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# DIRAC DELTA FUNCTIONS VIA NONSTANDARD ANALYSIS

BY

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1. **Introduction.** We recall that a Dirac delta function $\delta(x)$ in the real number system $\mathscr{R}$ is the idealization of a function that vanishes outside a "short" interval and satisfies $\int_{-\infty}^{\infty} \delta=1$. It is conceived as a function $\delta$ for which $\delta(0)=+\infty$, $\delta(t)=0$ if $t\neq 0$, and $\int_{-\infty}^{\infty} \delta=1$. This function should possess the "sifting property" $\int_{-\infty}^{\infty} f\delta=f(0)$ for any continuous function $f$. Even though certain sequences of functions are used, via a limit operation, to approximate a Dirac delta function (for details, see [3] and [4]), no function in $\mathscr{R}$ has these properties.

Based on these intuitive ideas we shall define a Dirac delta function in Robinson's nonstandard number system $^*\mathscr{R}$ (see [1]) and shall derive the sifting property as a consequence of the definition. (In [2], it is suggested that the sifting property must be included in the definition.).

2. **Dirac delta functions.** We now show that certain internal functions of $^*\mathscr{R}$ can be regarded as Dirac delta functions. Let $F$ be the set of all function in $\mathscr{R}$; so $^*F$ is the set of all internal functions in $^*\mathscr{R}$.

DEFINITION. $\delta \in {}^*F$ is called a Dirac delta function if

$$(1)\ \operatorname{dom} \delta = {*R};$$

$$(2)\ \delta(x) \geq 0, \quad \text{for all} \quad x \in {*R};$$

$$(3)\ \underset{*R>0}{\exists\varepsilon} (\varepsilon \simeq 0 \land \underset{*R}{\forall x} (x \not\simeq 0 \rightarrow \delta(x) < \varepsilon));$$

$$(4)\ \int_{-\kappa}^{\kappa} \delta \simeq 1, \quad \text{for each } \kappa \in {}^*N-N.$$

From (3) it is clear that for all $x \in {}^*R$, $x\not\simeq 0$ implies $\delta(x)\simeq 0$. This expresses the idea that a Dirac delta function vanishes outside a "short" interval. Condition (2) is required to prove the sifting property of Dirac delta functions. The classical idea that $\delta(0)=+\infty$ is partially expressed by Lemma 2 below.

LEMMA 1. *For each $h \in R$, $h>0$, $\int_{-h}^{h} \delta\simeq 1$, where $\delta$ is a Dirac delta function.*

**Proof.** For each $h \in R$, $h>0$, each $\kappa \in {}^*N-N$, $1\simeq\int_{-\kappa}^{\kappa} \delta=\int_{-\kappa}^{-h} \delta+\int_{-h}^{h} \delta+\int_{h}^{\kappa} \delta$; but $0\leq\int_{-\kappa}^{-h} \delta\leq\varepsilon(\kappa-h)$, where $\varepsilon \in {}^*R$, $\varepsilon\simeq 0$, such that $\forall_{*R}x(x\not\simeq 0\rightarrow\delta(x)<\varepsilon)$. Take $\kappa<1/\sqrt{\varepsilon}$. Then $\int_{-\kappa}^{-h} \delta\simeq 0\simeq\int_{h}^{\kappa} \delta$. It follows that $\int_{-h}^{h} \delta\simeq 1$.



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LEMMA 2. *For each $h \in R$, $h>0$, the least upper bound of the values of $\delta$ on $[-h, h]$ is infinite.*

**Proof.** If this lemma is false then there exists $h \in R$, $h>0$, such that $\ell\mathrm{ub}_{x\in[-h,h]}\, \delta(x)<p$, some $p \in R$, $p>0$. Then by lemma 1, for each $v \in R$, $v>0$

$$1 \simeq \int_{-h/v}^{h/v} \delta \leq \int_{-h/v}^{h/v} p = \frac{2hp}{v}$$

Taking $v=3ph$, we get $\frac{2}{3}\geq 1$, a contradiction.

We now present three different examples of Dirac delta functions to illustrate our definition. Throughout, $\omega$ is an infinite natural number.

EXAMPLE 1. Let $f: {}^*R\to {}^*R$ be defined by

$$f(x) = \frac{\omega}{2}, \; -\frac{1}{\omega} \leq x \leq \frac{1}{\omega}$$
$$= 0 \quad \text{otherwise}$$

Clearly, $f$ satisfies the four conditions of the Definition and hence is a Dirac delta function.

EXAMPLE 2. Let $g: {}^*R\to {}^*R$ be defined by $g(x)=(\omega/\pi(\omega^2x^2+1))$. Certainly (1) and (2) are trivially satisfied. We now show that (3) is satisfied. Indeed, for all $x \in {}^*R$, $x \not\simeq 0$, $g(x)<(1/\pi\omega x^2)<(1/\pi\sqrt{\omega})\simeq 0$. To establish (4) observe that for each $\kappa \in {}^*N-N$, by the Fundamental Theorem of Integral Calculus in ${}^*\mathscr{R}$,

$$\int_{-\kappa}^{\kappa} g = (1/\omega\pi)(\omega\, {*\arctan}(\kappa\omega) - \omega\, {*\arctan}(-\kappa\omega)),$$
$$= (2/\pi){*\arctan}(\kappa\omega)$$
$$\simeq (2/\pi)\cdot(\pi/2) = 1$$

EXAMPLE 3. Let $h: {}^*R\to {}^*R$ be defined by $h(x)=\omega/\sqrt{\pi}\,{*\exp}(\omega^2x^2)$ where $*\exp$ is the function in ${}^*F$ rooted in the exponential function.

To show (4) apply the Transfer Theorem (see [2]) to

$$(1-\exp(-n^2b^2))^{1/2} \leq \int_{-b}^{b} (n/\sqrt{\pi}\exp(n^2x^2))\,dx \leq (1-\exp(-2n^2b^2))^{1/2},$$

which is true in $\mathscr{R}$ for each $n, b \in N$. Thus

$$(1-{*\exp}(-\omega^2\kappa^2))^{1/2} \leq \int_{-\kappa}^{\kappa} (\omega\,dx/\sqrt{\pi}\,{*\exp}(\omega^2x^2)) \leq (1-{*\exp}(-2\omega^2\kappa^2))^{1/2},$$

is true in ${}^*\mathscr{R}$ for each $\kappa \in {}^*N-N$. But ${*\exp}(-\omega^2\kappa^2)\simeq 0\simeq {*\exp}(-2\omega^2\kappa^2)$ and so $1\leq {}^0(\int_{-\kappa}^{\kappa}(\omega\,dx/\sqrt{\pi}{*\exp}(\omega^2x^2))\leq 1$. It follows that $\int_{-\kappa}^{\kappa}(\omega\,dx/\sqrt{\pi}{*\exp}(\omega^2x^2))\simeq 1$.

**3. The sifting property.** To prove that each Dirac delta function possesses the sifting property we shall need the following lemma.

LEMMA 3. *For each* $\kappa \in {}^*N - N$, *each* $h \in R$, $h > 0$, $\int_{-\kappa}^{-h} \delta \simeq 0$ *and* $\int_{h}^{\kappa} \delta \simeq 0$.

**Proof.** For each $\kappa \in {}^*N - N$, each $h \in R$, $h > 0$

$$1 \simeq \int_{-\kappa}^{\kappa} \delta = \int_{-\kappa}^{-h} \delta + \int_{-h}^{h} \delta + \int_{h}^{\kappa} \delta$$

By Lemma 1 and the fact that $\delta \geq 0$, we have $\int_{-\kappa}^{-h} \delta \simeq 0 \simeq \int_{h}^{\kappa} \delta$.

## 4. The sifting property of dirac delta functions.

THEOREM. *For each* $\kappa \in {}^*N - N$, *each* $f \in F$, *such that* $f: R \to R$ *and* $f$ *is bounded and continuous on* $R$,

$$\int_{-\kappa}^{\kappa} {}^*f \delta \simeq f(0).$$

**Proof.** For each $\kappa \in {}^*N - N$, each $h \in R$, $h > 0$,

$$\int_{-\kappa}^{\kappa} {}^*f \delta = \int_{-\kappa}^{-h} {}^*f \delta + \int_{-h}^{h} {}^*f \delta + \int_{h}^{\kappa} {}^*f \delta$$

$$= {}^*f(t_1) \int_{-\kappa}^{-h} \delta + \int_{-h}^{h} {}^*f \delta + {}^*f(t_2) \int_{h}^{\kappa} \delta$$

by a Mean Value Theorem for integrals in ${}^*\mathscr{R}$, where $t_1 \in (-\kappa, -h)$ and $t_2 \in (h, \kappa)$. Since $f$ is bounded, there exists $m \in R$, such that $\forall_R x(|f(x)| \leq m)$ is true for $\mathscr{R}$; it follows that $\forall_{{}^*R} x(|{}^*f(x)| \leq m)$ is true for ${}^*\mathscr{R}$. Thus for each $x \in {}^*R$, ${}^*f(x)$ is a finite number. Therefore

$${}^*f(t_1) \int_{-\kappa}^{-h} \delta = {}^*f(t_1)\varepsilon \simeq 0$$

for some $\varepsilon \simeq 0$ (by Lemma 3). Similarly, ${}^*f(t_2) \int_{h}^{\kappa} \delta \simeq 0$. Therefore

$$\int_{-\kappa}^{\kappa} {}^*f \delta \simeq \int_{-h}^{h} {}^*f \delta = {}^*f(t_3) \int_{-h}^{h} \delta \simeq {}^*f(t_3)$$

where $t_3 \in (-h, h)$.

(by Lemma 1 and the fact that ${}^*f(t_3)$ is finite).

We claim that ${}^*f(t_3) \simeq {}^*f(0) = f(0)$. If possible, assume that ${}^*f(t_3) \not\simeq {}^*f(0)$. Then there is an $r \in R$, $r > 0$, such that

$$|{}^*f(t_3) - {}^*f(0)| > r. \tag{5}$$

Since $f$ is continuous at 0, ${}^*f$ is $S$-continuous at 0 (see [2]); i.e.,

$$\forall_{R>0} \varepsilon \, \exists p \, \forall_{{}^*R} x \, (|x| < p \to |{}^*f(x) - {}^*f(0)| < \varepsilon)$$

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is true for $^*\mathscr{R}$. Now let us take $\varepsilon=r/2$. Then there exists $p \in R$, $p>0$, such that

$$\forall_{*R} x\,(|x| < p \rightarrow |{*f}(x)-{*f}(0)| < r/2) \tag{6}$$

is true for $^*\mathscr{R}$. But

$$\int_{-\kappa}^{\kappa} {*f}\,\delta \simeq \int_{-p}^{p} {*f}\,\delta = {*f}(t_4)\int_{-p}^{p} \delta \simeq {*f}(t_4).$$

where $t_4 \in (-p, p)$. Hence $^*f(t_3)\simeq {^*f}(t_4)$. But $|t_4|<p$ and so by (6)

$$|{*f}(t_4)-{*f}(0)| < r/2. \tag{7}$$

From (5) and (7),

$$|{*f}(t_3)-{*f}(0)| - |{*f}(t_4)-{*f}(0)| > r-(r/2) = r/2.$$

Thus $|{^*f}(t_3)-{^*f}(t_4)|>r/2$ contradicting the fact that $^*f(t_3)\simeq {^*f}(t_4)$. Therefore $^*f(t_3)\simeq {^*f}(0)=f(0)$, hence $\int_{-\kappa}^{\kappa} {^*f}\delta\simeq f(0)$.

COROLLARY. *For each $\kappa \in {^*N}-N$, each $f\in F$ such that $f$ is continuous and bounded on $R$, $\int_{-\kappa}^{\kappa} {^*f}(t-x)\,\delta(x)\,dx\simeq f(t)$, each $t \in R$.*

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