o))$, $p = mav \in MA\tilde{U}$, and dm is the probability Haar measure on M.

Then for any $\psi \in C_c(G)$ and $g \in G$, we have

$$\tilde{m}^{\mathrm{BMS}}(\psi) = \int_{Pg} \int_U \psi(u_{\mathbf{t}} pg)\, d\mu^{\mathrm{PS}}_{pg}(\mathbf{t})\, d\nu(pg) \tag{25}$$

and

$$\tilde{m}^{\mathrm{BR}}(\psi) = \int_{Pg} \int_U \psi(u_{\mathbf{t}} pg)\, d\mathbf{t}\, d\nu(pg). \tag{26}$$

LEMMA 3.9. *There exists a constant $\lambda = \lambda(\Gamma) > 1$ such that for all $g \in \operatorname{supp} \tilde{m}^{\mathrm{BMS}}$ and all $0 < \varepsilon < \operatorname{inj}(g)$, we have*

$$\begin{aligned} &\lambda^{-1} \varepsilon^{\delta_\Gamma + 1/2(n-1)(n-2)+1} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))} \\ &\le \nu(P_\varepsilon g) \\ &\le \lambda \varepsilon^{\delta_\Gamma + 1/2(n-1)(n-2)+1} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))}, \end{aligned}$$

where $x = g\Gamma$ and $k_2(x, \varepsilon)$ is as defined in Proposition 3.4.

Proof. Let $x = g\Gamma$. By Proposition 3.4, there exists $\tilde{\lambda} > 1$ such that for all such ε,

$$\begin{aligned} \tilde{\lambda}^{-1} \varepsilon^{\delta_\Gamma} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))} &\le \mu^{\mathrm{PS}\,-}_g (B_{\tilde{U}}(\varepsilon)) \\ &\le \tilde{\lambda} \varepsilon^{\delta_\Gamma} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))}. \end{aligned} \tag{27}$$

From (24), if m denotes the probability Haar measure on M, we then have

$$\begin{aligned} \nu(P_\varepsilon g) &\le \int_{A_\varepsilon} \int_{M_\varepsilon} \mu^{\mathrm{PS}\,-}_g (B_{\tilde{U}}(\varepsilon))\, dm\, ds \\ &\le C\tilde{\lambda} \varepsilon^{\delta_\Gamma + 1/2(n-1)(n-2)+1} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))}, \end{aligned}$$

where C is determined by the scaling of the probability Haar measures on A and M. The lower bound follows similarly. Then, $\lambda = \max\{C\tilde{\lambda}, \tilde{\lambda}\}$ satisfies the conclusion of the lemma. □

3.4. *Admissible boxes and smooth partitions of unity.* Recall that for $\eta > 0$, we denoted by G_η the closed η-neighborhood of e in G.

Take $\varepsilon > 0$ such that the map

$$g \mapsto gx \text{ is injective on } G_\varepsilon \text{ for all } x \in \Omega.$$

For $x \in X$ and $\eta_1 > 0$, $\eta_2 \ge 0$ less than $\operatorname{inj}(x)$, we call

$$B = B_U(\eta_1) P_{\eta_2} x$$

an *admissible box* (with respect to the PS measure) if B is the injective image of $B_U(\eta_1) P_{\eta_2}$ in X under the map $h \mapsto hx$ and

$$\mu^{\mathrm{PS}}_{px}(B_U(\eta_1) px) \ne 0$$

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for all $p \in P_{\eta_2}$. For $g \in G$, we say that $B = B_U(\eta_1)P_{\eta_2}g$ is an admissible box if $B = B_U(\eta_1)P_{\eta_2}x$ is one.

Note that if $B_U(\eta_1)P_{\eta_2}g$ is an admissible box, then there exists $\varepsilon > 0$ such that $B_U(\eta_1 + \varepsilon)P_{\eta_2+\varepsilon}g$ is also an admissible box. Moreover, every point has an admissible box around it by [26, Lemma 2.17].

The error terms in our main theorems are in terms of Sobolev norms, which we define here. For $\ell \in \mathbb{N}$, $1 \le p \le \infty$, and $\psi \in C^\infty(X) \cap L^p(X)$, we consider the following Sobolev norm:

$$S_{p,\ell}(\psi) = \sum \|U\psi\|_p,$$

where the sum is taken over all monomials U in a fixed basis of $\mathfrak{g} = \mathrm{Lie}(G)$ of order at most ℓ, and $\|\cdot\|_p$ denotes the $L^p(X)$-norm. Since we will be using $S_{2,\ell}$ most often, we set

$$S_\ell = S_{2,\ell}.$$

Our proofs will require constructing smooth indicator functions and partitions of unity with controlled Sobolev norms. We prove such lemmas below.

LEMMA 3.10. *Let H be a horospherical subgroup of G (that is, U or $\tilde{U}$). For every $\xi_1, \xi_2 > 0$ and $g \in G$, there exists a non-negative smooth function χ_{ξ_1,ξ_2} defined on $H_{\xi_1+\xi_2}g$ such that $0 \le \chi_{\xi_1,\xi_2} \le 1$, $S_\ell(\chi_{\xi_1,\xi_2}) \ll_{n,\Gamma} \xi_1^{n-1}\xi_2^{-\ell-(n-1)/2}$, and*

$$\chi_{\xi_1,\xi_2}(h) = \begin{cases} 0 & \text{if } h \notin H_{\xi_1+\xi_2}g, \\ 1 & \text{if } h \in H_{\xi_1-\xi_2}g. \end{cases}$$

Proof. According to [17, Lemma 2.4.7(b)], there exists $c_1 = c_1(n) > 0$ such that for every $\xi > 0$, there exists a non-negative smooth function σ_ξ defined on H_ξ such that

$$\int_H \sigma_\xi(h)\, dm^{\mathrm{Haar}}(h) = 1, \quad S_\ell(\sigma_\xi) < c_1\xi^{-\ell-(n-1)/2}. \tag{28}$$

For $g \in \Omega$, let $\chi_{\xi_1,\xi_2} = \mathbf{1}_{H_{\xi_1}g} * \sigma_{\xi_2}$. Then for any $h \in H$, we have $0 \le \chi_{\xi_1,\xi_2}(h) \le 1$ and

$$\chi_{\xi_1,\xi_2}(h) = \begin{cases} 0 & \text{if } h \notin H_{\xi_1+\xi_2}g, \\ 1 & \text{if } h \in H_{\xi_1-\xi_2}g. \end{cases}$$

Since for some $c_2 = c_2(\Gamma) > 0$,

$$S_{1,0}(\mathbf{1}_{H_{\xi_1}g_0}) = m^{\mathrm{Haar}}(H_{\xi_1}) < c_2{\xi_1}^{n-1},$$

by the properties of the Sobolev norm and (28), we arrive at

$$S_\ell(\chi_{\xi_1,\xi_2}) \le S_{1,0}(\mathbf{1}_{H_{\xi_1}g_0})S_\ell(\sigma_{\xi_2}) < c_1c_2{\xi_1}^{n-1}{\xi_2}^{-\ell-(n-1)/2}. \qquad \square$$

LEMMA 3.11. *Let H be a horospherical subgroup of G, $r > 0$, $\ell \in \mathbb{N}$, and let $E \subset H$ be bounded. Then, there exists a partition of unity $\sigma_1, \ldots, \sigma_k$ of E in H_rE, that is,*

$$\sum_{i=1}^k \sigma_i(x) = \begin{cases} 0 & \text{if } x \notin H_rE, \\ 1 & \text{if } x \in E, \end{cases}$$

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such that for some $u_1, \ldots, u_k \in E$ *and all* $1 \leq i \leq k$,

$$\sigma_i \in C_c^\infty(H_r u_i), \quad S_\ell(\sigma_i) \ll_n r^{-\ell+n-1}.$$

Moreover, if there exists $R > r$ *such that* $E = H_R$, *then* $k \ll_n (R/r)^{n-1}$.

Proof. Let $\{u_1, \ldots, u_k\}$ be a maximal $r/4$-separated set in E. Then

$$E \subseteq \bigcup_{i=1}^{k} H_{r/2} u_i. \tag{29}$$

Let $1 \leq i \leq k$. According to [13, Theorem 1.4.2], there exists $\chi_i \in C_c^\infty(H_r u_i)$ such that $0 \leq \chi_i \leq 1$, $\chi_i(u) = 1$ for any $u \in H_{r/2} u_i$, and for $1 \leq m \leq \ell$,

$$|\chi_i^{(m)}| \ll r^{-m} \tag{30}$$

(where the implied constant depends only on n). Let σ_i be defined by

$$\sigma_i = \chi_i(1 - \chi_{i-1}) \cdots (1 - \chi_1).$$

Then, each $\sigma_i \in C_c^\infty(H_r u_i)$ and

$$1 - \sum_{i=1}^{k} \sigma_i = \prod_{i=1}^{k} (1 - \chi_i) = 0 \quad \text{on} \ \bigcup_{i=1}^{k} H_r u_i$$

implies that $\sum_{i=1}^{k} \sigma_i = 1$ on $\bigcup_{i=1}^{k} H_{r/2} u_i$.

By the rules for differentiating a product and (30) for $1 \leq m \leq \ell$, we have

$$|\sigma_i^{(m)}| \leq C r^{-m},$$

where C is the multiplicity of the cover in (29). By Besicovitch covering theorem, C is bounded by a constant which depends only on n. Using the definition of the Sobolev norm, we arrive at

$$S_\ell(\sigma_i) \ll_n r^{-\ell+n-1}.$$

Now, assume there exists $R > r$ such that $E = H_R$. Since the geometry of H is of an Euclidean space of dimension $\dim H$, we then have

$$k \ll_n \left(\frac{R}{r}\right)^{n-1}. \qquad \square$$

LEMMA 3.12. *Let* H *be either* U *or* G. *There exists* $\ell' = \ell'(H) > 0$ *such that for any integer* $\ell > \ell'$, $\eta > 0$, $H \in \{U, G\}$, *and* $f \in C_c^\infty(H)$, *there exist functions* $f_{\eta,\pm} \in C_c^\infty(H)$ *which are supported on an* 2η *neighborhood of* supp f, *and for any* $h \in H$, *satisfy:*

(1) $f_{\eta,-}(h) \leq \min_{w \in H_\eta} f(wh) \leq \max_{w \in H_\eta} f(wh) \leq f_{\eta,+}(h)$;

(2) $|f_{\eta,\pm}(h) - f(h)| \ll_{\text{supp}\, f} \eta S_\ell(f)$;

(3) $S_\ell(f_{\eta,\pm}) \ll_{H,\text{supp}\, f} \eta^{-2\ell} S_\ell(f)$.

Proof. First, according to [1], there exists $\ell' \in \mathbb{N}$ such that any $\ell > \ell'$ satisfies $S_{\infty,1}(\psi) \ll_{\text{supp}\, \psi} S_\ell(\psi)$ for any $\psi \in C_c^\infty(H)$.

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Let $f'_{\eta,\pm}$ be defined by

$$f'_{\eta,+}(h) := \sup_{w\in H_\eta} f(wh) \quad \text{and} \quad f'_{\eta,-}(h) := \inf_{w\in H_\eta} f(wh)$$

for any $h \in H$.

As before, we use [17, Lemma 2.4.7(b)] to deduce that there exist $c_1 = c_1(H) > 0$, $n_1 = n_1(H)$, and a non-negative smooth function σ_η supported on H_η such that

$$\int_H \sigma_\eta(h)\, dm^{\text{Haar}}(h) = 1, \quad S_\ell(\sigma_\eta) < c_1\eta^{-\ell-n_1}.$$

Define $f_{\eta,\pm}$ by

$$f_{\eta,\pm} := f'_{2\eta,\pm} * \sigma_\eta.$$

Then, $f_{\eta,\pm}$ are smooth functions which are supported on a 2η neighborhood of supp f. Moreover, for any $h \in H$,

$$\begin{aligned} f'_{\eta,+}(h) &= \int_{H_\eta} f'_{\eta,+}(h)\sigma_\eta(u^{-1})\, dm^{\text{Haar}}(u) \\ &\le \int_{H_\eta} f'_{2\eta,+}(uh)\sigma_\eta(u^{-1})\, dm^{\text{Haar}}(u) \quad \text{by definition of } f'_{2\eta,+} \\ &= f_{\eta,+}(h) \\ &\le \int_{H_\eta} f'_{3\eta,+}(h)\sigma_\eta(u^{-1})\, dm^{\text{Haar}}(u) \quad \text{by (31) and definition of } f'_{3\eta,+} \\ &= f'_{3\eta,+}(h). \end{aligned} \tag{31}$$

In a similar way, one can show

$$f'_{3\eta,-} \le f_{\eta,-} \le f'_{\eta,-},$$

proving the first inequality.

By the mean value theorem, for any $h \in H$, $w \in H_{3\eta}$,

$$|f(wh) - f(h)| \ll \eta S_{\infty,1}(f) \ll_{\text{supp } f} S_\ell(f).$$

Since $f'_{3\eta,-} \le f_{\eta,-} \le f_{\eta,+} \le f'_{3\eta,+}$, there exist some $w_+, w_- \in H_{3\eta}$ such that

$$|f_{\eta,\pm}(h) - f(h)| \le |f(w_\pm h) - f(h)|,$$

and we have the second inequality.

Now, we have

$$S_\ell(f_{\eta,\pm}) \le S_{\infty,1}(f'_{2\eta,\pm})S_\ell(\sigma_\eta) \ll_{H,\text{supp } f} S_\ell(f)\eta^{-\ell-n_1+1}.$$

By choosing $\ell' > n_1$, we may deduce the last inequality. □

4. *Quantitative non-divergence*

In this section, we prove a quantitative non-divergence result that is crucial in the following sections. We use the notation established in §3.2. The results in this section hold for any Γ that is geometrically finite, without need for Assumption 1.1.

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Recall from §1 that for $0 < \varepsilon < 1$ and $s_0 \geq 1$, we say that $x \in X$ is (ε, s_0)-*Diophantine* if for all $\tau > s_0$,

$$d(\mathcal{C}_0, a_{-\tau}x) < (1-\varepsilon)\tau, \tag{32}$$

where $\mathcal{C}_0$ is the compact set defined in §3.2. Let R_0 and q also be as defined in §3.2.

This section is dedicated to the proof of the following theorem, which says (in a quantitative way) that most of the U orbit of a Diophantine point is not in the cusp.

THEOREM 4.1. *There exists $\beta > 0$ satisfying the following: for every $0 < \varepsilon < 1$ and $s_0 \geq 1$, and for every (ε, s_0)-Diophantine element $x \in X$, every $R \geq R_0$, every $T \gg_{\Gamma,\varepsilon} s_0$, and every $0 < s \leq T^{\varepsilon}$, we have*

$$\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T/s)a_{-\log s}x \cap \mathcal{X}(R)) \ll_{n,\Gamma} \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T/s)a_{-\log s}x)e^{-\beta R}.$$

We now follow the notation of Mohammadi and Oh in [25, §6]. Equip $\mathbb{R}^{n+1}$ with the Euclidean norm. Recall from §3.2 that for $1 \leq i \leq q$, $g_i^- = \xi_i$. Without loss of generality, we may further assume that g_i satisfies $\|g_i^{-1}e_1\| = 1$. Let

$$v_i = g_i^{-1}e_1.$$

LEMMA 4.2. *For any $i = 1, \ldots, q$, Γv_i is a discrete subset of $\mathbb{R}^{n+1}$.*

Proof. Since ξ_i is assumed to be a bounded parabolic limit point, by definition, $(\Lambda(\Gamma) \setminus \{\xi_i\})/\Gamma_{\xi_i} = (\Lambda(\Gamma) \setminus \{\xi_i\})/\Gamma_{v_i}$ is compact, where

$$G_{v_i} = g_i^{-1}MUg_i \quad \text{and} \quad \Gamma_{v_i} = \Gamma \cap G_{v_i}.$$

If $\gamma \in \Gamma_{v_i}$, then $\mathcal{H}_i(R_0)\gamma = \mathcal{H}_i(R_0)$. Therefore, the visual map induces a homeomorphism between $\mathcal{H}_i(R_0)/\Gamma_{v_i}$ and $(\partial\mathbb{H}^n \setminus \{\xi_i\})/\Gamma_{v_i}$. It follows that the quotient of $\{g^+ \in \Lambda : g \in \mathcal{H}_i(R_0)\}$ by the action of Γ_{v_i} is compact. Using Iwasawa decomposition, it follows that there exists a compact set $U_0 \subset U$ such that for any $g = kaug_i \in \mathcal{H}_i(R_0)$ such that $g^+ \in \Lambda(\Gamma)$, $k \in K$, $a \in A$, and $u \in U$, there exist $\gamma \in \Gamma_{v_i}$, $k' \in K$, $u' \in U_0$ so that $g\gamma = k'au'g_i$.

Since ξ_i is assumed to be a parabolic limit point, there exists a parabolic element $\gamma_0 \in \Gamma_{\xi_i}$, that is, $\gamma_0 = g_i^{-1}mug_i$.

Assume for contradiction that there exists an infinite sequence $\{\gamma_j\} \in \Gamma$ such that $\{\gamma_j v_i\}$ converges. Note that translating by an element of γ allows us to assume, without loss of generality, that the limit of this sequence is 0. Using the Iwasawa decomposition, we get that for all j, there exist $a_{t_j} \in A$, $k_j \in K$, and uniformly bounded $u_j \in U$ such that $\gamma_j = k_j a_{t_j} u_j g_i$. Since

$$\|\gamma_j v_i\| = \|k_j a_{t_j} u_j e_1\| = e^{t_j},$$

we may deduce that $t_j \to -\infty$. In particular, $\gamma_j \in H_i(R_0)$ for all large enough j.

We have

$$\begin{aligned}\gamma_j\gamma_0\gamma_j^{-1} &= (k_j a_{t_j} u_j g_i)(g_i^{-1}mug_i)(g_i^{-1}u_j^{-1}a_{t_j}^{-1}k_j^{-1}) \\ &= k_j a_{t_j} u_j muu_j^{-1}a_{t_j}^{-1}k_j^{-1}.\end{aligned}$$

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Since $u_j m u_j^{-1} = m_j u'_j \in MU$, with u'_j uniformly bounded, and since M centralizes A, we have

$$\gamma_j \gamma_0 \gamma_j^{-1} = k_j m_j a_{t_j} u'_j u a_{t_j}^{-1} k_j^{-1}.$$

Since $u'_j u$ is in a bounded subset of U, we get that $a_{t_j} u'_j u a_{t_j}^{-1} \to e$ as $t_j \to -\infty$. Since K and M are compact, it then follows that the sequence $\gamma_j \gamma_0 \gamma_j^{-1}$ has a convergent subsequence. This contradicts the discreteness of Γ, since the γ_j terms were assumed to be distinct. □

For any $g \in G$, we have that $g\Gamma \in \mathcal{X}_i(R)$ if and only if there exists $\gamma \in \Gamma$ such that

$$\|g\gamma v_i\| \le e^{-R}. \tag{33}$$

Indeed, by the Iwasawa decomposition and (17), if $g\Gamma \in \mathcal{X}_i(R)$, then there exist $\gamma \in \Gamma$, $k \in K$, $s > R$, and $u \in U$, such that

$$\|g\gamma v_i\| = \|k a_{-s} u g_i v_i\| = \|a_{-s} e_i\| = e^{-s}.$$

Moreover, it follows from [**25**, Lemmas 6.4 and 6.5] that the γ in (33) is unique. Note that both lemmas are proved under the additional assumption that $n = 3$, but the proofs also hold without it.

However, by [**25**, Lemma 6.5] and Lemma 4.2, there exists a constant $\eta_0 = \eta_0(\Gamma) > 0$ such that if $g\Gamma \notin \mathcal{X}_i(R_0)$, then for any $\gamma \in \Gamma$,

$$\|g\gamma v_i\| > \eta_0. \tag{34}$$

LEMMA 4.3. *There exists $c = c(\Gamma) > 0$ which satisfies the following. Let $\varepsilon, s_0 > 0$ and let $g \in G$. If $x = g\Gamma$ is (ε, s_0)-Diophantine, then for any $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\sup_{\|\boldsymbol{t}\| \le T} \inf_{\gamma \in \Gamma} \inf_{i=1,\ldots,q} \|u_{\boldsymbol{t}} g\gamma v_i\| > cT^{\varepsilon}. \tag{35}$$

Proof. Fix $T > T_0 = \max\{s_0, \eta_0^{1/(\varepsilon-1)}\}$. We will first show that

$$\inf_{\gamma \in \Gamma} \inf_{i=1,\ldots,q} \|a_{-\log T} g\gamma v_i\| > cT^{\varepsilon-1}, \tag{36}$$

for some constant $1 > c = c(\Gamma) > 0$.

There are two cases to consider. If $a_{-\log T} x \notin \mathcal{X}_i(R_0)$, then (36) follows from (34) and the choice of T.

Otherwise, $a_{-\log T} x \in \mathcal{X}_i(R)$ for some maximal $R > R_0$. According to Lemma 3.6, we have

$$d(x, \mathcal{C}_0) \ge R - R_0.$$

Then, because x is (ε, s_0) Diophantine and $T > s_0$, by (32), we may deduce that

$$R - R_0 < (1 - \varepsilon) \log T.$$

Hence, $a_{-\log T} x \notin \mathcal{X}_i((1 - \varepsilon) \log T + R_0)$, so (33) implies (36).

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Now, fix $\gamma \in \Gamma$ and $1 \le i \le q$, and let

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = a_{-\log T} g \gamma v_i.$$

According to (36), there exists $1 \le k \le n$ such that $|x_k| > cT^{\varepsilon - 1}$. If $|x_1| > cT^{\varepsilon-1}$, then it follows from the action of $a_{-\log T}$ on $\mathbb{R}^{n+1}$ that

$$\|g\gamma v_i\| \ge |cTx_1| > cT^{\varepsilon}.$$

Otherwise, there exists $2 \le k \le n$ such that $|x_k| > cT^{\varepsilon-1}$. Then, for any $\mathbf{t} \in \mathbb{R}^{n-1}$, the first coordinate of $u_{\mathbf{t}} a_{-\log T} g \gamma v_i$ is

$$x_1 + \mathbf{t} \cdot \mathbf{x}' + \frac{1}{2}\|\mathbf{t}\|^2 x_{n+1} \quad \text{where } \mathbf{x}' = \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In particular, by taking $t_k = \pm T$ (the kth entry in $\mathbf{t}$) one can ensure that $\|a_{\log T} u_{\mathbf{t}} a_{-\log T} g \gamma v_i\| > cT^{\varepsilon}$. □

A measure μ is called *D-Federer* if for all $v \in \operatorname{supp}(\mu)$ and $0 < \eta \le 1$,

$$\mu(B(v, 3\eta)) \le D\mu(B(v, \eta)).$$

It is proved in §A (specifically Corollary A.9) that there exists $D = D(\Gamma) > 0$ such that

$$\text{if } x \in X \text{ satisfies } x^- \in \Lambda(\Gamma), \text{ then } \mu_x^{\mathrm{PS}} \text{ is } D\text{-Federer.} \tag{37}$$

Indeed, Corollary A.9 actually establishes that there exists a constant $\sigma = \sigma(\Gamma) > 0$ such that for all $x \in \operatorname{supp} m^{\mathrm{BMS}}$, $c > 1$, and $T > 0$,

$$\mu_x^{\mathrm{PS}}(B_U(cT)) \ll_\Gamma c^\sigma \mu_x^{\mathrm{PS}}(B_U(T)). \tag{38}$$

For $f : \mathbb{R}^d \to \mathbb{R}$ and $B \subset \mathbb{R}^d$, let

$$\|f\|_B := \sup_{x \in B} |f(x)|.$$

Recall that $U \cong \mathbb{R}^{n-1}$.

LEMMA 4.4. *Let $\sigma = \sigma(\Gamma) > 0$ be as in (38). Let $y \in \operatorname{supp} m^{\mathrm{BMS}}$ and let $f : B_U(\eta) \to \mathbb{R}^{n-1}$ be such that there exists $b \ne 0$ so that for every coordinate function $f_i : B_U(\eta) \to \mathbb{R}$, there exist $a_i \in \mathbb{R}$, such that*

$$f_i(\boldsymbol{t}) = a_i + b\boldsymbol{t}_i.$$

Then for $0 < \eta \le 1$ and $0 < \varepsilon < 1$, we have

$$\mu_y^{\mathrm{PS}}(\{\boldsymbol{t} \in B_U(\eta) : \|f(\boldsymbol{t})\| < \varepsilon\}) \ll_\Gamma \left(\frac{\varepsilon}{\|f\|_{B_U(\eta)}}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta)), \tag{39}$$

where $\|f(x)\|$ denotes the max norm.

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Proof. First, note that if $\|f\|_{B_U(\eta)} < 2\varepsilon$, then the result holds by assuming that the implied coefficient in (39) is bigger than 2^σ: in this case, the right-hand side is greater or equal to

$$2^\sigma\left(\frac{\varepsilon}{\|f\|_{B_U(\eta)}}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta)) \geq 2^\sigma\left(\frac{\varepsilon}{2\varepsilon}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta)) \\ \geq \mu_y^{\mathrm{PS}}(\{\mathbf{t} \in B_U(\eta) : \|f(\mathbf{t})\| < \varepsilon\}),$$

as desired. Thus, we now assume that

$$\|f\|_{B_U(\eta)} \geq 2\varepsilon. \tag{40}$$

If $\|f(\mathbf{t})\| \geq \varepsilon$ for all $\mathbf{t} \in B_U(\eta)$ such that $(u_{\mathbf{t}}y)^+ \notin \Lambda(\Gamma)$, then there is nothing to prove. So assume that $\|f(\mathbf{t})\| < \varepsilon$ and $(u_{\mathbf{t}}y)^+ \in \Lambda(\Gamma)$. Since each f_i is linear, for all $\mathbf{t}' \in B_U(\eta)$ with $\|f(\mathbf{t}')\| < \varepsilon$, we get that for all $1 \leq i \leq n-1$,

$$|f_i(\mathbf{t}')| = |a_i + b\mathbf{t}'| < \varepsilon,$$

$$|b(t_i' - t_i)| = |f_i(\mathbf{t}') - f_i(\mathbf{t})| < 2\varepsilon.$$

Therefore,

$$\|f(x)\| < \varepsilon \Longrightarrow x \in B_U(2\varepsilon/b)z.$$

Thus, by (38), we have that there exists $\sigma = \sigma(\Gamma) > 0$ so that

$$\begin{aligned}\mu_y^{\mathrm{PS}}(\{x \in B_U(\eta)y : \|f(x)\| < \varepsilon\}) &\leq \mu_z^{\mathrm{PS}}(B_U(2\varepsilon/b)) \\ &\ll_\Gamma \left(\frac{2\varepsilon}{b\eta}\right)^\sigma \mu_z^{\mathrm{PS}}(B_U(\eta)) \\ &\ll_\Gamma \left(\frac{2\varepsilon}{b\eta}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(3\eta)) \\ &\ll_\Gamma \left(\frac{6\varepsilon}{b\eta}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta)).\end{aligned}$$

Assuming $\|f(\mathbf{t})\| < \varepsilon$ for some $\mathbf{t} \in B_U(\eta)$ (otherwise, as before, there is nothing to prove), for any $\mathbf{t}'' \in B_U(\eta)$ and $1 \leq i \leq n-1$, we have

$$|f_i(\mathbf{t}'')| \leq |f_i(\mathbf{t}'') - f_i(\mathbf{t})| + |f_i(\mathbf{t})| < 2b\eta + \varepsilon.$$

Thus, $\|f\|_{B_U(\eta)y} - \varepsilon \leq 2b\eta$, so by (40),

$$\tfrac{1}{2}\|f\|_{B_U(\eta)} \leq 2b\eta,$$

which completes the proof. □

A function f which satisfies (39) with the implied constant C for any $\varepsilon > 0$ and any ball $B \subset U \subset \mathbb{R}^m$ is called (C, σ)-*good on U with respect to* μ. Observe that

$$\text{if } g \text{ is } (C,\sigma)\text{-good and if } |g(x)| \leq |f(x)| \text{ for } \mu\text{-almost every } x, \text{ then } f \text{ is } (C,\sigma)\text{-good.} \tag{41}$$

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In the proof of the following theorem, we use similar ideas to those which appear in the proof of [18, Lemma 5.2]. Note that the proof in this case is simplified by the third assumption, reflecting our rank-one setting.

PROPOSITION 4.5. *Given positive constants* $C, \beta, D,$ *and* $0 < \eta < 1$, *there exists* $C' = C'(C, \beta, D) > 0$ *with the following property. Suppose* μ *is a* D*-Federer measure on* $\mathbb{R}^m$, $f : \mathbb{R}^m \to \mathrm{SL}_k(\mathbb{R})$ *is a continuous map,* $0 \le \varrho \le \eta$, $z \in \operatorname{supp}\mu$, $\Lambda \subset \mathbb{R}^k$, $B = B(z, r_0) \subset \mathbb{R}^m$, *and* $\tilde{B} = B(z, 3r_0)$ *satisfy the following.*

(1) *For any* $v \in \Lambda$, *the function* $\boldsymbol{t} \mapsto \|f(\boldsymbol{t})v\|$ *is* (C, β)*-good on* $\tilde{B}$ *with respect to* μ.

(2) *For any* $v \in \Lambda$, *there exists* $\boldsymbol{t} \in B$ *such that* $\|f(\boldsymbol{t})v\| \ge \varrho$.

(3) *For any* $\boldsymbol{t} \in B$, *there is at most one* $v \in \Lambda$ *which satisfies* $\|f(\boldsymbol{t})v\| < \eta$.

Then, for any $0 < \varepsilon < \varrho$,

$$\mu(\{\boldsymbol{t} \in B : \text{there exists } v \in \Lambda \text{ such that } \|f(\boldsymbol{t})v\| < \varepsilon\}) \le C'\left(\frac{\varepsilon}{\varrho}\right)^{\beta} \mu(B).$$

Proof. For any $\mathbf{t} \in B$, denote

$$f_\Lambda(\mathbf{t}) = \min\{\|f(\mathbf{t})v\| : v \in \Lambda\}.$$

Let

$$E = \{\mathbf{t} \in B : f_\Lambda(\mathbf{t}) < \varrho\} \cap \operatorname{supp}\mu,$$

and for each $v \in \Lambda$, define

$$E_v = \{\mathbf{t} \in B : \|f(\mathbf{t})v\| < \varrho\} \cap \operatorname{supp}\mu.$$

Observe that by assumption (3), the E_v terms are a disjoint cover of E. For each $\mathbf{t} \in E_v$, define

$$r_{\mathbf{t},v} = \sup\{r : \|f(\mathbf{s})v\| < \varrho \text{ for all } \mathbf{s} \in B(\mathbf{t}, r)\}.$$

By assumption (2), we know that for every $\mathbf{t} \in E$, the set $B(\mathbf{t}, r_{\mathbf{t},v})$ does not contain B. Thus, since $\mathbf{t} \in B$, we deduce that $r_{\mathbf{t},v} < 2r_0$. For any fixed $r_{\mathbf{t},v} < r'_{\mathbf{t},v} < 2r_0$, we have that

$$B(\mathbf{t}, r'_{\mathbf{t},v}) \subset B(z, 3r_0) = \tilde{B}, \tag{42}$$

and by the definition of $r_{\mathbf{t},v}$, there exists $\mathbf{s} \in B(\mathbf{t}, r'_{\mathbf{t},v})$ such that

$$\|f(\mathbf{s})v\| \ge \varrho.$$

Note that $\{B(\mathbf{t}, r_{\mathbf{t},v}) : \mathbf{t} \in E, v \in \Lambda\}$ is a cover of E. According to the Besicovitch covering theorem, there exists a countable subset $I \subset E \times \Lambda$ such that $\{B(\mathbf{t}, r_{\mathbf{t},v}) : (\mathbf{t}, v) \in I\}$ is a cover of E with a covering number bounded by a constant which only depends on m. Thus,

$$\sum_{(\mathbf{t},v)\in I} \mu(B(\mathbf{t}, r_{\mathbf{t},v})) \ll_m \mu\bigg(\bigcup_{(\mathbf{t},v)\in I} B(\mathbf{t}, r_{\mathbf{t},v})\bigg). \tag{43}$$

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By assumption (3) and the continuity of f, for any $(\mathbf{t}, v) \in I$ and $\mathbf{s} \in E \cap B(\mathbf{t}, r_{\mathbf{t},v})$,

$$f_\Lambda(\mathbf{s}) = \|f(\mathbf{s})v\|.$$

Thus,

$$\begin{aligned}\mu(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) : f_\Lambda(\mathbf{s}) < \varepsilon\}) &= \mu(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) : \|f(\mathbf{s})v\| < \varepsilon\}) \\ &\leq \mu(\{\mathbf{s} \in B(\mathbf{t}, r'_{\mathbf{t},v}) : \|f(\mathbf{s})v\| < \varepsilon\}).\end{aligned}$$

Thus, assumption (1) and the assumption that μ is D-Federer together imply that

$$\begin{aligned}\mu(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) : f_\Lambda(\mathbf{s}) < \varepsilon\}) &\leq \mu(\{\mathbf{s} \in B(\mathbf{t}, r'_{\mathbf{t},v}) : \|f(\mathbf{s})v\| < \varepsilon\}) \\ &\leq C\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B(\mathbf{t}, r'_{\mathbf{t},v})) \\ &\leq CD\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B(\mathbf{t}, r_{\mathbf{t},v})). \end{aligned} \tag{44}$$

Since E covers the set of points for which f_Λ is less than ε, we may now conclude

$$\begin{aligned}&\mu(\{\mathbf{t} \in B : f_\Lambda(\mathbf{t}) < \varepsilon\}) \\ &\quad\leq \sum_{(\mathbf{t},v)\in I} \mu(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) : f_\Lambda(\mathbf{t}) < \varepsilon\}) \\ &\quad\leq CD \sum_{(\mathbf{t},v)\in I} \left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B(\mathbf{t}, r_{\mathbf{t},v})) \quad \text{by (44)} \\ &\quad\ll_m CD\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu\bigg(\bigcup_{(\mathbf{t},v)\in I} B(\mathbf{t}, r_{\mathbf{t},v})\bigg) \quad \text{by (43)} \\ &\quad\ll_m CD\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(\tilde{B}) \quad \text{by (42)} \\ &\quad\ll_m CD^2\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B) \quad \mu \text{ is } D\text{-Federer.}\end{aligned}$$

□

Remark 4.6. Fix $x \in X$ such that $x^- \in \Lambda(\Gamma)$. Since the PS measure μ_x^{PS} is supported on $Ux \cap \operatorname{supp} m^{\mathrm{BMS}}$, it follows from Lemma 4.4 and (37) that Proposition 4.5 holds for μ_x^{PS} and function f, which satisfies the assumption of Lemma 4.4.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $x_0 = a_{-\log s}x$ and fix $g \in G$ such that $x = g\Gamma$. By Lemma 4.3, for all $T \gg_{\Gamma,\varepsilon} s_0$, we have (35), that is, that

$$\sup_{\|\mathbf{t}\|\leq T} \inf_{\gamma\in\Gamma} \inf_{i=1,\ldots,q} \|u_\mathbf{t} g\gamma v_i\| > T^\varepsilon.$$

Let $f : \mathbb{R}^{n-1} \to \mathrm{SL}_{n+1}(\mathbb{R})$ be defined by

$$f(\mathbf{t}) = u_\mathbf{t} a_{-\log s} g.$$

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We first show that parts (1), (2), and (3) of Proposition 4.5 for $\mu = \mu_{x_0}^{\mathrm{PS}}, f, \varrho = 1, z = x_0$, $r = T/s$, $\eta = e^{-R_0}$, and

$$\Lambda = \Gamma\{v_1, \ldots, v_q\}.$$

Note that $0 \notin \Lambda$.

It follows from the action of $u_{\mathbf{t}}$ on $\mathbb{R}^{n+1}$ that for any $v \in \mathbb{R}^{n+1}$, there exists $v' = (v_1', \ldots, v_{n+1}')^T \in \mathbb{R}^{n+1}$ such that

$$f(\mathbf{t})v = \big(v_1' + \mathbf{t}\cdot v'' + \tfrac{1}{2}\|\mathbf{t}\|^2 v_{n+1}', v_2' - t_1 v_{n+1}', \ldots, v_n' - t_{n-1}v_{n+1}', v_{n+1}'\big)^T, \tag{45}$$

where $v'' = (v_2', \ldots, v_n')^T$. Thus, if $v_{n+1}' \neq 0$, then $\mathbf{t} \mapsto f(\mathbf{t})v$ is bounded from below by a function which satisfies the assumption of Lemma 4.4. Therefore, by (41), for any $v \in \Lambda$, the function $\mathbf{t} \mapsto \|f(\mathbf{t})v\|$ is (C, β)-good on $\tilde{B}$ with respect to $\mu_{x_0}^{\mathrm{PS}}$ for some $C = C(\Gamma) \geq 1$, $\beta = \beta(\Gamma) > 0$, which proves (1) of Proposition 4.5. Note that these constants are uniform across all v so that $v_{n+1}' \neq 0$.

However, if $v \neq 0$ and

$$v_{n+1}' = 0,$$

then $\mathbf{t} \mapsto f(\mathbf{t})v$ is bounded below by some positive constant, and since positive constant functions are (C, β)-good for any $C \geq 1, \beta > 0$, we conclude that so is this function by (41).

By (6), we have

$$u_{\mathbf{t}} a_{-\log s} = a_{-\log s} u_{s\mathbf{t}}.$$

Since multiplication by $a_{-\log s}$ only changes the matrix entries by scaling, using (35), for $i = 1, \ldots, q$, we get

$$\sup_{\|\mathbf{t}\| \leq T/s} \|a_{-\log s} u_{s\mathbf{t}} g\gamma v_i\| > s^{-1} \sup_{\|\mathbf{t}\| \leq T} \|u_{\mathbf{t}} g\gamma v_i\| > s^{-1} T^{\varepsilon}.$$

Thus, for any $s \leq T^{\varepsilon}$, $\|\mathbf{t}\| < T/s$, and $v \in \Lambda$,

$$\|f(\mathbf{t})v\| \geq 1,$$

which establishes (2) of Proposition 4.5.

Since $\eta = e^{-R_0}$ and $\mathcal{H}_i(R_0)$ terms are pairwise disjoint, part (3) of Proposition 4.5 follows from the uniqueness of γ in (33) and (34).

According to 37, the measure $\mu_{x_0}^{\mathrm{PS}}$ is D-Federer for any $D > 0$. Thus, we may now use (33) and Proposition 4.5 to deduce

$$\begin{aligned}
&\mu_{x_0}^{\mathrm{PS}}(B_U(T/s) \cap \mathcal{H}_i(R)) \\
&\quad = \mu_{x_0}^{\mathrm{PS}}(\{\mathbf{t} \in B_U(T/s) : \text{ there exists } \gamma \in \Gamma, 1 \leq i \leq q \text{ such that } \|f(\mathbf{t})\gamma v_i\| < e^{-R}\}) \\
&\quad \ll e^{-R\beta} \mu_{x_0}^{\mathrm{PS}}(B_U(T/s)x_0),
\end{aligned}$$

where the implied constant depends on n and Γ. □

5. *Friendliness properties of the PS measure*

In this section, we prove several key properties of the PS measure, including that slightly enlarging a ball does not increase the measure too much and that scaling the size of the ball has a bounded multiplicative increase on the measure. Note that the results in this section hold for any Γ that is geometrically finite; we do not require Assumption 1.1. In the setting that all cusps have maximal rank, stronger statements hold. See the appendix, specifically §A.3, for more details.

The main results in this section are the following, which both establish control over the measure of a slightly enlarged ball. Many technical details of the proofs are hidden in Proposition 5.4, which is proved in §A.

THEOREM 5.1. *There exists a constant $\alpha' = \alpha'(\Gamma) > 0$ such that for every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine, for every $0 < s \leq T^{\varepsilon/(1-\varepsilon)}$, every $0 < \xi \ll_\Gamma 1$, and every $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\frac{\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(\xi + T))}{\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T))} - 1 \ll_\Gamma \xi^{\alpha'}.$$

THEOREM 5.2. *There exist $\alpha' = \alpha'(\Gamma) > 0$, $\theta' = \theta'(\Gamma) \geq \alpha'$, $\omega' = \omega'(\Gamma) \geq 2\delta_\Gamma$, such that for any $g \in G$ with $g^- \in \Lambda(\Gamma)$ and $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(g\Gamma)}$, we have that*

$$\frac{\nu(P_{\xi+\eta}g)}{\nu(P_\eta g)} - 1 \ll_\Gamma e^{\omega' \operatorname{height}(g\Gamma)} \frac{\xi^{\alpha'}}{\eta^{\theta'}}.$$

Theorem 5.2 will be obtained as a corollary of the following.

PROPOSITION 5.3. *There exist constants $\alpha = \alpha(\Gamma) > 0$, $\theta = \theta(\Gamma) \geq \alpha$, and $\omega = \omega(\Gamma) \geq 2\delta_\Gamma$ such that for $x \in G/\Gamma$, which satisfies $x^+ \in \Lambda(\Gamma)$, and $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(x)}$, we have*

$$\frac{\mu^{\mathrm{PS}}_x(B_U(\xi + \eta))}{\mu^{\mathrm{PS}}_x(B_U(\eta))} - 1 \ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta}.$$

We first show how to obtain Theorem 5.2 from Proposition 5.3.

Proof of Theorem 5.2 assuming Proposition 5.3. Using the product structure of ν, we can write

$$\nu(P_\eta g) = \int_{A_\eta} \int_{M_\eta} \mu^{\mathrm{PS}\,-}_g(B_{\tilde{U}}(\eta))\, dm\, ds.$$

Then, by an analogous statement to Proposition 5.3 for $\mu^{\mathrm{PS}\,-}$, there exists a constant $c_0 = c_0(\Gamma) > 0$ such that

$$\begin{aligned}
\nu(P_{\eta+\xi}g) &= \int_{A_{\xi+\eta}} \int_{M_{\xi+\eta}} \mu^{\mathrm{PS}\,-}_g(B_{\tilde{U}}(\xi+\eta))\, dm\, ds \\
&\leq \int_{A_{\xi+\eta}} \int_{M_{\xi+\eta}} \mu^{\mathrm{PS}\,-}_g(B_{\tilde{U}}(\eta)) \left[1 + c_0 \frac{\xi^\alpha}{\eta^\theta} e^{\omega \operatorname{height}(g\Gamma)}\right] dm\, ds
\end{aligned}$$

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$$= \left[1 + c_0 \frac{\xi^\alpha}{\eta^\theta} e^{\omega \,\mathrm{height}(g\Gamma)}\right]\left[\frac{(\xi+\eta)^{1/2(n-1)(n-2)+1}}{\eta^{1/2(n-1)(n-2)+1}} \nu(P_\eta g)\right]$$
$$\leq \left[1 + c_0 \frac{\xi^\alpha}{\eta^\theta} e^{\omega \,\mathrm{height}(g\Gamma)}\right]\left[1 + c_1 \frac{\xi}{\eta}\right] \nu(P_\eta g),$$

where $c_1 > 0$ is an absolute constant (which depends only on n) arising from the binomial theorem. Therefore,

$$\frac{\nu(P_{\xi+\eta} g)}{\nu(P_\eta g)} - 1 \ll_\Gamma e^{\omega \,\mathrm{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta} \cdot \frac{\xi}{\eta} + \frac{\xi}{\eta} + e^{\omega \,\mathrm{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}. \tag{46}$$

Since $\xi < \eta$, the first term on the left-hand side of (46) is dominated by the last term, and so

$$\frac{\nu(P_{\xi+\eta} g)}{\nu(P_\eta g)} - 1 \ll_\Gamma \frac{\xi}{\eta} + e^{\omega \,\mathrm{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}.$$

Since $e^{\omega \,\mathrm{height}(g\Gamma)} \geq 1$, if we define

$$\alpha' = \min\{1, \alpha\}, \quad \theta' = \max\{1, \theta\},$$

then both terms are dominated by

$$e^{\omega \,\mathrm{height}(g\Gamma)} \frac{\xi^{\alpha'}}{\eta^{\theta'}},$$

which completes the proof. □

The following result, showing that the PS measure is not concentrated near hyperplanes, is proved in the appendix to improve the readability of this section. See Proposition A.11 for the proof. This result builds upon the work of Das *et al.* in [**5**], where it is shown that the PS density ν_o is *friendly* when Γ is geometrically finite.

For a hyperplane $L \subset U \cong \mathbb{R}^{n-1}$ and $\xi > 0$, define

$$\mathcal{N}_U(L, \xi) := \{u_{\mathbf{t}} y : y \in L, \mathbf{t} \in B_U(\xi)\}.$$

PROPOSITION 5.4. *Let Γ be geometrically finite and Zariski dense. There exist constants $\alpha = \alpha(\Gamma) > 0$, $\omega = \omega(\Gamma) \geq 0$, and $\theta = \theta(\Gamma) > \alpha$ satisfying the following: for any $x \in G/\Gamma$ with $x^+ \in \Lambda(\Gamma)$, and for every $\xi > 0$ and $0 < \eta \ll_\Gamma e^{-\mathrm{height}(x)}$, we have that for every hyperplane L,*

$$\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L, \xi) \cap B_U(\eta)) \ll_\Gamma e^{\omega \,\mathrm{height}(x)} \frac{\xi^\alpha}{\eta^\theta} \mu_x^{\mathrm{PS}}(B_U(\eta)).$$

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. It follows from the geometry of $B_U(\xi+\eta)x - B_U(\eta)x$ that there exist hyperplanes $L_1, \ldots, L_m$, where m only depends on n, such that

$$B_U(\xi+\eta)x - B_U(\eta)x \subseteq \bigcup_{i=1}^m \mathcal{N}_U(L_i, 2\xi).$$

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For any $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(x)}$, we have that

$$\begin{aligned}\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} - 1 &= \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta) - B_U(\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} \\ &\le \sum_{i=1}^{m} \frac{\mu_x^{\mathrm{PS}}(\mathcal{N}(L_i,\xi) \cap B(x,\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} \quad \text{by Proposition 5.4} \\ &\ll_\Gamma m e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta} \cdot \frac{\mu_x^{\mathrm{PS}}(B_U(2\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))}.\end{aligned}$$

By (37), μ_x^{PS} is D-Federer (see Corollary A.8 for more detail), in particular,

$$\mu_x^{\mathrm{PS}}(B_U(2\eta)) \ll_\Gamma \mu_x^{\mathrm{PS}}(B_U(\eta)).$$

Thus, we obtain

$$\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} - 1 \ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta},$$

and relabeling the constants completes the proof. □

In (37), we saw that μ_x^{PS} is Federer when $x \in \operatorname{supp} m^{\mathrm{BMS}}$. Below, we show that μ_x^{PS} satisfies a similar condition for sufficiently large balls when x is Diophantine, but not necessarily a BMS point.

COROLLARY 5.5. *There exists a constant $\sigma = \sigma(\Gamma) \ge \delta_\Gamma$ such that for every $c \ge 1$ and every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine, if $T \gg_{\Gamma,\varepsilon} s_0$, then*

$$\mu_x^{\mathrm{PS}}(B_U(cT)) \ll_\Gamma c^\sigma \mu_x^{\mathrm{PS}}(B_U(T)).$$

Proof. By Lemma 3.8, for some $T_0 \gg_{\Gamma,\varepsilon} s_0$, there exists

$$y \in B_U(T_0)x \cap \operatorname{supp} m^{\mathrm{BMS}}.$$

Then for $T \ge T_0$, we have

$$B_U(T - T_0)y \subseteq B_U(T)x \subseteq B_U(T + T_0)y.$$

Since $c \ge 1$, we therefore have that for $T \ge 2T_0$,

$$\begin{aligned}\mu_x^{\mathrm{PS}}(B_U(cT)) &\le \mu_y^{\mathrm{PS}}(B_U(cT + T_0)) \\ &\le \mu_y^{\mathrm{PS}}(B_U((c+1)T)) \\ &\ll_\Gamma (c+1)^\sigma \mu_y^{\mathrm{PS}}(B_U(T/2)) \quad \text{by (37)} \\ &\ll_\Gamma (c+1)^\sigma \mu_y^{\mathrm{PS}}(B_U(T - T_0)) \\ &\ll_\Gamma (c+1)^\sigma \mu_x^{\mathrm{PS}}(B_U(T)) \\ &\ll_\Gamma (2c)^\sigma \mu_x^{\mathrm{PS}}(B_U(T)) \quad \text{since } c \ge 1 \\ &\ll_\Gamma c^\sigma \mu_x^{\mathrm{PS}}(B_U(T)).\end{aligned}$$

□

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Remark 5.6. Observe that if x is (ε, s_0)-Diophantine and $T \gg_{\Gamma,\varepsilon} s_0$, then T is sufficiently large to use Corollary 5.5 on $a_{-s}x$ for $s > 0$. To see this, observe that in the notation of the proof of Corollary 5.5, T_0 is such that for any (ε, s_0)-Diophantine point x, there exists $y \in \operatorname{supp} m^{\mathrm{BMS}}$ and $\mathbf{t} \leq T_0$ so that $x = u_{\mathbf{t}}y$. Then

$$a_{-s}x = a_{-s}u_{\mathbf{t}}y = u_{e^{-s}\mathbf{t}}a_{-s}y.$$

Thus, the distance to the nearest BMS point in the U orbit shrinks, and so T is still sufficiently large.

PROPOSITION 5.7. *Let $H_R = \{y \in G/\Gamma : \operatorname{height}(y) \leq R\}$. There exist constants $\alpha = \alpha(\Gamma) > 0$, and $\omega = \omega(\Gamma) \geq 0$ such that for every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine and for every $0 < \xi < 1/2$, and $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\frac{\mu_x^{\mathrm{PS}}((B_U(\xi + T) \cap H_R) - (B_U(T) \cap H_R))}{\mu_x^{\mathrm{PS}}(B_U(T))} \ll_\Gamma e^{\omega R}\xi^\alpha.$$

Proof. Let $T_0 \gg_{\Gamma,\varepsilon} s_0$ satisfy the conclusion of Lemma 3.8. For $T \geq T_0$, let

$$E_T := \mathcal{N}(L, \xi) \cap B_U(T) \cap H_R \cap \operatorname{supp} m^{\mathrm{BMS}},$$

and observe that $\mu_x^{\mathrm{PS}}(E_T) = \mu_x^{\mathrm{PS}}(\mathcal{N}(L, \xi) \cap B_U(T) \cap H_R)$.

Let $c_1 = c_1(\Gamma) > 0$ be the implied constant in Proposition 5.4. Fix $r = c_1 e^{-R}$ and let $\{u_1, \ldots, u_k\}$ be a maximal $r/2$-separated set in $E_{T-r/4}$. Then,

$$E_T \subseteq \bigcup_{i=1}^{k} B_U(r)u_i.$$

Note also that by (37), we have that there exists a constant $c_2 = c_2(\Gamma) > 0$ such that for all u_i,

$$\mu_{u_i}^{\mathrm{PS}}(B_U(r)) = \mu_{u_i}^{\mathrm{PS}}(B_U(8(r/8)) \leq c_2\mu_{u_i}^{\mathrm{PS}}(B_U(r/8)). \tag{47}$$

Therefore,

$$\begin{aligned}
&\mu_x^{\mathrm{PS}}(\mathcal{N}(L, \xi) \cap B_U(T) \cap H_R) \\
&\quad \leq \sum_{i=1}^{k} \mu_{u_i}^{\mathrm{PS}}(\mathcal{N}(L, \xi) \cap B_U(r)) \\
&\quad \ll_\Gamma e^{\omega R}\frac{\xi^\alpha}{r^\theta} \sum_{i=1}^{k} \mu_{u_i}^{\mathrm{PS}}(B_U(r)) \quad \text{by Proposition 5.4} \\
&\quad \ll_\Gamma e^{(\omega+\theta)R}\xi^\alpha \sum_{i=1}^{k} \mu_{u_i}^{\mathrm{PS}}(B_U(r/8)) \quad \text{by (47)} \\
&\quad \ll_\Gamma e^{(\omega+\theta)R}\xi^\alpha \mu_x^{\mathrm{PS}}(B_U(T+1)) \quad \text{as the } 1/8 \text{ balls are disjoint.}
\end{aligned}$$

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By Corollary 5.5, there exists $\sigma = \sigma(\Gamma) \geq \delta_\Gamma$ so that

$$\mu_x^{\mathrm{PS}}(B_U(T+1)) \subseteq \mu_x^{\mathrm{PS}}(B_U(2T)) \ll_\Gamma 2^\sigma \mu_x^{\mathrm{PS}}(B_U(T)).$$

Let

$$\omega' = \omega + \theta.$$

It follows from the geometry of $B_U(\xi + T)x - B_U(T)x$ that there exist $L_1, \ldots, L_m$, where m only depends on n, such that

$$B_U(\xi + T)x - B_U(T)x \subseteq \bigcup_{i=1}^m \mathcal{N}_U(L_i, 2\xi).$$

Thus, we also have

$$(B_U(\xi + T)x - B_U(T)x) \cap H_R \subseteq \bigcup_{i=1}^m \mathcal{N}_U(L_i, 2\xi).$$

We arrive at

$$\begin{aligned}\frac{\mu_x^{\mathrm{PS}}((B_U(\xi+T)\cap H_R) - (B_U(T)\cap H_R))}{\mu_x^{\mathrm{PS}}(B_U(T))} &\leq \sum_{i=1}^m \frac{\mu_x^{\mathrm{PS}}(\mathcal{N}(L_i, 2\xi) \cap B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))} \\ &\ll_\Gamma m e^{\omega' R} \xi^\alpha \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))}.\end{aligned}$$

By Corollary 5.5 again, we conclude that

$$\frac{\mu_x^{\mathrm{PS}}((B_U(\xi+T)\cap H_R) - (B_U(T)\cap H_R))}{\mu_x^{\mathrm{PS}}(B_U(T))} \ll_\Gamma e^{\omega' R} \xi^\alpha,$$

which completes the proof. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Observe that by Lemma 3.6, for any $R > R_0$,

$$\begin{aligned}\mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T)) &= \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T) \cap H_{R-R_0}) + \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T) \cap \mathcal{X}(R)), \\ \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T+\xi)) &= \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T+\xi) \cap H_{R-R_0}) + \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T+\xi) \cap \mathcal{X}(R)).\end{aligned} \tag{48}$$

By Theorem 4.1, for $T \gg_{\Gamma,\varepsilon} s_0$, $0 < s \leq T^{\varepsilon/(1-\varepsilon)}$, and any $R \geq R_0$,

$$\begin{aligned}\mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T+\xi) \cap \mathcal{X}(R)) &= \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U((s(T+\xi)/s) \cap \mathcal{X}(R)) \\ &\ll_\Gamma \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T+\xi))e^{-\beta R} \\ &\ll_\Gamma \mu_{a_{-\log s}x}^{\mathrm{PS}}(B_U(T))e^{-\beta R} \quad \text{by Corollary 5.5.}\end{aligned}$$

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Observe that use of Corollary 5.5 is justified if $T \gg_{\Gamma,\varepsilon} s_0$ by Remark 5.6. Similarly, by Proposition 5.7 and the same reasoning as in Remark 5.6, for $T \gg_{\Gamma,\varepsilon} s_0$, we have

$$\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi) \cap H_{R-R_0}) - \mu^{\mathrm{PS}}_{a_{-s}x}(B_U(T) \cap H_{R-R_0}) \ll_\Gamma e^{\omega R}\xi^\alpha \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)).$$

Putting this together with (48), we conclude

$$\begin{aligned}
&\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi)) - \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)) \\
&\quad \ll_\Gamma e^{\omega R}\xi^\alpha \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)) + 2e^{-\beta R}\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)) \\
&\quad \ll_\Gamma (e^{\omega R}\xi^\alpha + 2e^{-\beta R})\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)).
\end{aligned}$$

Taking $R = -\alpha/(\omega+\beta)\log\xi$ implies the result, provided that ξ is sufficiently small so that this is larger than R_0. Note that since α, ω, β, R_0 are all constants depending only on Γ, this is equivalent to requiring $\xi \ll_\Gamma 1$. □

6. *Proof of Theorem 1.5*

In this section, we keep the notation of §3.2. In particular, d denotes the hyperbolic distance, height is the height of a point in the convex core into the cusps, and $\mathcal{C}_0$ is the fixed compact set in G/Γ which is defined in §3.2.

We will first prove the following proposition, which is a form of Theorem 1.5 for G. Theorem 1.5 will follow by a partition of unity argument.

PROPOSITION 6.1. *There exist $\kappa = \kappa(\Gamma)$ and $\ell = \ell(\Gamma)$ which satisfy the following: let $0 < r < 1$, $\psi \in C_c^\infty(G)$ supported on an admissible box, and $f \in C_c^\infty(B_U(r))$. Then, there exists $c = c(\Gamma, \operatorname{supp}\psi) > 0$ such that for any $g \in \operatorname{supp}\tilde{m}^{\mathrm{BMS}}$, and $s \gg_\Gamma$* height$(g\Gamma)$, *we have*

$$\begin{aligned}
&\left|\sum_{\gamma\in\Gamma}\int_U \psi(a_s u_{\boldsymbol{t}} g\gamma) f(\boldsymbol{t})\, d\mu^{\mathrm{PS}}_g(\boldsymbol{t}) - \mu^{\mathrm{PS}}_g(f)\tilde{m}^{\mathrm{BMS}}(\psi)\right| \\
&\quad < cS_\ell(\psi)S_\ell(f)e^{-\kappa s}\mu^{\mathrm{PS}}_g(B_U(1)).
\end{aligned}$$

Proof. Without loss of generality, assume that f and ψ are non-negative functions.
Step 1: Setup and approximations.

Let κ', ℓ' satisfy the conclusion of Assumption 1.1, and let $\ell > \ell'$ satisfy the conclusion of Lemma 3.12. Observe that ℓ can be increased if necessary while maintaining this property.

Because ψ is supported on an admissible box, there exists $0 < \eta_0 < 1/2$ (depending on supp ψ) such that $G_{3\eta_0}$ supp ψ is still an admissible box. For $0 < \eta < \eta_0$, let $\psi_{\eta,\pm}$ satisfy the conclusion of Lemma 3.12 for G, 3η, and ψ. In particular, for all small $\eta > 0$,

$$S_{\ell'}(\psi_{\eta,\pm}) \ll_{\operatorname{supp}\psi} \eta^{-2\ell} S_\ell(\psi). \tag{49}$$

Since ψ is uniformly continuous and the BMS measure is finite, we may deduce from Lemma 3.12(2) that

$$|\tilde{m}^{\mathrm{BMS}}(\psi_{\eta,\pm}) - \tilde{m}^{\mathrm{BMS}}(\psi)| \ll_{\operatorname{supp}\psi,\Gamma} \eta S_\ell(\psi). \tag{50}$$

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According to Lemma 3.1, for any $p \in P_\eta$, there exists $\rho_p : B_U(1) \to B_U(1 + O(\eta))$ that is a diffeomorphism onto its image and a constant $D = D(\eta) < 3\eta$ such that

$$u_{\mathbf{t}} p^{-1} \in P_D u_{\rho_p(\mathbf{t})}. \tag{51}$$

Step 2: Assuming that f is supported on a small ball.

We start by proving that there exists $\kappa > 0$ such that if $f \in C_c^\infty(B_U(r_1))$, where $r_1 \leq \mathrm{inj}(g)$, then for $s > 0$,

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_U \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \tilde{m}^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f)\\ \ll_{\Gamma,\mathrm{supp}\,\psi} S_\ell(\psi)S_\ell(f)e^{-2\kappa s}\mu_g^{\mathrm{PS}}(B_U(1)).\end{aligned} \tag{52}$$

For any $s > 0$ and $\gamma \in \Gamma$, from (51), we have that

$$\begin{aligned}&\int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t})\\ &= \frac{1}{\nu(P_\eta g)} \int_{P_\eta g} \int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} p^{-1} pg\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t})\, d\nu(pg)\\ &\leq \frac{1}{\nu(P_\eta g)} \int_{P_\eta g} \int_{B_U(r_1)} \psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})} pg\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t})\, d\nu(pg),\end{aligned}$$

where the last inequality follows since $a_s P_{3\eta} a_{-s} \subset P_{3\eta}$ for any positive s.

Step 2.1: Use the product structure of the BMS measure. For any $p \in P_\eta$, $(u_{\mathbf{t}} g)^+ = (u_{\rho_p(\mathbf{t})} pg)^+$, the measures $d\mu_g^{\mathrm{PS}}(\mathbf{t})$ and $d(\rho_{p*}\mu_{pg}^{\mathrm{PS}}(\mathbf{t})) = d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))$ are absolutely continuous with each other, and the Radon–Nikodym derivative at $\mathbf{t}$ is given by

$$\frac{d\mu_g^{\mathrm{PS}}(\mathbf{t})}{d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))} = e^{\delta_\Gamma \beta_{(u_{\mathbf{t}} g)^+}(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o))}. \tag{53}$$

Let $0 < \xi < \eta$. Let $\chi_{\eta,\xi}$ satisfy the conclusion of Lemma 3.10 for $H = P$, $\xi_1 = \eta - \xi$, $\xi_2 = \xi$, and g. Let $\varphi_{\eta,g}$ be the function defined on $B_U(1)P_\eta g$ given by

$$\varphi_{\eta,g}(u_{\rho_p(\mathbf{t})} pg) := \frac{f(\mathbf{t})\chi_{\eta,\xi}(pg)}{\nu(P_\eta g) e^{\delta_\Gamma \beta_{(u_{\mathbf{t}} g)^+}(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o))}}.$$

We will need a bound on $S_\ell(\varphi_{\eta,g})$. To that end, note that

$$\begin{aligned}|\beta_{(u_{\mathbf{t}} g)^+}(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o))| &\leq d(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o))\\ &= d(g(o), u_{-\mathbf{t}} u_{\rho_p(\mathbf{t})} pg(o)).\end{aligned}$$

Since $u_{-\mathbf{t}} u_{\rho_p(\mathbf{t})} p \in G_{5\eta}$, the above is bounded by some absolute constant (depending only on Γ) for all $\eta < \frac{1}{2}$. Observe that this bound holds on the support of $f(\mathbf{t})\chi_{\eta,\xi}(pg)$. Moreover, because the Busemann function is Lipschitz, all of its derivatives are bounded.

It then follows immediately from the product rule and the definition of the Sobolev norm that

$$S_\ell(f(\mathbf{t})\chi_{\eta,\xi}(pg) \exp(-\delta_\Gamma \beta_{(u_{\mathbf{t}} g)^+}(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o)))) \ll_{\Gamma,\ell} S_\ell(f(\mathbf{t})\chi_{\eta,\xi}(pg)). \tag{54}$$

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By [17, Lemma 2.4.7(a)], Lemma 3.10, (54), and Lemma 3.9, we have

$$\begin{aligned}
S_\ell(\varphi_{\eta,g}) &\ll_{\Gamma,\ell} \nu(P_\eta g)^{-1} S_\ell(f) S_\ell(\chi_{\eta,\xi}) \\
&\ll_{\Gamma,\ell} \eta^{-(\delta_\Gamma+1/2(n-1)(n-2)+1)} e^{(\delta_\Gamma - k_2(x,\eta)) d(\pi(\mathcal{C}_0),\pi(a_{\log\eta} g))} S_\ell(\chi_{\eta,\xi}) S_\ell(f) \\
&\ll_{\Gamma,\ell} e^{\delta_\Gamma(|\log\eta|+\mathrm{height}(g\Gamma))} \eta^{-(\delta_\Gamma+1/2(n-1)(n-2)+1)} \eta^{n-1} \xi^{-\ell-(n-1)/2} S_\ell(f) \\
&\ll_{\Gamma,\ell} e^{\delta_\Gamma\,\mathrm{height}(g\Gamma)} \eta^{-(2\delta_\Gamma+1/2(n-1)(n-2)+1)} \eta^{n-1} \xi^{-\ell-(n-1)/2} S_\ell(f) \\
&\ll_{\Gamma,\ell} e^{\delta_\Gamma\,\mathrm{height}(g\Gamma)} \eta^{4n-(1/2)n^2-3-2\delta_\Gamma} \xi^{-\ell-(n-1)/2} S_\ell(f).
\end{aligned} \tag{55}$$

Note that the dependence on ℓ arises from the exponential of the Busemann function in the denominator.

Also, using the product structure of $\tilde{m}^{\mathrm{BMS}}$ in (25), we get

$$\begin{aligned}
&\frac{1}{\nu(P_\eta g)} \int_{P_\eta g} \int_{B_U(r_1)} \psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})} pg\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t})\, d\nu(pg) \\
&= \frac{1}{\nu(P_\eta g)} \int_{P_\eta g} \int_{B_U(r_1)} \psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})} pg\gamma) f(\mathbf{t}) \frac{d\mu_g^{\mathrm{PS}}(\mathbf{t})}{d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))}\, d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))\, d\nu(pg) \\
&\le \int_G \psi_{\eta,+}(a_s h\gamma) \varphi_{\eta,g}(h)\, d\tilde{m}^{\mathrm{BMS}}(h).
\end{aligned}$$

Step 2.2: Use the exponential mixing assumption.

By defining $\Psi_{\eta,+}(h\Gamma) = \sum_{\gamma\in\Gamma} \psi_{\eta,+}(h\gamma)$ and $\Phi_{\eta,g}(h\Gamma) := \sum_{\gamma\in\Gamma} \varphi_{\eta,g}(h\gamma)$, we obtain

$$\sum_{\gamma\in\Gamma} \int_G \psi_{\eta,+}(a_s h\gamma)\varphi_{\eta,g}(h)\, d\tilde{m}^{\mathrm{BMS}}(h) \le \int_X \Psi_{\eta,+}(a_s x)\Phi_{\eta,g}(x)\, dm^{\mathrm{BMS}}(x)$$

for any positive s. Note that

$$S_{\ell'}(\Psi_{\eta,+}) = S_{\ell'}(\psi_{\eta,+}) \quad \text{and} \quad S_{\ell'}(\Phi_{\eta,g}) = S_{\ell'}(\varphi_{\eta,g}). \tag{56}$$

In particular, (49) and (55) imply

$$\begin{aligned}
&S_{\ell'}(\Psi_{\eta,+}) \ll_{\mathrm{supp}\,\psi} \eta^{-2\ell} S_\ell(\psi) \quad \text{and} \\
&S_{\ell'}(\Phi_{\eta,g}) \ll_\Gamma e^{\delta_\Gamma\,\mathrm{height}(g\Gamma)} \eta^{4n-(1/2)n^2-3-2\delta_\Gamma} \xi^{-\ell-(n-1)/2} S_\ell(f).
\end{aligned} \tag{57}$$

By Assumption 1.1,

$$\begin{aligned}
&\int \Psi_{\eta,+}(a_s x)\Phi_{\eta,g}(x)\, dm^{\mathrm{BMS}}(x) - m^{\mathrm{BMS}}(\Psi_{\eta,+}) m^{\mathrm{BMS}}(\Phi_{\eta,g}) \\
&\quad \ll_\Gamma S_{\ell'}(\Psi_{\eta,+}) S_{\ell'}(\Phi_{\eta,g}) e^{-\kappa' s}.
\end{aligned}$$

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Then, by (57), there exists $c_1 = c_1(\Gamma, \operatorname{supp} \psi)$ such that

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} &\psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &< m^{\mathrm{BMS}}(\Psi_{\eta,+}) m^{\mathrm{BMS}}(\Phi_{\eta,g}) \\ &\quad + c_1 e^{\delta_\Gamma \operatorname{height}(g\Gamma)} \eta^{4n-(1/2)n^2-3-2\delta_\Gamma} \xi^{-\ell-(n-1)/2} S_\ell(\psi) S_\ell(f) e^{-\kappa' s}.\end{aligned}$$

Step 2.3: Rewrite in terms of ψ and f.

Using Lemma 3.10 and (53), one can calculate

$$\begin{aligned}m^{\mathrm{BMS}}(\Phi_{\eta,g}) &= \int_G \varphi_{\eta,g}(h)\, d\tilde{m}^{\mathrm{BMS}}(h) \\ &= \frac{1}{\nu(P_\eta g)} \int_{Pg} \int_U \frac{f(\mathbf{t})\chi_{\eta,\xi}(p)}{e^{\delta_\Gamma \beta_{(u_{\mathbf{t}} g)^+}(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o))}}\, d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))\, d\nu(pg) \\ &= \frac{1}{\nu(P_\eta g)} \int_{Pg} \int_U f(\mathbf{t})\chi_{\eta,\xi}(p)\, d\mu_g^{\mathrm{PS}}(\mathbf{t})\, d\nu(pg) \\ &\le \frac{\nu(P_{\eta+\xi} g)}{\nu(P_\eta g)} \int_{B_U(r_1)} f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}).\end{aligned}$$

Thus, by Theorem 5.2, there exist $\alpha, \theta, \omega, c_0 > 0$ depending only on Γ such that for any $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(g\Gamma)}$,

$$\begin{aligned}m^{\mathrm{BMS}}(\Phi_{\eta,g}) &\le \bigg(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\bigg) \int_{B_U(r_1)} f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &= \bigg(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\bigg) \int_{B_U(r_1)} f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &= \bigg(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\bigg) \mu_g^{\mathrm{PS}}(f).\end{aligned}$$

Using (50), we get that there exists $c_3 = c_3(\Gamma, \operatorname{supp} \psi)$ such that

$$\begin{aligned}m^{\mathrm{BMS}}(\Psi_{\eta,+}) &\le \int_G \psi_{\eta,+}(g)\, d\tilde{m}^{\mathrm{BMS}}(g) \\ &< \tilde{m}^{\mathrm{BMS}}(\psi) + c_3 \eta S_\ell(\psi).\end{aligned}$$

To summarize, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} &\psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &\le \frac{1}{\nu(P_\eta g)} \sum_{\gamma\in\Gamma} \int_{P_\eta g} \int_{B_U(r_1)} \psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})} pg\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t})\, d\nu(pg) \\ &\le \sum_{\gamma\in\Gamma} \int_G \psi_{\eta,+}(a_s h\gamma)\varphi_{\eta,g}(h)\, d\tilde{m}^{\mathrm{BMS}}(h)\end{aligned}$$

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$$\leq \int_X \Psi_{\eta,+}(a_s x)\Phi_{\eta,g}(x)\, dm^{\mathrm{BMS}}(x)$$
$$< m^{\mathrm{BMS}}(\Psi_{\eta,+})m^{\mathrm{BMS}}(\Phi_{\eta,g}) + c_1\eta^{4n-(1/2)n^2-3-\delta_\Gamma-2\ell}\xi^{-\ell-(n-1)/2}S_\ell(\psi)S_\ell(f)e^{-\kappa' s}$$
$$< (\tilde{m}^{\mathrm{BMS}}(\psi) + c_3\eta S_\ell(\psi))\bigg(\bigg(1 + c_2 e^{\omega\ \mathrm{height}(g\Gamma)}\frac{\xi^\alpha}{\eta^\theta}\bigg)\mu_g^{\mathrm{PS}}(f)\bigg)$$
$$+ c_1 e^{\delta_\Gamma\ \mathrm{height}(g\Gamma)}\eta^{4n-(1/2)n^2-3-2\delta_\Gamma}\xi^{-\ell-(n-1)/2}S_\ell(\psi)S_\ell(f)e^{-\kappa' s}.$$

It follows from the proof of Lemma 3.12 that $\tilde{m}^{\mathrm{BMS}}(\psi) \ll_{\mathrm{supp}\,\psi} S_\ell(\psi)$ and $\mu_g^{\mathrm{PS}}(f) \ll S_\ell(f)\mu_g^{\mathrm{PS}}(B_U(1))$. Then, using Proposition 3.4, we arrive at

$$\sum_{\gamma\in\Gamma}\int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \mu_g^{\mathrm{PS}}(f)\tilde{m}^{\mathrm{BMS}}(\psi)$$
$$\ll_\Gamma \bigg(e^{\omega\ \mathrm{height}(g\Gamma)}\frac{\xi^\alpha}{\eta^\theta} + e^{\delta_\Gamma\ \mathrm{height}(g\Gamma)}\eta^{4n-(1/2)n^2-3-2\delta_\Gamma}\xi^{-\ell-(n-1)/2}e^{-\kappa' s}\bigg)$$
$$\cdot S_\ell(\psi)S_\ell(f)\mu_g^{\mathrm{PS}}(B_U(1)).$$

Define

$$\kappa = \frac{3\alpha\theta\kappa'}{2\theta(2\ell+n-1) + 9\alpha(2\delta_\Gamma + 3 + n^2/2 - 4n)},$$

and note that by making ℓ larger if necessary, we guarantee $\kappa > 0$.

Recall from (22) that

$$e^{-\ \mathrm{height}(g\Gamma)} \ll_\Gamma \mathrm{inj}(g).$$

For $s \geq \max\{\theta,\omega\}\ \mathrm{height}(g\Gamma)/\kappa$, choose

$$\eta = e^{-\kappa s/\theta}, \quad \xi = e^{-4\kappa s/\alpha}. \tag{58}$$

Note that $\eta < \mathrm{inj}(g\Gamma)$ by choice of s, $\omega\,\mathrm{height}(g\Gamma) \leq \kappa s$, and $\xi < \eta$ since by Proposition 5.2, $\alpha < \theta$. By Proposition 5.2, we have $\omega > \delta_\Gamma$; therefore, δ_Γ $\mathrm{height}(g\Gamma) \leq \kappa s$. Note also that $\max\{\theta,\omega\}\ \mathrm{height}(g\Gamma)/\kappa \ll_\Gamma \mathrm{height}(g\Gamma)$.

With these choices, we obtain

$$e^{\omega\ \mathrm{height}(g\Gamma)}\bigg(\frac{\xi}{\eta^{\theta'}}\bigg)^{\alpha'} + e^{\delta_\Gamma\ \mathrm{height}(g\Gamma)}\eta^{4n-(1/2)n^2-3-2\delta_\Gamma}\xi^{-\ell-(n-1)/2}e^{-\kappa' s} \leq 2e^{-2\kappa s}. \tag{59}$$

In a similar way, using $\psi_{\eta,-}$, one can show a lower bound, proving (52).

Step 3: Covering argument for general *f*.

We now deduce the claim by decomposing f into a sum of functions, each defined on a ball of radius r_1 in U.

Let $u_1,\ldots,u_k$ and $\sigma_1,\ldots,\sigma_k \in C_c^\infty(B_U(r))$ satisfy the conclusion of Lemma 3.11 for $E = B_U(r)$ and r_1. For $1 \leq i \leq k$, let

$$f_i := f\sigma_i.$$

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Then, $f \leq \sum_{i-1}^{k} f_i$, and by Lemma 3.11 and [17, Lemma 2.4.7(a)],

$$S_\ell(f_i) \ll_\Gamma S_\ell(f) S_\ell(\sigma_i) \ll_\Gamma r_1^{-\ell+n-1} S_\ell(f). \tag{60}$$

Since each f_i is supported on $B_U(r_1)u_i$ for some $u_i \in B_U(1)$, by (52), we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} &\psi(a_s u_{\mathbf{t}} g\gamma) f_i(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \tilde{m}^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f_i) \\ &\ll_{\Gamma,\mathrm{supp}\,\psi} \mu_g^{\mathrm{PS}}(B_U(1)) S_\ell(\psi) S_\ell(f_i) e^{-2\kappa s}.\end{aligned}$$

Summing the above expressions for $i = 1, \ldots, k$, we get

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_{B_U(r)} &\psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t})\, d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \tilde{m}^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f) \\ &\ll k r_1^{-\ell+n-1} S_\ell(\psi) S_\ell(f) e^{-2\kappa s} \mu_g^{\mathrm{PS}}(B_U(1)) \\ &\ll \left(\frac{r}{r_1}\right)^{n-1} r_1^{-\ell+n-1} S_\ell(\psi) S_\ell(f) e^{-2\kappa s} \mu_g^{\mathrm{PS}}(B_U(1)) \\ &\ll r_1^{-\ell} S_\ell(\psi) S_\ell(f) e^{-2\kappa s} \mu_g^{\mathrm{PS}}(B_U(1)) \\ &\ll S_\ell(\psi) S_\ell(f) e^{-\kappa s} \mu_g^{\mathrm{PS}}(B_U(1)),\end{aligned}$$

where the first inequality is by Lemma 3.11, the second inequality follows from $r_1 = \mathrm{inj}(g) > e^{-\kappa s/\ell}$, the third is by (58) and because $r < 1$, and the implied constants depend on Γ and supp ψ.

As before, using similar arguments, one can show a lower bound, proving the claim. □

We will now use a partition of unity argument to prove Theorem 1.5. For the reader's convenience, we restate it in the following.

THEOREM 6.2. *There exist $\kappa = \kappa(\Gamma)$ and $\ell = \ell(\Gamma)$ which satisfy the following: for any $\psi \in C_c^\infty(X)$, there exists $c = c(\Gamma, \mathrm{supp}\,\psi) > 0$ such that for any $f \in C_c^\infty(B_U(r))$, $0 < r < 1$, $x \in \mathrm{supp}\, m^{\mathrm{BMS}}$, and $s \gg_\Gamma \mathrm{height}(x)$, we have*

$$\left| \int_U \psi(a_s u_t x) f(t)\, d\mu_x^{\mathrm{PS}}(t) - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BMS}}(\psi) \right| < c S_\ell(\psi) S_\ell(f) e^{-\kappa s}.$$

Proof. According to [26, Lemma 2.17], there exists an admissible box B_y around y for any $y \in X$. Then, $\{B_y : y \in \mathrm{supp}\,\psi\}$ is an open cover of the compact set supp ψ. Hence, there exists a minimal sub-cover $B_{y_1}, \ldots, B_{y_k}$. Using a similar construction to that in Lemma 3.11, there exist $\sigma_1, \ldots, \sigma_k$, a partition of unity for supp ψ, such that for $i = 1, \ldots, k$, we have $\sigma_i \in C_c^\infty(B_{y_i})$ and for $i = 1, \ldots, k$ and $m = 1, \ldots, \ell$,

$$|\sigma_i^{(m)}| \ll_{\mathrm{supp}\,\psi,\Gamma} 1 \tag{61}$$

(the implied constant depends on the chosen sub-cover).

Define $\psi_i = \psi\sigma_i$. Then

$$\psi = \sum_{i=1}^{k} \psi_i, \tag{62}$$

and by (61) and the product rule, we have

$$S_\ell(\psi_i) \ll_{\operatorname{supp}\psi,\Gamma} S_\ell(\psi). \tag{63}$$

According to Proposition 6.1 and Proposition 3.4, there exist $c = c(\Gamma, \operatorname{supp}\psi) > 0$, $\lambda = \lambda(\Gamma) > 1$ such that for $s \gg_\Gamma \operatorname{height}(x)$,

$$\begin{aligned}
&\int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} \\
&\quad = \sum_{i=1}^{k} \int_{B_U(r)} \psi_i(a_s u_{\mathbf{t}} x) f(\mathbf{t})\, d\mathbf{t} \\
&\quad \le \sum_{i=1}^{k} m^{\mathrm{BMS}}(\psi_i)\mu_x^{\mathrm{PS}}(f) + cS_\ell(\psi_i)S_\ell(f)e^{-\kappa s}\mu_x^{\mathrm{PS}}(B_U(1)) \\
&\quad \le \sum_{i=1}^{k} m^{\mathrm{BMS}}(\psi_i)\mu_x^{\mathrm{PS}}(f) + c\lambda S_\ell(\psi_i)S_\ell(f)e^{-\kappa s+(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))} \\
&\quad \ll_{\Gamma,\operatorname{supp}\psi} m^{\mathrm{BMS}}(\psi)\mu_x^{\mathrm{PS}}(f) + c\lambda S_\ell(\psi)S_\ell(f)e^{-\kappa s+(n-1-\delta_\Gamma)\operatorname{height}(x)},
\end{aligned}$$

where the last line follows by the definition of $\operatorname{height}(x)$, and (62) and (63). Moreover, we may assume that $s \ge 2(n-1-\delta_\Gamma)/\kappa \operatorname{height}(x)$ without changing the assumption $s \gg_\Gamma \operatorname{height}(x)$. Then,

$$e^{-\kappa s+(n-1-\delta_\Gamma)\operatorname{height}(x)} \ll_\Gamma e^{-\kappa s/2},$$

as desired. □

We will now use Theorem 1.5 to prove a similar result for the Haar measure. This will be necessary for the proof of Theorem 1.4. Note that such a result is proven in [23] under a spectral gap assumption on Γ, but we show here how to prove it whenever the frame flow is exponentially mixing.

THEOREM 6.3. *There exists $\kappa = \kappa(\Gamma) < 1$ and $\ell = \ell(\Gamma)$ that satisfy the following: let $0 < r < 1$, let $f \in C_c^\infty(B_U(r))$, and let $\psi \in C_c^\infty(X)$ be supported on an admissible box. Then there exists $c = c(\Gamma, \operatorname{supp}\psi) > 0$ such that for every $x \in \operatorname{supp} m^{\mathrm{BMS}}$ and $s \gg_{\Gamma,\operatorname{supp}\psi} \operatorname{height}(x)$,*

$$\left| e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t})\, d\mathbf{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \right| < cS_\ell(\psi)S_\ell(f)e^{-\kappa s}.$$

Proof
Step 1: Setup and approximations. Assume $s \gg_\Gamma \operatorname{height}(x)$, and let κ, ℓ' satisfy the conclusion of Theorem 1.5 and $\ell > \ell'$ satisfy the conclusion of Lemma 3.12.

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Since ψ is assumed to be supported on an admissible box, there exist $r_0, \eta, \varepsilon_0, \varepsilon_1 > 0$ (depending only on supp ψ) and $z \in X$ such that

$$\operatorname{supp} \psi = B_U(r_0)P_\eta z,$$

and

$$G_{\varepsilon_0} \operatorname{supp} \psi \subset B_U(r_0+\varepsilon_1)P_{\eta+\varepsilon_1}z,$$

where $B_U(r+\varepsilon_1)P_{\eta+\varepsilon_1}z$ is also an admissible box. Denote $\eta' = \eta + \varepsilon_1$ and $r_0' = r_0 + \varepsilon_1$.

Without loss of generality, assume that f is a non-negative function. Continuously extend ψ to $P_{\eta'}$ by defining $\psi = 0$ on $P_{\eta'} \setminus P_\eta$.

For $0 < \varepsilon < \varepsilon_0$, let $\psi_{\varepsilon,\pm}$ and $f_{\varepsilon,\pm}$ for Lemma 3.12 for G, ε, ψ and U, ε, f respectively. By Lemma 3.12,

$$S_{\ell'}(\psi_{\varepsilon,\pm}) \ll_{\Gamma,\operatorname{supp}(\psi)} \varepsilon^{-2\ell} S_\ell(\psi) \quad \text{and} \quad S_{\ell'}(f_{\varepsilon,\pm}) \ll_\Gamma \varepsilon^{-2\ell} S_\ell(f). \tag{64}$$

Moreover, by Lemma 3.12(2),

$$\|f_{\varepsilon,\pm} - f\|_\infty \leq \varepsilon S_\ell(f). \tag{65}$$

For $p \in P_{\eta'}$, define

$$\varphi(p) := \mu_{pz}^{\mathrm{PS}}(B_U(r_0')pz). \tag{66}$$

Step 1.1: Construct a smooth approximation to $1/\varphi$. Since the Busemann function is smooth and φ is bounded below by a positive quantity on $P_{\eta'}$ by Corollary 3.3, the mean value theorem implies that for any $0 < \varepsilon < \varepsilon_0$ and all $p, p' \in P_\varepsilon$, there exists a constant $d = d(\Gamma, \operatorname{supp} \psi)$ such that

$$\left| \frac{1}{\varphi(p)} - \frac{1}{\varphi(p')} \right| \leq \frac{d\varepsilon}{\varphi(p)}. \tag{67}$$

By Lemma 3.10, for any $\xi > 0$, there exists a non-negative smooth function χ_ξ with

$$1_{P_{\varepsilon-\xi}} \leq \chi_\xi \leq \mathbf{1}_{P_\varepsilon} \tag{68}$$

and $S_{\ell'}(\chi_\xi) \ll_{\Gamma,n} (\varepsilon - \xi/2)^{n-1}(\xi/2)^{-\ell'-(n-1)/2}$. Define

$$\sigma(p) := \frac{1}{\varphi} * \frac{\chi_\xi}{m(P_{\varepsilon-\xi})}, \tag{69}$$

where m denotes the probability Haar measure on P. Then, assuming $\varepsilon_0 < 1/2$ and $\xi \leq \varepsilon^2$, by (67), (68), and (69), we have that

$$\begin{aligned} \frac{1-d\varepsilon}{\varphi(p)} &\leq \frac{1}{m(P_{\varepsilon-\xi})} \int_{pP_{\varepsilon-\xi}} \frac{1}{\varphi(p')}\, dp' \\ &\leq \sigma(p) \\ &\leq \frac{1}{m(P_{\varepsilon-\xi})} \int_{pP_\varepsilon} \frac{1+d\varepsilon}{\varphi(p')}\, dp' \end{aligned} \tag{70}$$

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$$\leq \left(\frac{\varepsilon}{\varepsilon - \xi}\right)^n \frac{1 + d\varepsilon}{\varphi(p)}$$
$$\leq \frac{1 + d'\varepsilon}{\varphi(p)}, \tag{71}$$

for some absolute constant $d' > 0$.

For $upz \in B_U(r_0')P_{\eta'}z$ and $0 < \varepsilon < \varepsilon_0$, let

$$\Psi_{\varepsilon,\pm}(upz) = \sigma(p) \int_{Upz} \psi_{c_1\varepsilon,\pm}(u_{\mathbf{t}}pz)\, d\mathbf{t}.$$

Then, by (67),

$$\sup_{w \in G_\varepsilon} \Psi_{\varepsilon,\pm}(wupz) = \sup_{w \in P_\varepsilon} \sigma(wp) \int_{Uwpz} \psi_{c_1\varepsilon,+}(u_{\mathbf{t}}wpz)\, d\mathbf{t}$$
$$\leq (1 + d'\varepsilon)\Psi_{2\varepsilon,\pm}. \tag{72}$$

Step 2: Bounding with PS measure.

Let

$$P(f, \psi, x; s) = \{p \in P_\eta : a_s \operatorname{supp}(f)x \cap B_U(r_0)pz \neq \emptyset\}.$$

By [**23**, Lemma 6.2], there exists an absolute constant $c_1 > 0$ such that

$$e^{(n-1)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t})\, d\mathbf{t} \tag{73}$$
$$\leq (1 + c_1\varepsilon) \sum_{p \in P(f,\psi,x;s)} f_{c_1 e^{-s}\eta}(a_{-s}pz) \int_{Upz} \psi_{c_1\varepsilon,+}(u_{\mathbf{t}}pz)\, d\mathbf{t}.$$

It now follows from [**23**, Lemma 6.5], (70), and (72) that there exists an absolute constant $c_2 > 0$ such that

$$e^{-\delta_\Gamma s} \sum_{p \in P(f,\psi,x;s)} f_{c_1 e^{-s}\eta}(a_{-s}pz) \int_{Upz} \psi_{c_1\varepsilon,+}(u_{\mathbf{t}}pz)\, d\mathbf{t}$$
$$\leq \frac{(1 + c_2\varepsilon)(1 + d'\varepsilon)}{1 - d\varepsilon} \int_U \Psi_{2c_2\varepsilon,+}(a_s u_{\mathbf{t}} x) f_{(c_1+c_2)e^{-s}\varepsilon_0,+}(\mathbf{t})\, d\mu_x^{\mathrm{PS}}(\mathbf{t}).$$

Note that (70) is needed because our definition of $\Psi_{\varepsilon,+}$ is not identical to Ψ as defined in [**23**, Lemma 6.5]. The latter is bounded above by $1/(1 - d\varepsilon)\Psi_{\varepsilon,+}$ by (70).

Combining the above with (73), we get that there exist constants $c_3, c_4 = c_4(\Gamma, \operatorname{supp} \psi) > 0$ such that

$$e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t})\, d\mathbf{t}$$
$$\leq (1 + c_4\varepsilon) \int_U \Psi_{c_3\varepsilon,+}(a_s u_{\mathbf{t}} x) f_{c_3 e^{-s}\varepsilon_0,+}(\mathbf{t})\, d\mu_x^{\mathrm{PS}}(\mathbf{t}).$$

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It follows from Theorem 1.5 that for some constant $c_5 = c_5(\Gamma, \operatorname{supp} \psi) > 0$,

$$e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) \, d\mathbf{t} \\ \leq (1 + c_4\varepsilon)(\mu_x^{\mathrm{PS}}(f_{c_3 e^{-s}\varepsilon_0,+}) m^{\mathrm{BMS}}(\Psi_{c_3\varepsilon,+}) + c_5 S_{\ell'}(\Psi_{c_3\varepsilon,+}) S_{\ell'}(f_{c_3e^{-s}\varepsilon_0,+}) e^{-\kappa s}). \tag{74}$$

Step 3: Bounding the error terms. We now show how to bound the various error terms to obtain the desired conclusion.

To compute $m^{\mathrm{BMS}}(\Psi)$, we use (25), (65), and (71) to deduce that for some $c_6 = c_6(\Gamma, \operatorname{supp} \psi)$, if $\xi = \varepsilon^2$,

$$m^{\mathrm{BMS}}(\Psi_{c_3\varepsilon,+}) \\ \leq (1 + d'\varepsilon) \int_{P_{\eta'} z} \int_{B_U(r_0')} \frac{1}{\mu_{pz}^{\mathrm{PS}}(B_U(r_0') pz)} \int_{B_U(r_0') pz} \psi_{c_1\varepsilon,\pm}(u_{\mathbf{t}} pz) \, d\mathbf{t} \, d\mu_{pz}^{\mathrm{PS}}(\mathbf{t}) \, d\nu(pz) \\ \leq (1 + d'\varepsilon) \int_{P_{\eta'} z} \int_{B_U(r_0') pz} \psi_{c_1\varepsilon,\pm}(u_{\mathbf{t}} pz) \, d\mathbf{t} \, d\nu(pz) \\ \leq (1 + d'\varepsilon)(m^{\mathrm{BR}}(\psi) + c_6 \varepsilon S_\ell(\psi)). \tag{75}$$

By Proposition 3.4, if s is sufficiently large so that $r + c_3 e^{-s}\varepsilon_0 \leq 1$ (note that this requirement on s depends only on Γ and $\operatorname{supp} \psi$), we have that

$$\mu_x^{\mathrm{PS}}(B_U(r + c_3 e^{-s}\varepsilon_0)) \leq \mu_x^{\mathrm{PS}}(B_U(1)) \ll_\Gamma e^{(n-1-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(x))}. \tag{76}$$

Hence, by (65) and (76), we have

$$\mu_x^{\mathrm{PS}}(f_{c_3e^{-s}\varepsilon_0,+}) - \mu_x^{\mathrm{PS}}(f) \ll_\Gamma e^{-s}\varepsilon_0 S_\ell(f) \mu_x^{\mathrm{PS}}(B_U(r + c_3 e^{-s}\varepsilon_0)) \\ \ll_\Gamma e^{-s}\varepsilon_0 S_\ell(f) e^{(n-1-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(x))}. \tag{77}$$

According to [17, Lemma 2.4.7(a)] and (64), if $\xi = \varepsilon^2$ and

$$\varepsilon = e^{-\kappa s/2(n+4\ell)}, \tag{78}$$

then

$$S_{\ell'}(\Psi_{c_3\varepsilon,+}) \ll_\Gamma S_{\ell'}(\psi_{c_3\varepsilon,+}) S_{\ell'}(\sigma) \\ \ll_\Gamma (m(P_{\varepsilon-\xi}))^{-1} (\varepsilon - \xi/2)^{n-1} \xi^{-\ell'-(n-1)/2} \varepsilon^{-2\ell} S_\ell(\psi) \\ \ll \varepsilon^{-1-2\ell} \xi^{-\ell'-(n-1)/2} S_\ell(\psi) \\ \leq e^{\kappa s/2} S_\ell(\psi). \tag{79}$$

Using (74), (75), (77), and (79), we obtain

$$e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) \, d\mathbf{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \\ \leq (1 + c_4\varepsilon)[d'\varepsilon \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) + (1 + d'\varepsilon)\{c_6 \varepsilon \mu_x^{\mathrm{PS}}(f) S_\ell(\psi)$$

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$$+ (e^{-s}\varepsilon_0 m^{\mathrm{BR}}(\psi)S_\ell(f) + c_6 e^{-s}\varepsilon_0\varepsilon S_\ell(f)S_\ell(\psi))e^{(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}\}$$
$$+ c_8 S_\ell(\psi)S_\ell(f)e^{-\kappa s/2}]. \tag{80}$$

These remaining error terms can be controlled as follows. Using (76), we can deduce

$$\mu_x^{\mathrm{PS}}(f) \le \|f\|_\infty \mu_x^{\mathrm{PS}}(B_U(r)) \ll_\Gamma S_\ell(f)e^{(n-1-\delta_\Gamma)(\pi(\mathcal{C}_0),\pi(x))}. \tag{81}$$

We also have that

$$m^{\mathrm{BR}}(\psi) \ll_{\Gamma,\mathrm{supp}\,\psi} S_\ell(\psi). \tag{82}$$

Combining (80), (81), and (82) implies

$$e^{(n-1-\delta_\Gamma)s}\int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t})\, d\mathbf{t} - \mu_x^{\mathrm{PS}}(f)m^{\mathrm{BR}}(\psi)$$
$$\ll_{\Gamma,\mathrm{supp}\,\psi} S_\ell(\psi)S_\ell(f)\cdot[d'\varepsilon + c_8 e^{-\kappa s/2} +$$
$$(1+d'\varepsilon)(c_6\varepsilon(1+e^{-s}\varepsilon_0) + e^{-s}\varepsilon_0)e^{(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}]. \tag{83}$$

Finally, by the choice of ε in (78) and because we may assume without loss of generality that $\kappa < 1$, we obtain from (83) that there exists $\kappa' < 1$ such that

$$e^{(n-1-\delta_\Gamma)s}\int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t})\, d\mathbf{t} - \mu_x^{\mathrm{PS}}(f)m^{\mathrm{BR}}(\psi)$$
$$\ll_{\Gamma,\mathrm{supp}\,\psi} S_\ell(\psi)S_\ell(f)e^{-\kappa' s+(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}.$$

Recall that $d(\pi(\mathcal{C}_0),\pi(x)) = \mathrm{height}(x)$. Thus, if we assume that $s \ge 2(n-1-\delta_\Gamma)/\kappa'$ $\mathrm{height}(x)$ (which means $s \gg_\Gamma \mathrm{height}(x)$), then

$$e^{-\kappa' s+(n-1-\delta_\Gamma)\,\mathrm{height}(x)} \ll_\Gamma e^{-\kappa'/2s},$$

which completes the proof. □

7. *Proof of Theorem 1.3*

In this section, we prove Theorem 1.3, which is restated in the following for the reader's convenience. The proof relies on the quantitative non-divergence result in Theorems 4.1 and 1.5.

THEOREM 7.1. *For any $0 < \varepsilon < 1$ and $s_0 \ge 1$, there exist constants $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying: for every $\psi \in C_c^\infty(G/\Gamma)$, there exists $c = c(\Gamma, \mathrm{supp}\,\psi)$ such that every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine, and for every T with $T^{1-\varepsilon/2} \gg_\Gamma s_0$,*

$$\left|\frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))}\int_{B_U(T)} \psi(u_{\boldsymbol{t}} x)\, d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - m^{\mathrm{BMS}}(\psi)\right| \le c S_\ell(\psi)T^{-\kappa},$$

where $S_\ell(\psi)$ is the ℓ-Sobolev norm.

Proof. Let $\beta > 0$ satisfy the conclusion of Theorem 4.1 for ε and s_0. Let $\kappa' > 0$, $\ell \in \mathbb{N}$ satisfy the conclusion of Theorem 1.5.

Since x is (ε, s_0)-Diophantine, by Theorem 4.1, for $T_0 \gg_\Gamma s_0$ and $R \geq R_0$,

$$\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0 \cap \mathcal{X}(R)) \ll \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0)e^{-\beta R}, \tag{84}$$

where

$$s_\varepsilon := \frac{\varepsilon}{2}\log T, \quad T_0 := Te^{-s_\varepsilon} = T^{1-\varepsilon/2}, \quad x_0 := a_{-s_\varepsilon}x. \tag{85}$$

By (6) and (14), we have

$$\frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))}\int_{B_U(T)} \psi(u_{\mathbf{t}}x)\,d\mu_x^{\mathrm{PS}}(\mathbf{t}) = \frac{1}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))}\int_{B_U(T_0)} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)\,d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}).$$

Fix $R > R_0$ and define

$$Q_0 = B_U(T_0)x_0 \cap \mathcal{C}(R).$$

By the definition of $\mathcal{C}(R)$,

$$Q_0 \subseteq \operatorname{supp} m^{\mathrm{BMS}}.$$

Let $\rho > 0$ be smaller than half of the injectivity radius of Q_0.

First, by Lemma 3.11, there exist $\{y : y \in I_0\} \subseteq Q_0$ and $f_y \in C_c^\infty(B_U(2\rho)y)$ satisfying

$$S_\ell(f_y) \ll \rho^{-\ell+n-1} \tag{86}$$

and

$$\sum_y f_y = 1 \text{ on } E_1 := \bigcup_{y\in I_0} B_U(\rho)y \supseteq Q_0,$$

which are 0 outside of

$$E_2 = \bigcup_{y\in I_0} B_U(2\rho)y.$$

Observe that

$$Q_0 \subseteq E_1 \subseteq E_2 \subseteq B_U(T_0 + 2\rho)x_0. \tag{87}$$

Thus,

$$\int_{u_{\mathbf{t}}x_0\in E_1} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)\,d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}) \leq \sum_{y\in I_0}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)\,d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}).$$

Because $Q_0 \subseteq \operatorname{supp} m^{\mathrm{BMS}}$, we may use Proposition 3.4 to deduce that there exists $\lambda = \lambda(\Gamma) \geq 1$ such that for any $y \in I_0$, we have

$$\begin{aligned}
\mu_y^{\mathrm{PS}}(B_U(\rho)) &\geq \lambda^{-1}\rho^{\delta_\Gamma}e^{(k(y,\rho)-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(a_{-\log\rho}y))}\\
&\geq \lambda^{-1}\rho^{\delta_\Gamma}e^{-\delta_\Gamma d(\pi(\mathcal{C}_0),\pi(a_{-\log\rho}y))}\\
&\geq \lambda^{-1}\rho^{\delta_\Gamma}e^{-\delta_\Gamma(-\log\rho)}e^{-\delta_\Gamma\,\mathrm{height}(y)} \quad \text{since } \rho < 1
\end{aligned}$$

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$$\gg_\Gamma \lambda^{-1}\rho^{2\delta_\Gamma}e^{-\delta_\Gamma \operatorname{height}(y)}$$
$$\geq \lambda^{-1}\rho^{2\delta_\Gamma}e^{-\delta_\Gamma R},$$

where the last line follows by Lemma 3.6.

Since $e^{s_\varepsilon} = T^{\varepsilon/2}$, it follows from (86) and the above, that if we choose ρ and R such that

$$e^{\delta_\Gamma R}\rho^{n-1-\ell-2\delta_\Gamma} \ll_\Gamma T^{\varepsilon\kappa'/4}, \tag{88}$$

then, by the choice of f_y, we have

$$S_\ell(f_y) \ll \mu_y^{\mathrm{PS}}(B_U(\rho))e^{\kappa' s_\varepsilon/2} \ll \mu_y^{\mathrm{PS}}(f_y)e^{\kappa' s_\varepsilon/2}, \tag{89}$$

where the implied constant is absolute.

If we further assume that

$$T \gg_\Gamma e^{2R/\varepsilon} \tag{90}$$

(with the implied constant coming from Theorem 1.5), then $s_\varepsilon \gg_\Gamma R$, and by (89), Theorem 1.5, and Lemma 3.6, there exist $c_1, c_2 > 0$ which depend only on Γ and supp ψ such that

$$\sum_{y\in I_0}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y}\psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)\,d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t})$$
$$\leq \sum_{y\in I_0}(m^{\mathrm{BMS}}(\psi)\mu_y^{\mathrm{PS}}(f_y) + c_1 S_\ell(\psi)S_\ell(f_y)e^{-\kappa' s_\varepsilon})$$
$$\leq \sum_{y\in I_0}\mu_y^{\mathrm{PS}}(f_y)(m^{\mathrm{BMS}}(\psi) + c_2 S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}).$$

By Theorem 5.1, there exists $c_3 = c_3(\Gamma) > 0$ such that if $T_0 \gg s_0$, then there exist $\alpha = \alpha(\Gamma) > 0$, $c_3 = c_3(\Gamma) > 0$ such that

$$\sum_{y\in I_0}\mu_y^{\mathrm{PS}}(f_y) \leq \mu_{x_0}^{\mathrm{PS}}(B_U(T_0+2\rho))$$
$$\ll_\Gamma (1 + c_3(2\rho)^\alpha)\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)).$$

If $m^{\mathrm{BMS}}(\psi) + c_2 S_\ell(\psi)e^{-\kappa' s_\varepsilon/2} \geq 0$, we arrive at

$$\sum_{y\in I_0}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y}\psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)\,d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t})$$
$$\leq \mu_{x_0}^{\mathrm{PS}}(B_U(T_0))(1 + c_3(2\rho)^\alpha)(m^{\mathrm{BMS}}(\psi) + c_2 S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}). \tag{91}$$

However, if $m^{\mathrm{BMS}}(\psi) + c_2 S_\ell(\psi)e^{-\kappa' s_\varepsilon/2} < 0$, by (84), there exists $c_6 = c_6(\Gamma) > 0$ so that

$$\sum_{y\in I_0}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y}\psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)\,d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t})$$

$$\leq \mu^{\text{PS}}_{x_0}(Q_0)(m^{\text{BMS}}(\psi) + c_2 S_\ell(\psi)e^{-\kappa' s_\varepsilon/2})$$
$$\leq \mu^{\text{PS}}_{x_0}(B_U(T_0))(1 - c_6 e^{-\beta R})(m^{\text{BMS}}(\psi) + c_2 S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}). \tag{92}$$

Fix

$$\kappa := \kappa'\varepsilon/4, \quad R > \frac{\kappa}{\beta}\log T, \quad \rho < T^{-\frac{\kappa}{\alpha}}, \tag{93}$$

such that ρ also satisfies the assumption of Theorem 5.1, and R which satisfies (88) and (90). Thus, (91) and (93) imply in either case that

$$\frac{1}{\mu^{\text{PS}}_{x_0}(B_U(T_0))}\int_{u_{\mathbf{t}}x_0\in E_1} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)\, d\mu^{\text{PS}}_{x_0}(\mathbf{t}) - m^{\text{BMS}}(\psi)$$
$$\ll_{\Gamma,\text{supp}\,\psi} S_\ell(\psi)T^{-\kappa}, \tag{94}$$

where we have used that by [1], $\|\psi\|_\infty \ll_{\text{supp}\,\psi} S_\ell(\psi)$, so $m^{\text{BMS}}(\psi) \ll_{\text{supp}\,\psi} S_\ell(\psi)$.

By (84),

$$\int_{B_U(T_0)x_0\setminus E_1} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)\, d\mu^{\text{PS}}_{x_0}(\mathbf{t}) \leq \|\psi\|_\infty \mu^{\text{PS}}_{x_0}(B_U(T_0)\setminus E_1)$$
$$\ll_{\text{supp}\,\psi} S_\ell(\psi)\mu^{\text{PS}}_{x_0}(B_U(T_0)x_0)e^{-\beta R}$$
$$\ll_{\text{supp}\,\psi} S_\ell(\psi)\mu^{\text{PS}}_{x_0}(B_U(T_0)x_0)T^{-\kappa},$$

where we have again used that by [1], $\|\psi\|_\infty \ll_{\text{supp}\,\psi} S_\ell(\psi)$. Combining the above with (94) implies that

$$\frac{1}{\mu^{\text{PS}}_{x_0}(B_U(T_0))}\int_{B_U(T_0)} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)\, d\mu^{\text{PS}}_{x_0}(\mathbf{t}) - m^{\text{BMS}}(\psi)$$
$$\ll_{\Gamma,\text{supp}\,\psi} S_\ell(\psi)T^{-\kappa}.$$

For the lower bound, define

$$Q_1 := B_U(T_0 - 2\rho)x_0 \cap \mathcal{C}(R).$$

As before, according to Lemma 3.11, there exist $\{y : y \in I_1\} \subseteq Q_1$ and $f_y \in C_c^\infty(B_U(2\rho)y)$ satisfying

$$S_\ell(f_y) \ll \rho^{-\ell+n-1}$$

and

$$\sum_{y\in I_1} f_y = 1 \text{ on } E_4 := \bigcup_{y\in I_1} B_U(\rho)y \supseteq Q_1,$$

and which are 0 outside of

$$\bigcup_{y\in I_1} B_U(2\rho)y \subseteq B_U(T_0)x_0. \tag{95}$$

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Hence,

$$\int_{u_{\mathbf{t}}x_0 \in B_U(T_0)x_0} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mathbf{t} \geq \sum_{y\in I_1} \int_{u_{\mathbf{t}}x_0 \in B_U(2\rho)y} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) f_y(u_{\mathbf{t}} x_0)\, d\mathbf{t}.$$

Moreover, by the same argument as in (89), we deduce that

$$S_\ell(f_y) \ll \mu_y^{\mathrm{PS}}(f_y) e^{\kappa' s_\varepsilon/2}. \tag{96}$$

By Theorem 1.5 and (96), we have that

$$\begin{aligned}\sum_{y\in I_1} \int_{u_{\mathbf{t}}x_0 \in B_U(2\rho)y} & \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) f_y(u_{\mathbf{t}} x_0)\, d\mathbf{t} \\ &\geq \sum_{y\in I_1} (\mu_{x_0}^{\mathrm{PS}}(f_y) m^{\mathrm{BMS}}(\psi) - c_4 S_\ell(\psi) S_\ell(f_y) e^{-\kappa' s_\varepsilon}) \\ &\geq \sum_{y\in I_1} \mu_{x_0}^{\mathrm{PS}}(f_y)(m^{\mathrm{BMS}}(\psi) - c_5 S_\ell(\psi) e^{-\kappa' s_\varepsilon/2}),\end{aligned}$$

where c_5 arises from c_4 and the implied constant in (96).

Note that by replacing ψ with $-\psi$ if necessary, we may assume that $m^{\mathrm{BMS}}(\psi) - c_5 S_\ell(\psi) e^{-\kappa' s_\varepsilon/2} < 0$. Thus, by observing that

$$\sum_{y\in I_1} \mu_y^{\mathrm{PS}}(f_y) \leq \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)),$$

we immediately conclude that

$$\begin{aligned}\frac{1}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))} \int_{u_{\mathbf{t}}x_0\in E_1} & \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}) - m^{\mathrm{BMS}}(\psi) \\ &\gg_{\Gamma, \mathrm{supp}\,\psi} S_\ell(\psi) e^{-\kappa' s_\varepsilon/2}.\end{aligned}$$

Hence, (85) and (93) imply that

$$\frac{1}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))} \int_{u_{\mathbf{t}}x_0\in E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}) - m^{\mathrm{BMS}}(\psi) \gg_{\Gamma, \mathrm{supp}\,\psi} -S_\ell(\psi) T^{-\kappa}.$$

□

8. *Proof of Theorem 1.4*

In this section, we will prove Theorem 1.4 using Theorem 6.3. We will use a partition of unity argument for a cover of the intersection of $B_U(r)x$ with a fixed compact set by small balls centered at PS points.

We will need the following lemma.

LEMMA 8.1. *There exists an absolute constant $c > 0$ satisfying the following: for $x \in X$, $y \in Ux$, $\psi \in C_c^\infty(X)$ supported on an admissible box of diameter smaller than 1, $0 < \rho < \mathrm{inj}(y)$, $f \in C_c^\infty(B_U(\rho)y)$ such that for $0 \leq f \leq 1$ and $s > c$, we have*

$$e^{(n-1-\delta_\Gamma)s} \int_{Ux} \psi(a_s u_t y) f(u_t y)\, dt \ll_{\Gamma, \mathrm{supp}\,\psi} S_\ell(\psi) \mu_y^{\mathrm{PS}}(B_U(2\rho)y),$$

where $\ell \in \mathbb{N}$ satisfies the conclusion of Lemma 3.12.

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Proof. Assume that for $0 < \varepsilon_0, \varepsilon_1 < 1$, ψ is supported on the admissible box $B_U(\varepsilon_0)P_{\varepsilon_1}z$ for $z \in X$. Without loss of generality, we may assume that ψ is non-negative. Fix $y \in Ux$.

For small $\eta > 0$, $h \in G_\eta \operatorname{supp}(\psi)$, and $p \in P$, let

$$\psi_{\eta,+}(h) := \sup_{w \in G_\eta} \psi(wh), \quad \Psi_{\eta,+}(ph) := \int_{Uph} \psi_{\eta,+}(u_{\mathbf{t}}ph)\, d\mathbf{t},$$

and for $upz \in B_U(\varepsilon_0)P_{\varepsilon_1}z$, let

$$\tilde{\Psi}_{\eta,+}(upz) := \frac{1}{\mu^{\mathrm{PS}}_{pz}(B_U(\varepsilon_0)pz)} \Psi_{\eta,+}(pz).$$

By the choice of ℓ, for any $\eta > 0$ and $h \in G_\eta \operatorname{supp}(\psi)$,

$$|\psi_{\eta,+}(h) - \psi(h)| \ll \eta S_\ell(\psi),$$

and

$$|\psi(z)| \leq S_{\infty,0}(\psi) \ll S_\ell(\psi),$$

where the implied constants depend on $\operatorname{supp}\psi$. Since the diameter of $\operatorname{supp}\psi$ is smaller than 1, we may assume that the implied constants in the above are absolute. Then, for any $u \in U$ such that $a_s uy = u'pz \in B_U(\varepsilon_0)P_{\varepsilon_1}z$ and $0 < \eta < 1$, we have

$$\begin{aligned} |\tilde{\Psi}_{\eta,+}(a_s uy)| &= \left| \frac{1}{\mu^{\mathrm{PS}}_{pz}(B_U(\varepsilon_0)pz)} \int_{Upz} \psi_{\eta,+}(u_{\mathbf{t}}pz)\, d\mathbf{t} \right| \\ &= \frac{\mu^{\mathrm{Leb}}_{pz}(B_U(\varepsilon_0)pz)}{\mu^{\mathrm{PS}}_{pz}(B_U(\varepsilon_0)pz)} S_\ell(\psi) \\ &\ll S_\ell(\psi), \end{aligned} \tag{97}$$

where the implied constant depends only on $\operatorname{supp}\psi$.

For small $\eta > 0$ and $uy \in B_U(\eta + \varepsilon_0)y$, let

$$f_{\eta,+}(uy) := \sup_{w \in B_U(\eta)} f(wuy).$$

Using Lemmas 6.2 and 6.5 in [23], we get that for some absolute constant $c' > 0$,

$$\begin{aligned} e^{(n-1-\delta_\Gamma)s} \int_{B_U(\rho)y} \psi(a_s u_{\mathbf{t}} y) f(u_{\mathbf{t}} y)\, d\mathbf{t} &\ll \int_U \tilde{\Psi}_{c'\rho,+}(a_s u_{\mathbf{t}} y) f_{c'e^{-s}\rho,+}(u_{\mathbf{t}} y)\, d\mu^{\mathrm{PS}}_y(\mathbf{t}) \\ &\leq \int_{B_U(\rho + c'e^{-s}\rho)y} \tilde{\Psi}_{c'\rho,+}(a_s u_{\mathbf{t}} y)\, d\mu^{\mathrm{PS}}_y(\mathbf{t}), \end{aligned}$$

where the implied constant is absolute. Then, by (97), we get

$$\begin{aligned} e^{(n-1-\delta_\Gamma)s} \int_{B_U(\rho)y} \psi(a_s u_{\mathbf{t}} y) f(u_{\mathbf{t}} y)\, d\mathbf{t} &\ll_{\operatorname{supp}\psi} \mu^{\mathrm{PS}}_y(B_U(\rho + c'e^{-s}\rho)) S_\ell(\psi) \\ &\leq \mu^{\mathrm{PS}}_y(B_U(2\rho)) S_\ell(\psi). \end{aligned}$$

Choosing $c := \log c'$, we may conclude the claim. □

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We are now ready to prove Theorem 1.4. For the reader's convenience, we restate that theorem in the following.

THEOREM 8.2. *For any $0 < \varepsilon < 1$ and $s_0 \geq 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying: for every $\psi \in C_c^\infty(G/\Gamma)$, there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that for every $x \in G/\Gamma$ that is (ε, s_0)-Diophantine, and for all T such that $T^{1-\varepsilon/2} \gg_{\Gamma, \operatorname{supp}\psi} s_0$,*

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(u_{\mathbf{t}}x)\, d\mathbf{t} - m^{\mathrm{BR}}(\psi) \right| \leq c S_\ell(\psi) T^{-\kappa},$$

where $S_\ell(\psi)$ is the ℓ-Sobolev norm.

Proof. We keep the notation of §4. By an argument similar to the proof of Theorem 1.5, we may assume that ψ is supported on an admissible box. Because ψ is compactly supported, we may also assume $\psi \geq 0$ by using a translation.

Let $\beta > 0$ satisfy the conclusion of Theorem 4.1 for ε and s_0. Let $\kappa' > 0$, $\ell \in \mathbb{N}$ satisfy the conclusion of Theorem 6.3.

Since x is (ε, s_0)-Diophantine, by Theorem 4.1, for $T_0 \gg_\Gamma s_0$ and $R \geq R_0$, we have

$$\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0 \cap \mathcal{X}(R)) \ll \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0) e^{-\beta R}, \tag{98}$$

where

$$s_\varepsilon := \frac{\varepsilon}{2} \log T, \quad x_0 := a_{-s_\varepsilon} x, \quad \text{and} \quad T_0 = T^{1-\varepsilon/2}. \tag{99}$$

Observe that by (6), (13), and (14),

$$\frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(u_{\mathbf{t}}x)\, d\mathbf{t} = \frac{e^{(n-1-\delta)s_\varepsilon}}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))} \int_{B_U(T_0)} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mathbf{t}.$$

Fix $R > R_0$ and define

$$Q_0 := B_U(T_0)x_0 \cap \mathcal{C}(R).$$

Since for any $R \geq R_0$, the set $\mathcal{C}(R)$ is in the convex core of $\mathbb{H}^n/\Gamma$,

$$Q_0 \subseteq \operatorname{supp} m^{\mathrm{BMS}}. \tag{100}$$

Let $\rho > 0$ be smaller than half of the injectivity radius of Q_0.

First, by Lemma 3.11, there exist $\{y : y \in I_0\} \subseteq Q_0$ and $f_y \in C_c^\infty(B_U(2\rho)y)$ satisfying

$$S_\ell(f_y) \ll \rho^{-\ell+n-1} \tag{101}$$

and

$$\sum_y f_y = 1 \text{ on } E_1 := \bigcup_{y \in I_0} B_U(\rho)y \supseteq Q_0,$$

which are 0 outside of

$$E_2 = \bigcup_{y \in I_0} B_U(2\rho)y.$$

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By replacing references to Theorem 1.5 with references to Theorem 6.3, the exact same argument as in the proof of Theorem 1.3 will establish that for $T \gg_\Gamma e^{2R/\varepsilon}$ and

$$\kappa = \frac{\beta\varepsilon}{2}, \quad R = \frac{\kappa \log T}{\beta}, \quad \rho \leq T^{-\kappa/\alpha}. \tag{102}$$

We get that if we assume without loss of generality that $\kappa' < 2\beta$ and also that $T \gg_\Gamma 1$,

$$\frac{e^{(n-1-\delta_\Gamma)s_\varepsilon}}{\mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0)} \int_{u_{\mathbf{t}}x_0 \in E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mathbf{t} - m^{\mathrm{BR}}(\psi) \ll_{\Gamma, \mathrm{supp}\,\psi} S_\ell(\psi) T^{-\kappa}. \tag{103}$$

We now want to bound the integral over $B_U(T_0)x_0 \setminus E_1$. Using Lemma 3.11 again, we may deduce that there exist $\{y : y \in I_1\} \subseteq B_U(T_0)x_0 \setminus E_1$ and $f_y \in C_c^\infty(B_U(\rho/4)y)$ satisfying $\sum_{y \in I_1} f_y = 1$ on $\bigcup_{y \in I_1} B_U(\rho/8)y$ and 0 outside of

$$\bigcup_{y \in I_1} B_U(\rho/4)y.$$

In particular, by the definition of E_1, we have

$$E_3 := \bigcup_{y \in I_1} B_U(\rho/2)y \subseteq (B_U(T_0)x_0 \setminus Q_0) \cup (B_U(T_0 + \rho/2)x_0 \setminus B_U(T_0))x_0.$$

Using Lemma 8.1, we arrive at

$$\begin{aligned} e^{(n-1-\delta_\Gamma)s_\varepsilon} &\int_{B_U(T_0)\setminus E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mathbf{t} \\ &\leq e^{(n-1-\delta_\Gamma)s_\varepsilon} \sum_{y \in I_1} \int_{B_U(\rho/2)y} \psi(a_{s_\varepsilon} u_{\mathbf{t}} y) f_y(u_{\mathbf{t}} y)\, d\mathbf{t} \\ &\ll \sum_{y \in I_1} S_\ell(\psi) \mu^{\mathrm{PS}}_y(B_U(\rho/2)y) \\ &\leq S_\ell(\psi)(\mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0 \setminus Q_0) + \mu^{\mathrm{PS}}_{x_0}((B_U(T_0 + \rho/2) \setminus B_U(T_0))x_0)). \end{aligned}$$

Thus, by Theorem 5.1, there exists $\alpha = \alpha(\Gamma) > 0$ such that using (98) and (102), we arrive at

$$\begin{aligned} e^{(n-1-\delta_\Gamma)s_\varepsilon} &\int_{B_U(T_0)\setminus E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mathbf{t} \\ &\ll_{\Gamma, \mathrm{supp}\,\psi} S_\ell(\psi) \mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0)(e^{-\beta R} + \rho^\alpha) \\ &\ll_{\Gamma, \mathrm{supp}\,\psi} S_\ell(\psi) \mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0) T^{-\kappa}. \end{aligned}$$

Using (103), we may now deduce

$$\frac{e^{(n-1-\delta_\Gamma)s_\varepsilon}}{\mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0)} \int_{B_U(T_0)} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mathbf{t} - m^{\mathrm{BR}}(\psi) \ll_{\Gamma, \mathrm{supp}\,\psi} S_\ell(\psi) T^{-\kappa}.$$

The lower bound follows similarly, as in the proof of Theorem 1.3. □

Remark 8.3. The dependence of T on $\mathrm{supp}\,\psi$ in the previous proof arises from Theorem 6.3, through the quantity s_ε. Upon closer inspection, one can verify that this means T depends on $\mathrm{supp}\,\psi$ through the maximum height of elements in $\mathrm{supp}\,\psi$. In

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particular, we may choose a larger compact set containing supp ψ and have T depend on that compact set, rather than supp ψ specifically.

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A. *Appendix. Friendliness of the PS measure*

For simplicity, in this section, we work in the Poincaré ball models of hyperbolic geometry $\mathbb{D}^n$, instead of $\mathbb{H}^n$. Recall that $\mathbb{D}^n$ and $\mathbb{H}^n$ are isometric via the Cayley transform.

Denote by d_E the Euclidean metric on $\mathbb{R}^m$. For a subset $S \subseteq \mathbb{R}^m$ and $\xi > 0$, let

$$\mathcal{N}(S, \xi) = \{x \in \mathbb{R}^m : d_E(x, S) \le \xi\}.$$

For $v \in \mathbb{R}^m$ and $r > 0$, let

$$B(v, r) = \{u \in \mathbb{R}^m : d_E(u, v) \le r\}$$

be the Euclidean ball of radius r around v.

Definition A.1. Let μ be a measure defined on $\mathbb{R}^m$.

(1) μ is called *Federer* (respectively *doubling*) if for any $c > 1$, there exists $k_1 > 0$ such that for all $v \in \operatorname{supp}(\mu)$ and $0 < \eta \le 1$ (respectively $\eta > 0$),

$$\mu(B(v, c\eta)) \le k_1 \mu(B(v, \eta)).$$

(2) μ is called *decaying and non-planar* if there exist $\alpha, c_2 > 0$ such that for all $v \in \operatorname{supp} \mu$, $\xi > 0$, $0 < \eta \le 1$, and every affine hyperplane $L \subseteq \mathbb{R}^n$,

$$\mu(\mathcal{N}(L, \xi \|d_L\|_{\mu, B(v,\eta)}) \cap B(v, \eta)) \le c_2 \xi^\alpha \mu(B(v, \eta)),$$

where

$$\|d_L\|_{\mu, B(v,\eta)} := \sup\{d(\mathbf{y}, L) : \mathbf{y} \in B(v, \eta) \cap \operatorname{supp} \mu\}.$$

(3) μ is called *friendly* if it is Federer, decaying, and non-planar.

In the case that all cusps have maximal rank (which vacuously includes the case of convex cocompact Γ), a stronger statement holds, see §A.3.

THEOREM A.2. [5, Theorem 1.9] *Assume Γ is geometrically finite and Zariski dense. Then the PS densities $\{\nu_x\}_{x \in \mathbb{D}^n}$ are friendly. Moreover, in this case, the constants in Definition A.1 only depend on Γ.*

Note that, as in [5, Definition 1.1(1.3)], using closed thickenings, one obtains Definition A.1(2) by combining the separate definitions of decaying and of non-planar from [5]. The above result for the case Γ is convex cocompact was proved in [34, Theorem 2].

In this section, we will prove the results in §5. In particular, because of the shadow lemma, Proposition 3.4, we will see that the leafwise PS measures $\{\mu_x^{\mathrm{PS}}\}$ satisfy a stronger condition than that of friendliness. In general, we will begin by proving a statement for ν_o, then for μ_x^{PS} when $x^+ \in \Lambda(\Gamma)$, and then finally a nicer statement for $x \in \operatorname{supp} m^{\mathrm{BMS}}$.

The next lemma and subsequent corollaries are necessary to move between these measures.

As in §3.1, we fix $o \in \mathbb{D}^n$. For any $x \in \mathbb{D}^n$, define the *Gromov distance* at x of $\xi, \eta \in \partial\mathbb{D}^n$ by

$$d_x(\xi, \eta) = \exp\big(-\tfrac{1}{2}\beta_\xi(x, y) - \tfrac{1}{2}\beta_\eta(x, y)\big),$$

where y is on the ray joining ξ and η. For any $x \in \mathbb{D}^n$, $\xi \in \partial\mathbb{D}^n$, and $r > 0$, let

$$B_x(\xi, r) := \{\eta \in \partial\mathbb{D}^n : d_x(\xi, \eta) \leq r\}.$$

For $v \in \mathrm{T}^1(\mathbb{D}^n)$, denote by $\mathrm{Pr}_{v^-} : Uv \to \partial\mathbb{D}^n \setminus \{v^-\}$ the projection $w \mapsto w^+$.

The next lemma follows from §1.6 in [14], and [33, Lemma 2.5 and Theorem 3.4].

LEMMA A.3. *There exist constants $\alpha_0 > 0$, $c > 1$ such that for all $g \in G$ and $0 < \eta \leq \alpha_0$, we have*

$$B_{\pi(g)}(g^+, c^{-1}\eta) \subseteq \mathrm{Pr}_{g^-}(B_U(\eta)g) \subseteq B_{\pi(g)}(g^+, c\eta).$$

According to [6, Lemma 3.5.1] for any $\xi, \eta \in \partial\mathbb{D}^n$,

$$d_o(\xi, \eta) = \tfrac{1}{2} d_E(\xi, \eta). \tag{A.1}$$

Using the triangle inequality on the hyperbolic distance and the definition of the Busemann function, one can show that for any $x \in \mathbb{D}^n$ and $\xi, \eta \in \partial\mathbb{D}^n$,

$$e^{-d(o,x)} \leq \frac{d_x(\xi, \eta)}{d_o(\xi, \eta)} \leq e^{d(o,x)}. \tag{A.2}$$

The following is a direct corollary of (A.1), (A.2), and Lemma A.3.

COROLLARY A.4. *There exist constants $\alpha_0 > 0$, $c > 1$ such that for all $g \in G$ and $0 < \eta \leq \alpha_0$, we have*

$$B(g^+, c^{-1}e^{-d(o,\pi(g))}\eta) \subseteq \mathrm{Pr}_{g^-}(B_U(\eta)g) \subseteq B(g^+, ce^{d(o,\pi(g))}\eta).$$

The next corollary will be necessary to obtain a non-planarity result for μ_x^{PS}. It follows from Corollary A.4 by covering the hyperplane with small balls using the fact that $\eta \leq 1$ to uniformly bound the $d(o, \pi(g'))$ terms with $d(o, \pi(g))$, where g' is the center of one of the balls in this cover.

COROLLARY A.5. *Let α_0 be as in Corollary A.4. There exists a constant $c > 1$ so that for every $g \in G$, every $0 < \xi < \eta \leq \alpha_0$, and every hyperplane L in $\mathbb{R}^{n-1}$, there exists a*

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hyperplane L' in $\partial(\mathbb{H}^n)$ so that

$$\begin{aligned}\mathcal{N}(L', c^{-1}e^{-d(o,\pi(g))}\xi) &\cap B(g^+, c^{-1}e^{-d(o,\pi(g))}\eta)\\ &\subseteq \mathrm{Pr}_{g^-}(\mathcal{N}(L,\xi)\cap B_U(\eta)g)\\ &\subseteq \mathcal{N}(L', ce^{d(o,\pi(g))}\xi)\cap B(g^+, ce^{d(o,\pi(g))}\eta).\end{aligned}$$

Proof. Let $\{g_i\}_{i\in I}$ be chosen so that

$$\mathcal{N}(L,\xi) = \bigcup_{i\in I} B_U(\xi)g_i.$$

By Corollary A.4,

$$B(g_i^+, c^{-1}e^{-d(o,\pi(g_i))}\xi) \subseteq \mathrm{Pr}_{g_i^-}(B_U(\xi)g_i) \subseteq B(g_i^+, ce^{d(o,\pi(g_i))}\xi).$$

Let

$$I_2 = \{i : B_U(\xi)g_i \cap B_U(\eta)g \neq \emptyset\}.$$

Observe that for $i \in I_2$, $g_i \in B_U(2)g$. Then,

$$d(o,\pi(g_i)) \leq d(o,\pi(g)) + d(\pi(g),\pi(g_i)),$$

and $g_i = u_{\mathbf{t}}g$ for $|\mathbf{t}| \leq 2$. Thus,

$$\begin{aligned}d(\pi(g),\pi(g_i)) &= d(\pi(g),\pi(u_{\mathbf{t}}g))\\ &= d(g(o), u_{\mathbf{t}}g(o)).\end{aligned}$$

This implies that there exists a constant $\hat{c} > 1$ that is uniform for all $g \in G$ so that for all $i \in I_2$,

$$\hat{c}^{-1}e^{-d(o,\pi(g))} \ll_\Gamma e^{-d(o,\pi(g_i))} \ll_\Gamma e^{d(o,\pi(g_i))} \ll_\Gamma \hat{c}e^{d(o,\pi(g))}.$$

Thus, for all $i \in I_2$, we have

$$B(g^+, \hat{c}^{-1}c^{-1}e^{-d(o,\pi(g))}\xi) \subseteq \mathrm{Pr}_{g_i^-}(B_U(\xi)g_i) \subseteq B(g^+, \hat{c}ce^{d(o,\pi(g))}\xi).$$

Hence, for every $i \in I_2$,

$$\begin{aligned}B(g^+, \hat{c}^{-1}c^{-1}e^{-d(o,\pi(g))}\xi) &\cap B(g^+, \hat{c}^{-1}c^{-1}e^{-d(o,\pi(g))}\eta)\\ &\subseteq \mathrm{Pr}_{g^-}(B_U(\xi)g_i \cap B_U(\eta)g)\\ &\subseteq B(g^+, \hat{c}ce^{d(o,\pi(g))}\xi)\cap B(g^+, \hat{c}ce^{d(o,\pi(g))}\eta).\end{aligned}$$

The result then follows by taking the union over all $i \in I_2$. □

A.1. *The PS measure is Federer.* In this section, we prove more specific Federer statements for ν_o and μ_x^{PS}.

LEMMA A.6. *There exists a constant $\sigma \geq \delta_\Gamma$ depending only on Γ such that for any $\lambda \in \Lambda(\Gamma)$, $\eta > 0$ and $c \geq 1$, we have that*

$$\nu_o(B(\lambda, c\eta)) \ll_\Gamma c^\sigma \nu_o(B(\lambda,\eta)).$$

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Proof. We will prove this for the balls $B_o(\lambda, c\eta)$, and $B_o(\lambda, \eta)$ using the Gromov distance. It then immediately follows for the Euclidean balls $B(\lambda, c\eta)$ and $B(\lambda, \eta)$ by the Federer condition and (A.1).

Let $\{\lambda_t\}_{t\geq 0}$ be a geodesic ray joining o to λ. By the shadow lemma for ν_o [35, Theorem 2] (see also [33, Theorem 3.2]), we have that for any $\eta > 0$,

$$\eta^{\delta_\Gamma} e^{(k(\lambda_{-\log\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log\eta})} \ll_\Gamma \nu_o(B_o(\lambda, \eta)) \ll_\Gamma \eta^{\delta_\Gamma} e^{(k(\lambda_{-\log\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log\eta})}. \tag{A.3}$$

Here, $k(\lambda_{-\log\eta})$ denotes the rank of the cusp that $\lambda_{-\log\eta}$ lies in; if it is in $\pi(\mathcal{C}_0)$, it is defined to be zero. (Recall the definition of $\mathcal{C}_0$ from §3.2.) Note also that we have absorbed a constant depending on diam $\pi(\mathcal{C}_0)$ (hence only on Γ) to write the distance from $\pi(\mathcal{C}_0)$ rather than from the fixed reference point o.

It follows from (A.3) that it is enough to show that for some $\sigma \geq \delta_\Gamma$,

$$\begin{aligned}\nu_o(B_o(\lambda, c\eta)) &\ll_\Gamma (c\eta)^{\delta_\Gamma} e^{(k(\lambda_{-\log c\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log c\eta})}\\ &\ll_\Gamma c^\sigma \eta^{\delta_\Gamma} e^{(k(\lambda_{-\log\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log\eta})}\\ &\ll_\Gamma c^\sigma \nu_o(B_o(\lambda, \eta)).\end{aligned}$$

Equivalently, it is enough to show that

$$(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) \ll_\Gamma (\sigma - \delta_\Gamma)\log c. \tag{A.4}$$

Case 1: Assume $k(\lambda_{-\log c\eta}) \leq k(\lambda_{-\log\eta})$.

Then

$$\begin{aligned}&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta})\\ &\quad\leq (k(\lambda_{-\log\eta}) - \delta_\Gamma)(d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}))\\ &\quad\leq (k(\lambda_{-\log\eta}) - \delta_\Gamma)\log c\\ &\quad\leq (n - 1 - \delta_\Gamma)\log c.\end{aligned}$$

Case 2: $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log\eta})$ and $k(\lambda_{-\log\eta}) = 0$.

Then, $d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) = 0$ and

$$0 < d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) + d(\lambda_{-\log\eta}, \lambda_{-\log c\eta}) \leq \log c.$$

Therefore,

$$\begin{aligned}&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta})\\ &\quad\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)d((\pi(\mathcal{C}_0), \lambda_{-\log c\eta})\\ &\quad\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)\log c\\ &\quad\leq (n - 1 - \delta_\Gamma)\log c.\end{aligned}$$

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Case 3: Assume $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log \eta})$ and $k(\lambda_{-\log \eta}) > 0$. In particular, $\lambda_{-\log \eta}$ and $\lambda_{-\log c\eta}$ are in two different cusps, and hence there exists $1 < r < c$ such that $\lambda_{-\log r\eta} \in \pi(\mathcal{C}_0)$. Then,

$$d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log \eta}) \leq \log r \leq \log c$$

$$d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log c\eta}) \leq \log(c/r) \leq \log c.$$

Note that since $k(\lambda_{-\log c\eta}) \geq 2$, we have $\delta_\Gamma > 1$, because $\delta_\Gamma > k/2$, where k is the maximal cusp rank. We arrive at

$$\begin{aligned}(k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) &\geq (1 - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta})\\ &\geq (1 - \delta_\Gamma) \log c\\ (k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0, \lambda_{-\log c\eta}) &\leq (n - 1 - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta})\\ &\leq (n - 1 - \delta_\Gamma) \log c.\end{aligned}$$

It follows that

$$\begin{aligned}&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta})\\ &\quad\leq (n - 1 - \delta_\Gamma) \log c - (1 - \delta_\Gamma) \log c\\ &\quad\leq (n - 2) \log c.\end{aligned}$$

Thus, choosing

$$\sigma = \max\{n - 1 - \delta_\Gamma, n - 2\} + \delta_\Gamma$$

completes the proof. □

When $c < 1$, we obtain a similar result, with a slightly more involved argument.

LEMMA A.7. *There exists a constant $\sigma > 0$ depending only on Γ such that for any $\lambda \in \Lambda(\Gamma)$, $\eta > 0$ and $0 < c < 1$, we have that*

$$\nu_o(B(\lambda, c\eta)) \ll_\Gamma c^\sigma \nu_o(B(\lambda, \eta)).$$

Proof. The proof is extremely similar to that of Lemma A.6.

By the shadow lemma, as in the proof of Lemma A.6, it is enough to show that for some $\sigma > 0$,

$$\begin{aligned}&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta})\\ &\quad\ll_\Gamma (\delta_\Gamma - \sigma)|\log c|.\end{aligned}\tag{A.5}$$

Case 1: Assume $k(\lambda_{-\log c\eta}) \leq k(\lambda_{-\log \eta})$.

Then

$$\begin{aligned}&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta})\\ &\quad\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)(d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) - d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}))\\ &\quad\leq |k(\lambda_{-\log \eta}) - \delta_\Gamma||\log c|.\end{aligned}$$

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Let k be the maximal cusp rank. Since $|k - \delta_\Gamma| < \delta_\Gamma$, we get that

$$\sigma := \delta_\Gamma - |k - \delta_\Gamma| > 0$$

satisfies the claim.

Case 2: $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log \eta})$ and $k(\lambda_{-\log \eta}) = 0$.

Then, $d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) = 0$ and

$$0 < d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) + d(\lambda_{-\log \eta}, \lambda_{-\log c\eta}) \leq |\log c|.$$

Therefore,

$$\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\quad \leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \\
&\quad \leq |k(\lambda_{-\log c\eta}) - \delta_\Gamma||\log c|,
\end{aligned}$$

and the claim follows as in Case 1.

Case 3: Assume $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log \eta})$ and $k(\lambda_{-\log \eta}) > 0$. In particular, $\lambda_{-\log \eta}$ and $\lambda_{-\log c\eta}$ are in two different cusps, and hence there exists $c < r < 1$ such that $\lambda_{-\log r\eta} \in \pi(\mathcal{C}_0)$. Then since $r < 1$,

$$d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log \eta}) \leq |\log r| \tag{A.6}$$

$$d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log c\eta}) \leq \log(r/c). \tag{A.7}$$

Note that since $k(\lambda_{-\log c\eta}) \geq 2$, we have $\delta_\Gamma > 1$. By (A.6) and (A.7), we arrive at

$$(k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \geq (1 - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \tag{A.8}$$

$$\geq (\delta_\Gamma - 1) \log c \tag{A.9}$$

$$(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq \max\{0, \log(r/c)(k(\lambda_{-\log c\eta}) - \delta_\Gamma)\}. \tag{A.10}$$

We now have two cases. First, assume that $\log(r/c)(k(\lambda_{-\log c\eta}) - \delta_\Gamma) \leq 0$. Then $k(\lambda_{-\log c\eta}) - \delta_\Gamma \leq 0$, so by (A.8), we have that

$$\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\quad \leq -(\delta_\Gamma - 1) \log c \\
&\quad = (\delta_\Gamma - 1)|\log c|.
\end{aligned}$$

Now, assume that $\log(r/c)(k(\lambda_{-\log c\eta}) - \delta_\Gamma) > 0$, that is, that $k(\lambda_{-\log c\eta}) - \delta_\Gamma > 0$. Then it follows from (A.7) that

$$\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\quad \leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma) \log(r/c) - (\delta_\Gamma - 1) \log r.
\end{aligned} \tag{A.11}$$

Now, consider two further cases: $k(\lambda_{-\log c\eta}) - \delta_\Gamma > \delta_\Gamma - 1$ or $k(\lambda_{-\log c\eta}) - \delta_\Gamma \leq \delta_\Gamma - 1$. In the first case, (A.11) is bounded above by

$$(\delta_\Gamma - 1) \log(r/c) - (\delta_\Gamma - 1) \log r = -(\delta_\Gamma - 1) \log c = (\delta_\Gamma - 1)|\log c|.$$

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In the second case, note that (A.11) is equal to

$$(k - 2\delta_\Gamma + 1)\log r - (k(\lambda_{-\log c\eta} - \delta_\Gamma)\log c,$$

and our assumption implies that the first term is negative. Thus, an upper bound is

$$-(k(\lambda_{-\log c\eta} - \delta_\Gamma)\log c = (k(\lambda_{-\log c\eta} - \delta_\Gamma)|\log c| \leq (k - \delta_\Gamma)|\log c|,$$

where k is the maximal cusp rank, as before. Note that $k - \delta_\Gamma < \delta_\Gamma$ because $\delta_\Gamma > 2k$ always holds.

Thus, choosing

$$\sigma = \min\{\delta_\Gamma - |k - \delta_\Gamma|, 1\}$$

completes the proof. □

Using Lemma A.3, we obtain the following quantitative Federer-like statement for $\{\mu_x^{\mathrm{PS}}\}_{x^+\in\Lambda(\Gamma)}$.

COROLLARY A.8. *There exists constants $\sigma_1 = \sigma_1(\Gamma) \geq \delta_\Gamma$, $\sigma_2 = \sigma_2(\Gamma) > 0$ which satisfy the following: let $x \in G$ be such that $x^+ \in \Lambda(\Gamma)$. Then for $c > 0$ and $\eta \ll_\Gamma c^{-1}e^{-\operatorname{height}(x)}$, we have that*

$$\mu_x^{\mathrm{PS}}(B_U(c\eta)) \ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\}e^{2(\delta_\Gamma+\sigma_1)\operatorname{height}(x)}\mu_x^{\mathrm{PS}}(B_U(\eta)).$$

Proof. Fix $g \in G$ which satisfies $x = g\Gamma$ and $\operatorname{height}(x) = d(\pi(\mathcal{C}_0), \pi(g))$. By (22), $\operatorname{inj}(x)$ and $\operatorname{height}(x)$ are related, so that for $\eta \ll_\Gamma c^{-1}\operatorname{height}(x)$,

$$\mu_g^{\mathrm{PS}}(B_U(c\eta)) = \mu_x^{\mathrm{PS}}(B_U(c\eta)).$$

For any $0 < \eta \leq 1$ and $u_{\mathbf{t}} \in B_U(\eta)$, we have that

$$\begin{aligned}|\beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))| &\leq d(u_{\mathbf{t}}^{-1}(o), g(o))\\ &\leq d(u_{\mathbf{t}}^{-1}(o), o) + d(o, g(o))\\ &\leq 2\operatorname{diam}(B_U(1)\pi(\mathcal{C}_0)) + \operatorname{height}(x).\end{aligned}$$

The above gives a bound on the Busemann function for the following when $\eta \leq 1$:

$$e^{-\delta_\Gamma\operatorname{height}(x)}\nu_o(\mathrm{Pr}_{g^-}(B_U(\eta)))$$
$$\ll_\Gamma \mu_g^{\mathrm{PS}}(B_U(\eta)) = \int_{\mathbf{t}\in B_U(\eta)} e^{\delta_\Gamma\beta_{(u_{\mathbf{t}}g)^+}(o,u_{\mathbf{t}}g(o))}\,d\nu_o((u_{\mathbf{t}}g)^+) \tag{A.12}$$
$$\ll_\Gamma e^{\delta_\Gamma\operatorname{height}(x)}\nu_o(\mathrm{Pr}_{g^-}(B_U(\eta))). \tag{A.13}$$

Assume $c \geq 1$. By Lemmas A.3 and A.6, we have that

$$\begin{aligned}\nu_o(B(g^+, \eta)) &= \nu_o(B(g^+, (\tilde{c}e^{\operatorname{height}(x)}\tilde{c}^{-1}e^{-\operatorname{height}(x)}\eta))\\ &\ll_\Gamma (\tilde{c}e^{\operatorname{height}(x)})^{\sigma_1}\nu_o(B(g^+, \tilde{c}e^{-\operatorname{height}(x)}\eta)).\end{aligned} \tag{A.14}$$

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Let $\tilde{c} > 1$ be as in Corollary A.4. Then as long as

$$\eta \leq \tilde{c}^{-1} c^{-1} e^{-\operatorname{height}(x)},$$

we have the following:

$$\begin{aligned}
\mu_g^{\mathrm{PS}}(B_U(c\eta)) &\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o(\operatorname{Pr}_{g^-}(B_U(c\eta)) \quad \text{by (116)} \\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o(B(g^+, \tilde{c} e^{\operatorname{height}(x)} c\eta)) \quad \text{by Corollary 9.4} \\
&\ll_\Gamma c^{\sigma} e^{(\delta_\Gamma + \sigma_1) \operatorname{height}(x)} \nu_o(B(g^+, \eta)) \quad \text{by Lemma 9.6} \\
&\ll_\Gamma c^{\sigma_1} e^{(\delta_\Gamma + 2\sigma_1) \operatorname{height}(x)} \nu_o(B(g^+, \tilde{c}^{-1} e^{-\operatorname{height}(x)} \eta) \quad \text{by (117)} \\
&\ll_\Gamma c^{\sigma_1} e^{(\delta_\Gamma + 2\sigma_1) \operatorname{height}(x)} \nu_o(\operatorname{Pr}_{g^-}(B_U(\eta))) \quad \text{by Corollary 9.4} \\
&\ll_\Gamma c^{\sigma_1} e^{2(\delta_\Gamma + \sigma_1) \operatorname{height}(x)} \mu_g^{\mathrm{PS}}(B_U(\eta)) \quad \text{by (115),}
\end{aligned}$$

which completes the proof in this case.

The case $0 < c < 1$ can be shown in a similar way using Lemma A.7. □

When $x \in \operatorname{supp} m^{\mathrm{BMS}}$, a flowing argument with $\{a_{-s} : s \geq 0\}$ allows us to remove the restriction that η must be small in a way that depends on $\operatorname{height}(x)$. More precisely, we obtain the corollary.

COROLLARY A.9. *If Γ is geometrically finite and Zariski dense, then for any $x \in \operatorname{supp} m^{\mathrm{BMS}}$, the measure μ_x^{PS} is doubling, and the constants only depend on Γ. More precisely, there exist constants $\sigma_1 = \sigma_1(\Gamma) \geq \delta_\Gamma$, $\sigma_2 = \sigma_2(\Gamma) > 0$ such that for every $c > 0$, every $x \in \operatorname{supp} m^{\mathrm{BMS}}$, and every $T > 0$,*

$$\mu_x^{\mathrm{PS}}(B_U(cT)) \ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\} \mu_x^{\mathrm{PS}}(B_U(T)).$$

Proof. On a geometrically finite quotient, there exists a compact set $\Omega_0 \subset X$ such that for every $x \in X$ with $x^- \in \Lambda_r(\Gamma)$, there exists a sequence $s_n \to \infty$ such that $a_{-s_n} x \in \Omega_0$.

Because Ω_0 depends only on Γ, the height of any point in Ω_0 is bounded by a constant depending only on Γ. Thus, by Corollary A.8, for all $x \in \Omega_0 \cap \operatorname{supp} m^{\mathrm{BMS}}$ with $x^- \in \Lambda_r(\Gamma)$ and for all $\eta \ll_\Gamma c^{-1}$, we have that

$$\mu_x^{\mathrm{PS}}(B_U(c\eta)) \ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\} \mu_x^{\mathrm{PS}}(B_U(\eta)). \tag{A.15}$$

Now, fix $x \in \operatorname{supp} m^{\mathrm{BMS}}$ with $x^- \in \Lambda_r(\Gamma)$. Let $T \geq 0$ and let $s > 0$ be sufficiently large so that $e^{-s} T \ll_\Gamma c^{-1}$ and $a_{-s} x \in \Omega_0$. Then,

$$\begin{aligned}
\mu_x^{\mathrm{PS}}(B_U(cT)) &= e^{\delta_\Gamma s} \mu_{a_{-s}x}^{\mathrm{PS}}(B_U(ce^{-s} T)) \\
&\ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\} e^{\delta_\Gamma s} \mu_{a_{-s}x}^{\mathrm{PS}}(B_U(e^{-s} T)) \quad \text{by (118)} \\
&\ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\} \mu_{a_{-s}x}^{\mathrm{PS}}(B_U(T)),
\end{aligned}$$

so the result holds for $x^- \in \Lambda_r(\Gamma)$.

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Since $x \mapsto \mu_x^{\mathrm{PS}}$ is continuous (see Lemma 3.2) and the set of x with $x^- \in \Lambda_r(\Gamma)$ is dense in the set of points $y \in X$ which satisfy $y^- \in \Lambda(\Gamma)$, the result then follows for all $x \in \operatorname{supp} m^{\mathrm{BMS}}$. □

A.2. *Non-planarity of the PS measure.* For a subset $S \subseteq \mathbb{R}^{n-1}$ and $\xi > 0$, let

$$\mathcal{N}_U(S, \xi) = \{u_{\mathbf{t}} \in U : \text{there exists } \mathbf{s} \in S \text{ such that } \|\mathbf{t} - \mathbf{s}\| < \xi\}.$$

In the following, we use the shadow lemma for ν_o to obtain a stronger version of non-planarity than that in Definition A.1. From this, we will see that the PS measures when Γ is geometrically finite satisfies a non-planarity-like property. More specifically, the bound we get is independent of the hyperplane, but the size of η must be restricted in a way that depends on height(x), and a factor of height(x) will appear.

THEOREM A.10. *There exist $\theta = \theta(\Gamma) \geq 1$, $\alpha = \alpha(\Gamma) > 0$ which satisfy the following. For any $w \in \mathbb{H}^n$, $\lambda \in \Lambda(\Gamma)$, $0 < \eta \leq 1$, and $\xi > 0$, we have*

$$\nu_w(\mathcal{N}(L, \xi\eta^{\theta}) \cap B(\lambda, \eta)) \ll_{\Gamma} e^{2\delta_{\Gamma} d(o,w)} \xi^{\alpha} \nu_w(B(\lambda, \eta)).$$

Proof. First, we show the result for o.

According to [5, Lemma 3.8], there exists $\beta > 0$ such that for any $\eta > 0$ and any affine hyperplane $L \subset \mathbb{R}^n$, we have

$$\nu_o(\mathcal{N}(L, \eta)) \ll_{\Gamma} \eta^{\beta}. \tag{A.16}$$

For $\lambda \in \Lambda(\Gamma)$ and for $t \in \mathbb{R}$, let λ_t be the unit speed geodesic ray from o to λ. It follows from the shadow lemma for ν_o (see [35, Theorem 2], also [33, Theorem 3.2]) that for any $\eta > 0$, we have

$$\nu_o(B_o(\lambda, \eta)) \gg_{\Gamma} \eta^{\delta_{\Gamma}} e^{(k(\lambda_{-\log\eta}) - \delta_{\Gamma}) d(o, \lambda_{-\log\eta})},$$

where $k(\lambda_{-\log\eta})$ is the rank of the cusp containing $\lambda_{-\log\eta}$ (see §3.2). It follows from the fact that $k(\lambda_{-\log\eta}) \geq 0$ and $d(o, \lambda_{-\log\eta}) \leq -\log\eta$, that

$$\nu_o(B_o(\lambda, \eta)) \gg_{\Gamma} \eta^{2\delta_{\Gamma}}.$$

Since ν_o is Federer (by Theorem A.2), using (A.1), we arrive at the same bound for Euclidean balls (with the implied constant changing):

$$\nu_o(B(\lambda, \eta)) \gg_{\Gamma} \eta^{2\delta_{\Gamma}}. \tag{A.17}$$

Note that by the definition of $\|d_L\|_{\nu_o, B(\lambda,\eta)}$,

$$B(\lambda, \eta) \cap \operatorname{supp} \nu_o \subset \mathcal{N}(L, \|d_L\|_{\nu_o, B(\lambda,\eta)}).$$

It then follows from (A.16) and (A.17) that

$$\eta^{\delta_{\Gamma}} \ll_{\Gamma} (\|d_L\|_{\nu_o, B(\lambda,\eta)})^{\beta}.$$

Hence,

$$\|d_L\|_{\nu_o, B(\lambda,\eta)} \gg_{\Gamma} \eta^{2\delta_{\Gamma}/\beta}. \tag{A.18}$$

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According to Theorem A.2, the PS density is friendly. In particular, it is decaying and non-planar, so there exists $\alpha > 0$ such that for all $\lambda \in \Lambda(\Gamma)$, $0 < \eta \leq 1$, $\xi > 0$, an affine hyperplane $L \subset \mathbb{R}^n$, and $B = B(\lambda, \eta)$, we have

$$\nu_o(\mathcal{N}(L, \xi \|d_L\|_B) \cap B) \ll_\Gamma \xi^\alpha \nu_o(B). \tag{A.19}$$

The claim now follows for o from (A.18) and (A.19) by taking $\theta = 2\delta_\Gamma/\beta$.

Second, we show the result for a general $w \in \mathbb{H}^n$. Note that

$$e^{-\delta_\Gamma d(o,w)} \ll_\Gamma e^{-\delta_\Gamma \beta_\lambda(w,o)} \ll_\Gamma e^{\delta_\Gamma d(o,w)}.$$

Thus, using this and the fact that $\{\nu_w\}_{w \in \mathbb{H}^n}$ is a conformal density satisfying (10), we arrive at

$$\begin{aligned}\nu_w(\mathcal{N}(L, \xi\eta^\theta) \cap B(\lambda, \eta)) &\ll_\Gamma e^{\delta_\Gamma d(o,w)} \nu_o(\mathcal{N}(L, \xi\eta^\theta) \cap B(\lambda, \eta)) \\ &\ll_\Gamma e^{\delta_\Gamma d(o,w)} \xi^\alpha \nu_o(B(\lambda, \eta)) \\ &\ll_\Gamma e^{2\delta_\Gamma d(o,w)} \xi^\alpha \nu_w(B(\lambda, \eta)).\end{aligned}$$

Last, note that by taking $\xi = \eta^{1-\theta}$, we conclude that $\theta \geq 1$. □

PROPOSITION A.11. *Let Γ be geometrically finite and Zariski dense. There exist constants $\alpha = \alpha(\Gamma) > 0$, $\omega = \omega(\Gamma) \geq 0$, and $\theta = \theta(\Gamma) > \alpha$ satisfying the following: for any $x \in G/\Gamma$ with $x^+ \in \Lambda(\Gamma)$, and for every $\xi > 0$ and $0 < \eta \ll_\Gamma e^{-\operatorname{height}(x)}$, we have that for every hyperplane L,*

$$\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L, \xi) \cap B_U(\eta)) \ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta} \mu_x^{\mathrm{PS}}(B_U(\eta)).$$

Proof. Let $\alpha = \alpha(\Gamma)$, $\theta = \theta(\Gamma) > 0$ satisfy the conclusion of Theorem A.10, and $c' > 1$ satisfy the conclusion of Corollary A.4. Fix $g \in G$ which satisfies $x = g\Gamma$ and $\operatorname{height}(x) = d(\pi(\mathcal{C}_0), \pi(g))$.

By the same argument as in the proof of Corollary A.8 to bound the Busemann function when $\eta \leq 1$, we obtain

$$\begin{aligned}&e^{-\delta_\Gamma \operatorname{height}(x)} \nu_o(\mathrm{Pr}_{g^-}(\mathcal{N}(L, \xi)x \cap B_U(\eta)x)) \\ &\quad\ll_\Gamma \mu_g^{\mathrm{PS}}(\mathcal{N}(L, \xi) \cap B_U(\eta)) = \int_{\mathbf{t} \in \mathcal{N}(L,\xi) \cap B_U(\eta)} e^{\delta_\Gamma \beta_{(u_\mathbf{t} g)^+}(o, u_\mathbf{t} g(o))}\, d\nu_o((u_\mathbf{t} g)^+) \\ &\quad\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o(\mathrm{Pr}_{g^-}(\mathcal{N}(L, \xi)x \cap B_U(\eta)x)).\end{aligned}$$

Thus, for $\eta \ll_\Gamma e^{-\operatorname{height}(x)}$ (so that $ce^{d(o,\pi(x))}\eta \leq 1$ in the following, and we stay within the injectivity radius at x, using (22)), we have that

$$\begin{aligned}&\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L, \xi) \cap B_U(\eta)) \\ &\quad\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o(\mathrm{Pr}_{g^-}(\mathcal{N}(L, \xi) \cap B_U(\eta))) \\ &\quad\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o(\mathcal{N}(L', ce^{d(o,\pi(x))}\xi) \cap B(g^+, ce^{d(o,\pi(x))}\eta)) \quad \text{by Corollary 9.5}\end{aligned}$$

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$$\ll_\Gamma e^{\delta_\Gamma \text{ height}(x)} \bigg(\frac{\xi (c e^{d(o,\pi(x))})^{1-\theta}}{\eta^\theta}\bigg)^\alpha \nu_o(B(g^+, c e^{d(o,\pi(x))}\eta)) \quad \text{by Theorem 9.10}$$

$$\ll_\Gamma e^{\delta_\Gamma \text{ height}(x)} \bigg(\frac{\xi (e^{d(o,\pi(x))})^{1-\theta}}{\eta^\theta}\bigg)^\alpha e^{d(o,\pi(x))\sigma} \nu_o(B(g^+, \eta)) \quad \text{by Lemma 9.6}$$

$$\ll_\Gamma e^{\delta_\Gamma \text{ height}(x)} \bigg(\frac{\xi (e^{d(o,\pi(x))})^{1-\theta}}{\eta^\theta}\bigg)^\alpha$$

$$e^{2d(o,\pi(x))\sigma} \nu_o(B(g^+, c^{-1} e^{-d(o,\pi(x))}\eta)) \quad \text{by Corollary 9.4}$$

$$\ll_\Gamma e^{2\delta_\Gamma \text{ height}(x) + (\sigma + (1-\theta)\alpha) d(o,\pi(x))} \bigg(\frac{\xi}{\eta^\theta}\bigg)^\alpha \mu_x^{\text{PS}}(B_U(\eta))$$

$$\ll_\Gamma e^{(2\delta_\Gamma + \sigma + (1-\theta)\alpha) \text{ height}(x)} \bigg(\frac{\xi}{\eta^\theta}\bigg)^\alpha \mu_x^{\text{PS}}(B_U(\eta))$$

$$\ll_\Gamma e^{\omega \text{ height}(x)} \frac{\xi^\alpha}{\eta^{\theta'}} \mu_x^{\text{PS}}(B_U(\eta)),$$

where

$$\omega = \max\{2\delta_\Gamma + \sigma + (1-\theta)\alpha, 0\}, \quad \theta' = \theta\alpha. \qquad \square$$

A.3. *Absolute friendliness of the PS measure.* When all cusps are of maximal rank, the PS measure is *absolutely friendly*, and stronger results hold. Note that if Γ is convex cocompact, then there are no cusps, so this additional assumption is vacuously true.

Definition A.12. Let μ be a measure defined on $\mathbb{R}^m$.

(1) μ is called *absolutely decaying* (respectively *globally absolutely decaying*) if there exist $\alpha, c_2 > 0$ such that for all $v \in \operatorname{supp} \mu$, all $0 < \xi < \eta \le 1$ (respectively $0 < \xi < \eta$), and every affine hyperplane $L \subseteq \mathbb{R}^n$,

$$\mu(\mathcal{N}(L, \xi) \cap B(v, \eta)) \le c_2 \bigg(\frac{\xi}{\eta}\bigg)^\alpha \mu(B(v, \eta)).$$

(2) μ is called *absolutely friendly* (respectively *globally friendly*) if it is Federer (respectively doubling) and absolutely decaying (respectively globally absolutely decaying).

It is easy to see that if a measure μ is globally friendly, then it is also absolutely friendly.

According to [**34**, Theorem 2], if Γ is convex cocompact or [**5**, Theorem 1.12], if Γ is geometrically finite, ν_o is absolutely friendly if and only if all cusps have maximal rank.

THEOREM A.13. *Assume that Γ is Zariski dense and either convex cocompact or geometrically finite with all cusps having maximal rank. Then the PS measures $\{\mu_x^{\text{PS}}\}_{x^- \in \Lambda(\Gamma)}$ are globally friendly, and the constants in Definition A.12 only depend on Γ (in particular, they do not depend on x).*

This follows by a flowing argument, similar to the doubling results for $x \in \operatorname{supp} m^{\text{BMS}}$ proven before. The key difference is observed by contrasting Definition A.12(1) with

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Theorem A.10: when the powers of ξ, η match, a flowing argument may be used for BMS points. When they do not match, one introduces a power corresponding to how far one flows with a_{-s}.

COROLLARY A.14. *Assume that Γ is Zariski dense and either convex cocompact or geometrically finite with all cusps having maximal rank. There exists $0 < \alpha = \alpha(\Gamma) < 1$ such that for any $x \in \operatorname{supp} m^{\mathrm{BMS}}$, $T > 0$, and $0 < \xi \leq T$, we have*

$$\frac{\mu_x^{\mathrm{PS}}(B_U(T+\xi))}{\mu_x^{\mathrm{PS}}(B_U(T))} - 1 \ll_\Gamma \left(\frac{\xi}{T}\right)^\alpha.$$

Proof. Let $c_1 = c_1(\Gamma), c_2 = c_2(\Gamma) > 0$ and $\alpha = \alpha(\Gamma) > 0$ satisfy the conclusion of Definition A.12 for μ_x^{PS} and $k = 2$.

It follows from the geometry of $B_U(\xi + \eta)x - B_U(\eta)x$ that there exist $L_1, \ldots, L_m$, where m only depends on n, such that

$$B_U(\xi + T)x - B_U(T)x \subseteq \bigcup_{i=1}^m \mathcal{N}_U(L_i, 2\xi).$$

Then, by Definition A.12, we have

$$\begin{aligned}\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))} - 1 &= \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T) - B_U(T))}{\mu_x^{\mathrm{PS}}(B_U(T))} \\ &\leq mc_2\left(\frac{\xi}{T}\right)^\alpha \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))} \\ &\leq mc_1c_2\left(\frac{\xi}{T}\right)^\alpha. \end{aligned}$$

□

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