POINTS OF SPHERICAL MAXIMA AND SOLVABILITY
OF SEMILINEAR ELLIPTIC EQUATIONS

MARTIN SCHECHTER AND KYRIL TINTAREV

ABSTRACT. We give mild sufficient conditions on a nonlinear functional to have
eigenvalues. These results are intended for the study of boundary value problems for
semilinear elliptic equations.

1. Introduction. Let \( g(u) \) be a differentiable functional on a real Hilbert space \( H \).
We are interested in finding eigenvalues and eigenelements of \( g \), i.e., solutions \((\rho, u)\) of
\[
\tag{1.1} g'(u) = \rho u
\]
where \( \rho \in \mathbb{R} \) and \( u \in H \) (for the applications we are considering, it is important that
\( \rho \neq 0 \) and \( u \neq 0 \)). Following an idea used in [4], we make use of the fact that an element
\( u_0 \in H \) which satisfies
\[
\tag{1.2} \|u_0\|^2 = t_0 > 0, \quad g(u_0) = \max_{\|v\|^2=t_0} g(v)
\]
is a solution of (1.1) with
\[
\tag{1.3} \rho = \left( g'(u_0), u_0 \right) / t_0.
\]
Our goal is to locate such elements \( u_0 \). In the present paper we assume as little on the
the functional \( g(u) \) as necessary to obtain the existence of these elements. Our only reg­
ularity assumption on \( g \) is that it be weakly continuous, i.e., that \( u_k \rightharpoonup u \) weakly in \( H \)
implies \( g(u_k) \rightarrow g(u) \). This allows us to obtain solutions of semilinear partial differential
equations under weaker conditions than normally assumed.

We have shown elsewhere [6] that the function
\[
\tag{1.4} \gamma(t) = \sup_{\|v\|^2=t} g(v)
\]
plays an important role in the study of (1.1). We have shown that it is a continuous
nondecreasing function of \( t \). In Section 2 we show that if \( \gamma(t_0) < \gamma(t_1) \), then there is an
infinite number of solution of (1.1) satisfying \( t_0 < \|u\|^2 < t_1 \) with at least one of these
solutions satisfying
\[
\tag{1.5} \rho \geq 2[\gamma(t_1) - \gamma(t_0)] / (t_1 - t_0).
\]
Moreover, we show that if \(0 < t_0 < t_1 < t_2\) and \(\rho\) is any number satisfying
\[
(\gamma(t_2) - \gamma(t_0))/(t_2 - t_1) < \rho/2 < (\gamma(t_1) - \gamma(t_0))/(t_1 - t_0)
\]
then (1.1) has a solution \(u\) satisfying \(t_0 < \|u\|^2 < t_2\). As a corollary we see that if
\[
(1.7) \quad \alpha_\infty := \liminf_{t \to \infty} \gamma(t) / t < \alpha := \sup_{t_1, t_2} (\gamma(t_2) - \gamma(t_1))/(t_2 - t_1)
\]
then for any \(\rho \in (2\alpha_\infty, 2\alpha)\) there is a solution \(u \in H \setminus \{0\}\) of (1.1). Further results are given in Section 2.

Application of these results to boundary value problems for semilinear elliptic equations might be inferred from [6]. If \(A\) is a linear elliptic operator of order \(2m\) and \(f(x, t)\) is a Caratheodory function, (1.1) has a meaning of
\[
(1.8) \quad Au = \lambda f(x, u), \quad u \in H_0^m(\Omega),
\]
holding in a semi-strong sense. Here
\[
(1.9) \quad g(u) = \int_\Omega F(x, u(x)) \, dx,
\]
with
\[
(1.10) \quad F(x, s) = \int_0^s f(x, \sigma) \, d\sigma.
\]
Weak continuity of \(g\) and estimates on \(\gamma\) involved in theorems of this paper result in milder conditions on \(F\) rather than on \(f\).

2. Existence of Eigenvalues. In this section we shall be concerned with proving the existence of eigenvalues assuming only
(i) \(g(u)\) is a weakly continuous Frechet differentiable map from an infinite dimensional real Hilbert space \(H\) to \(\mathbb{R}\).

We define for \(t \geq 0\)
\[
(2.1) \quad S_t = \{x \in H \mid \|x\|^2 = t\}
\]
\[
(2.2) \quad \gamma(t) = \sup_{u \in S_t} g(u)
\]
\[
(2.3) \quad \Sigma_t = \{u \in S_t \mid g(u) = \gamma(t)\}.
\]
It was shown in [6] that \(\gamma(t)\) is a continuous nondecreasing function of \(t\). First we have

**Theorem 2.1.** If \(\gamma(t) > \gamma(t_0)\) for \(t > t_0\), then there are sequences \(\{s_k\} \subset \mathbb{R}\), \(\{u_k\} \subset H\), \(\{\rho_k\} \subset \mathbb{R}\) such that \(s_k \downarrow t_0, u_k \in \Sigma_{s_k}, \rho_k > 0\) and
\[
(2.4) \quad g'(u_k) = \rho_k u_k.
\]

**Corollary 2.2.** If \(\gamma(t_0) < \gamma(t_1)\), then there is an infinite number of solutions \((u, \rho)\) of
\[
(2.5) \quad g'(u) = \rho u.
\]
POINTS OF SPHERICAL MAXIMA AND SOLVABILITY

In proving Theorem 2.1 we shall make use of the following results from [6].

**Lemma 2.3.** If \( \Sigma_t = \phi \), then there is a \( t_- < t \) such that \( \gamma(s) = \gamma(t) \) for \( t_- \leq s \leq t \).

**Lemma 2.4.** If \( \varphi(t) \in C^1(0, \infty) \) is such that \( \varphi(t) - \gamma(t) \) has a local minimum at \( t_0 \) and \( \varphi'(t_0) \neq 0 \), then there is a \( u \in \Sigma_{t_0} \) such that

\[
(2.7) \quad g'(u) = 2\varphi'(t_0)u. 
\]

**Proof of Theorem 2.1.** Let \( t_1 \) be any number > \( t_0 \), and \( \hat{t} = (t_0 + t_1)/2 \). For \( t_0 < s < \hat{t} \) let \( \varphi_s(t) = A(t - s)^2 + B \), where the constants \( A, B \) are chosen so that \( \varphi_s(t_i) = \gamma(t_i), i = 0, 1 \). Note that \( A > 0 \) as long as \( s < \hat{t} \). Let \( \psi_s(t) = \varphi_s(t) - \gamma(t) \). Then \( \psi_s(t_i) = 0, i = 0, 1 \). Since \( \varphi_s(t) \) is decreasing for \( t_0 < t < s \) and \( \gamma(t) \) is nondecreasing, \( \psi_s(t) \) has a negative minimum in \( [t_0, t_1] \) for every \( s \). Let \( t_s \) be a point where

\[
\psi_s(t_s) = \min_{t_0 \leq t \leq t_1} \psi_s(t). 
\]

Clearly \( t_s \geq s \) for every \( s < \hat{t} \). We claim that there cannot be a \( \delta > 0 \) such that \( t_s = s \) for \( t_0 < s < t_0 + \delta \). For then

\[
\varphi_s(t) - \gamma(t) \geq \varphi_s(s) - \gamma(s), \quad t_0 \leq t \leq t_1, \quad t_0 < s < t_0 + \delta. 
\]

This implies

\[
\gamma(t) - \gamma(s) \leq \varphi_s(t) - \varphi_s(s) = A(t - s)^2 
\]

for such \( s, t \). In turn this implies that \( \gamma'(s) \) exists and vanishes for \( t_0 < s < t_0 + \delta \). This would mean that \( \gamma(t) = \gamma(t_0) \) for \( t_0 \leq t < t_0 + \delta \), contrary to assumption. Hence for each \( \delta > 0 \) there is an \( s \) such that \( t_0 < s < t_0 + \delta \) and \( t_s > s \). Consequently \( \psi_s(t) \) has a local minimum at \( t_s \) while \( \varphi_s'(t_s) = 2A(t_s - s) > 0 \). Thus there is a \( u \in \Sigma_{t_s} \) such that \( g'(u) = 2\varphi_s'(t_s)u \) (Lemma 2.4). This means that (2.5) has a solution satisfying (2.6). Since \( t_1 > t_0 \) was arbitrary, the result follows.

**Proof of Corollary 2.2.** Let \( \tilde{t}_0 \) be the largest number such that \( \gamma(t) = \gamma(t_0) \) for \( t_0 \leq t < \tilde{t}_0 < t_1 \) and \( \gamma(t) > \gamma(t_0) \) for \( t > \tilde{t}_0 \). Apply Theorem 2.1.

**Theorem 2.5.** If \( \gamma(t_0) < \gamma(t_1) \), then for each \( \rho_0 \) satisfying \( 0 < \rho_0 < \sigma_0 := 2[\gamma(t_1) - \gamma(t_0)]/ (t_1 - t_0) \) the following alternative holds: either

a) (2.5) has a solution \( u \in S_{t_1} \) with \( \rho \geq \rho_0 \)

or

b) (2.5) has a solution \( u \) with \( t_0 < ||u||^2 < t_1 \) and \( \rho = \rho_0 \).

In proving Theorem 2.5 we shall make use of
LEMMA 2.6. Assume that \( \varphi(t) \in C^1[t_0, t_1] \), \( \varphi'(t_1) > 0 \) and
\[
\varphi(t) - \gamma(t) \geq \varphi(t_1) - \gamma(t_1), \quad t_0 \leq t \leq t_1
\]
Then \( \Sigma_{t_1} \neq \emptyset \) and every \( u \in \Sigma_{t_1} \) is a solution of (2.5) with \( \rho \geq 2\varphi'(t_1) \).

PROOF. Clearly \( \gamma(t) < \gamma(t_1) \) for \( t < t_1 \). For otherwise (2.9) will imply that \( \varphi'(t_1) \leq 0 \). By Lemma 2.3 we see that \( \Sigma_{t_1} \neq \emptyset \). For \( u \in S_t \), \( t_0 \leq t \leq t_1 \), let
\[
I(u) = \varphi(\|u\|^2) - g(u).
\]
Then if \( u_1 \in \Sigma_{t_1} \), \( u \in S_t \),
\[
I(u_1) = \varphi(t_1) - g(u_1) = \varphi(t_1) - \gamma(t_1)
\]
\[
\leq \varphi(t) - \gamma(t) \leq \varphi(t) - g(u) = I(u),
\]
Thus \( I(u_1) \) is a minimum of \( I(u) \) for \( u \) satisfying \( t_0 \leq \|u\|^2 \leq t_1 \). Consequently there is a \( \beta \geq 0 \) such that
\[
I'(u_1) = -\beta u_1.
\]
Thus
\[
2\varphi'(t_1)u_1 - g'(u_1) = -\beta u_1
\]
or
\[
g'(u_1) = [2\varphi'(t_1) + \beta]u_1.
\]
This gives the desired result.

PROOF OF THEOREM 2.5. Let \( \psi(t) = \frac{1}{2}\rho_0(t - t_0) + \gamma(t_0) - \gamma(t) \). Then \( \psi(t_0) = 0 \) and \( \psi(t_1) \leq 0 \). Assume first that \( \psi(t) \geq \psi(t_1) \) for \( t_0 \leq t \leq t_1 \). Then by Lemma 2.6, \( \Sigma_{t_1} \neq \emptyset \) and every \( u \in \Sigma_{t_1} \) is a solution of (2.5) with \( \rho \geq \rho_0 \). This is alternative (a). Otherwise there is a \( t \) between \( t_0 \) and \( t_1 \) such that \( \psi(t) < \psi(t_1) \). This means that \( \psi \) has a minimum in \( (t_0, t_1) \). We can now apply Lemma 2.4 to conclude that (2.5) has a solution satisfying \( t_0 < \|u\|^2 < t_1 \) and \( \rho = \rho_0 \).

COROLLARY 2.7. If \( \gamma(t_0) < \gamma(t_1) \), then (2.5) has a solution satisfying
\[
t_0 < \|u\|^2 \leq t_1, \quad \rho \leq 2[\gamma(t_1) - \gamma(t_0)]/ (t_1 - t_0).
\]

PROOF. Let \( \rho_0 \) equal the right hand side in (2.9).

THEOREM 2.8. If \( 0 \leq t_0 < t_1 < t_2 \) and
\[
[\gamma(t_2) - \gamma(t_0)]/(t_2 - t_0) < \rho / 2 < [\gamma(t_1) - \gamma(t_0)]/ (t_1 - t_0)
\]
then (2.5) has a solution satisfying \( t_0 < \|u\|^2 < t_2 \).

PROOF. Let \( \varphi(t) = \frac{1}{2}\rho(t - t_0) + \gamma(t_0) \) and \( \psi(t) = \varphi(t) - \gamma(t) \). Then \( \psi(t_1) < 0 \) while \( \psi(t_2) > 0 \). Thus there is a point \( t_3 \) such that \( t_1 < t_3 < t_2 \) and \( \psi(t_3) = 0 \). Since \( \psi(t_0) = 0 \) and \( \psi(t_1) < 0 \), \( \psi(t) \) must have a negative minimum in the interval \( (t_0, t_2) \). We can now apply Lemma 2.4 to conclude that (2.5) has a solution for \( u \in \Sigma_t \) for some \( t \) satisfying \( t_0 < t < t_2 \).
COROLLARY 2.9.  If $0 < t_1 < t_2$ and
\begin{equation}
[\gamma(t_2) - \gamma(t_1)] / (t_2 - t_1) < \rho / 2 < D^- \gamma(t_1),
\end{equation}
then for each $\epsilon > 0$ there is a solution $u \in H$ of (2.5) satisfying $t_1 - \epsilon < \|u\|^2 < t_2$.

PROOF.  By (2.11) there is a $t_0 < t_1$ such that $t_1 - t_0 < \epsilon$ and (2.10) holds. Apply Theorem 2.8.

COROLLARY 2.10.  If $0 \leq t_0 < t_1$ and
\begin{equation}
D^+ \gamma(t_1) < \rho / 2 < [\gamma(t_1) - \gamma(t_0)] / (t_1 - t_0)
\end{equation}
then for every $\epsilon > 0$ there is a solution $u \in H$ of (2.5) such that $t_0 < \|u\|^2 < t_1 + \epsilon$.

PROOF.  By (2.12), there is a $t_2 > t_1$ such that $t_2 - t_1 < \epsilon$ and (2.10) holds. Apply Theorem 2.8.

COROLLARY 2.11.  If
\begin{equation}
\alpha_\infty := \liminf_{t \to \infty} \gamma(t) / t < \alpha := \sup_{t_1, t_2} \frac{\gamma(t_2) - \gamma(t_1)}{(t_2 - t_1)}
\end{equation}
then for any $\rho \in (2\alpha_\infty, 2\alpha)$ there is a solution $u \in H \setminus \{0\}$ of (2.5).

PROOF.  By (2.13) one can pick $t_0, t_1, t_2$ such that (2.10) holds and apply Theorem 2.8.

COROLLARY 2.12.  If $\alpha \neq 0, \infty$ then for every $\epsilon > 0$ there is a solution $(u, \rho)$ of (2.5) such that $u \neq 0$ and $\rho \geq \alpha - \epsilon$.

PROOF.  Apply Corollary 2.7.

COROLLARY 2.13.  If either $D^- \gamma(t_1)$ or $D^+ \gamma(t_1)$ is positive, then there are sequences
\begin{equation}
\{\rho_k\} \subset \mathbb{R}, \{u_k\} \subset H \setminus \{0\}
\end{equation}
such that $\rho_k > 0$, $\|u_k\|^2 \to t_1$ and (2.4) holds.

PROOF.  If $D^- \gamma(t_1) > 0$, then for every $\epsilon > 0$ there is a $t_0 < t_1$ such that $t_1 - t_0 < \epsilon$ and $\gamma(t_0) < \gamma(t_1)$. By Corollary 2.2 there is an infinite number of solutions of (2.5), (2.6). A similar argument works if $D^+ \gamma(t_1) > 0$.

LEMMA 2.14.  If $u_0 \in \Sigma_{t_0}$, then
\begin{equation}
D^- \gamma(t_0) \leq \left( g'(u_0), u_0 \right) / 2t_0 \leq D^+ \gamma(t_0).
\end{equation}

PROOF.  We have, modulo $o(t - t_0)$,
\begin{equation}
\gamma(t) - \gamma(t_0) \geq g(t^{1/2}t_0^{-1/2}u_0) - g(u_0) - \left( g'(u_0), u_0 \right) (t^{1/2}t_0^{-1/2} - 1) - \frac{\left( g'(u_0), u_0 \right) (t - t_0)}{2t_0}.
\end{equation}
If $t < t_0$, this gives the first inequality in (2.14); if $t > t_0$, it gives the second.
COROLLARY 2.15. If $D^\gamma(t_0) > 0$, then $\Sigma_0 \neq \emptyset$. Thus (2.5) has a solution $(u, \rho)$ with $u \in \Sigma_0$ and $D^\gamma(t_0) \leq \rho / 2$.

PROOF. If $\Sigma_0 = \emptyset$, then $D^\gamma(t_0) = 0$ by Lemma 2.3. If $u \in \Sigma_0$, then it is a solution of (2.5) for some $\rho \geq 0$.

COROLLARY 2.16. If $D^\gamma(t_0) > 0$, then the conclusion of Theorem 2.1 holds with

\begin{equation}
\liminf_k \rho_k \geq D^\gamma(t_0).
\end{equation}

PROOF. For each $k$ there is a point $t_k > t_0$ such that $t_k - t_0 < 1/k$ and

$$
\frac{\gamma(t_k) - \gamma(t_0)}{(t_k - t_0)} \geq D^\gamma(t_0) - (1/k).
$$

By Corollary 2.7 there is a solution $(u_k, \rho_k)$ of (2.4) satisfying

$$
t_0 < \|u_k\|^2 \leq t_k, \quad \rho_k \geq D^\gamma(t_0) - (1/k).
$$

This gives the desired result. ■

COROLLARY 2.17. If $\gamma(t_0) < \gamma(t_1)$, then one of the following alternatives holds: either

(a) there is a solution $(u, \rho)$ of (2.5) such that $u \in S_t$ and $\rho \geq \sigma_0 = 2[\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0)$ or

(b) there is a sequence $(u_k, \rho_k)$ of solutions of (2.4) such that $u_k \in S_t$ and $\rho_k \not\nearrow \sigma_0$, or

(c) there is a $\bar{\rho} < \sigma_0$ such that for each $\rho$ satisfying $\bar{\rho} < \rho < \sigma_0$ there is a solution $(u, \rho)$ of (2.5) with $t_0 < \|u\|^2 < t_1$.

PROOF. Suppose (a) does not hold, and let $\bar{\rho}$ be the supremum of all $\rho$ such that (2.5) has a solution $(u, \rho)$ with $u \in S_t$. If $\bar{\rho} = \sigma_0$, then (b) must hold. If $\bar{\rho} < \sigma_0$, then (c) must hold by Theorem 2.5. ■

COROLLARY 2.18. If $\gamma(t_0) < \gamma(t_1)$, then the following alternative holds: either

(a) there is a $\tilde{t} < t_1$ such that for every $t$ satisfying $\tilde{t} < t < t_1$ there is a $u \in S_t$ and a $\rho > 0$ such that (2.5) holds, or

(b) there is a $\rho_0 > 0$ such that for each $\rho$ satisfying $0 < \rho < \rho_0$ there is a $u$ such that $t_0 < \|u\|^2 < t_1$ and (2.5) holds.

PROOF. Let $\tilde{t}$ be the infimum of all $t$ in $[t_0, t_1]$ such that $\gamma(t_0) < \gamma(t)$. If for each $t > \tilde{t}$ there is a $u \in S_t$ and a $\rho > 0$ such that (2.5) holds, then option (a) is true. Otherwise there is a $\tilde{t}$ in $(\tilde{t}, t_1)$ such that (2.5) has no solution in $S_t$ for any $\rho > 0$. In this case Theorem 2.5 tells us that for any $\rho$ satisfying $0 < \rho < \rho_0 := 2[\gamma(\tilde{t} - \gamma(t_0))] / (\tilde{t} - t_0)$ there is a solution $u$ of (2.5) satisfying $t_0 < \|u\|^2 < \tilde{t}$. ■
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University of California
Irvine, California 92717
U.S.A.