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RESEARCH ARTICLE

Reengineering of interbank networks

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Abstract
We investigate the reengineeering of interbank networks with a specific focus on capital increase. We consider a scenario where all other components of the network's infrastructure remain stable (a practical assumption for short-term situations). Our objective is to assess the impact of raising capital on the network's robustness and to address the following key aspects. First, given a predefined target for network robustness, our aim is to achieve this goal optimally, minimizing the required capital increase. Second, in cases where a total capital increase has been determined, the central challenge lies in distributing this increase among the banks in a manner that maximizes the stability of the network. To tackle these challenges, we begin by developing a comprehensive theoretical framework. Subsequently, we formulate an optimization model for the network's redesign. Finally, we apply this framework to practical examples, highlighting its applicability in real-world scenarios.

Keywords: Decision analysis; networks; system dynamics; complexity theory

1. Introduction

Interbank markets play a crucial role in the functioning of modern economic systems by facilitating the flow of funds, enabling financial transactions and ensuring liquidity. Especially, during periods of financial stress or crises, the stability and functioning of bank institutions become crucial to maintaining overall financial stability. As a result, the problem of default contagion in an interbank market has emerged as one of the most important topics in economic studies.

Many academics and researchers have utilized network theory in order to study interbank systems and the emergence of systemic risk through the interconnections of the institutions. In such network structure, every node represents a bank and the capital flows (loans) between banks are represented by the edges of the network. The flourishing literature of the recent years has addressed several issues of default contagion. Many topics have been considered and studied extensively, for instance:

1. the role of the structure and the topology of the network (see Acemoglu et al., 2015; Allen & Gale, 2000; Anand et al., 2015; Bardoscia et al., 2017; Cohen-Cole et al., 2015; Elliott et al., 2014; Gai & Kapadia, 2010; Iori et al., 2006; Mistrulli, 2011);
2. the role of interconnectedness among banks (see Acemoglu et al., 2015; Allen & Gale, 2000; Amini et al., 2016; Ladley, 2013);
3. the role of heterogeneity in degree distributions, balance sheet size, and degree correlations between banks (see Caccioli et al., 2012);

4. the importance of the large institutions and of institutions which may be smaller, but they have a lot of connections in the interbank market (see Battiston et al., 2012; Caccioli et al., 2012; Chinazzi et al., 2015);
5. how we can relate the propagation of a crisis to measurable features of the network (see Amini et al., 2016; Caccioli et al., 2014; Cifuentes et al., 2005; Cont et al., 2010);
6. to introduce accurate indicators which objectively assess the network and the stability of each bank (see Leventides et al., 2022);
7. to study the channels of contagion (see Upper, 2011);
8. to assess the systemic risk (see Barucca et al., 2016; De Souza et al., 2016; Cimini et al., 2015).

Although not explicitly stated, it can be inferred that the ultimate objective underlying all the previously mentioned studies and research endeavors is the *reengineering of interbank networks*. This concept refers to the process of making improvements to the structure, functioning, and overall design of the interbank network. This procedure aims to enhance the network's efficiency, resilience, and stability and to mitigate the systemic risk. Networks reengineering is a main topic in mechanics; however, to the best of our knowledge, it has not found important applications in the field of interbank networks (see, however, Wishnick, 2021).

Despite our knowledge on the interbank networks and the mechanism of default contagion, the development of efficient policies for the reengineering of a network and the elimination of systemic risk remains a challenge. The propagation of a crisis through the network depends on a plethora of factors (interconnectedness, capital levels, exposures, etc) which are difficult to take into account simultaneously and whose evolution and role in a crisis are difficult to predict.

In this article, our precise focus is directed toward reengineering of interbank networks. To be more specific, we embrace the following scenario. We assume that all the other features of the network's infrastructure (interconnectedness, exposures, balance sheet size, deposits, etc.) remain constant, and we only allow changes to the capital levels of the institutions. This approach has two advantages. First, it corresponds to realistic situations, especially in short-time scenarios, for instance, in the case of increase in share capital or intervention of the central bank. Second, it enables us to develop a theoretical framework for studying the effect of capital increase in the robustness of the network. (More complicated scenarios involving alteration in several aspects of the networks seem that can be handled only with computer simulations.)

The problems and questions we examine can be categorized in three directions. First, we are interested in analyzing how sensitive is the robustness of the network to the capital raise of any single bank. Second, we assume that some specific target for the stability of the network has been determined in terms of some index which should be increased to a desirable level. We then study how this goal can be achieved with the minimum cost in capitals. Finally, suppose that some increase in capital levels has been determined for the total set of banks. The problem then is to decide how the new capitals should be distributed among banks so that to achieve the greatest possible result for the stability of the interbank network.

The rest of the paper is organized as follows. In Section 2, we describe some preliminaries that are used in the paper. The most important parts are the contagion graph of an interbank network and the indicators for the robustness of the network. In Section 3, we introduce the liabilities ordering, which determines the groups of institutions that are "poisonous" for a given a bank i (in the sense that their failure would lead the bank i to collapse). In Section 4, we study the sensitivity of the network to the capital raise of each bank. Sections 5 and 6 are devoted to the study of the changes in the indicators and the contagion graph that follow the capital raise. In Section 7, we consider in a systematic way the problem of capital raise in an interbank market, and in Section 8, we illustrate the ideas and techniques of the paper with an example. Finally, in Section 9, we utilize the concepts presented in the previous sections to formulate a Mixed Integer

Non-Linear Programming (MINLP) to determine the optimal allocation of new capital between the banks of the network to ensure its stability.

2. Preliminaries

In this section, we gather and briefly present some fundamental results that we shall use in the main part of the article. We start with the interbank networks and the mechanism of default contagion. Then, we describe the contagion graph which demonstrates the propagation of the crisis through the interbank market for all possible initial shocks. Finally, we revise various indicators measuring the robustness of the network that can be obtained by the contagion graph.

2.1 Interbank networks and default contagion

An interbank market can be depicted as a directed weighted graph. This approach is standard, and it has been utilized by several researchers (for instance, Acemoglu et al., 2015; Amini et al., 2016; Cont et al., 2010). More specifically, an interbank market consisting of n banks is described by a triplet (X, E, C), where $X = \{1, 2, \ldots, n\}$ is the vertex set of the graph whose elements represent the financial institutions. Furthermore, $C = (c_1, c_2, \ldots, c_n)$ is the vector of capitals, that is, c_i is the capital of the institution i indicating the ability of i to absorb any losses. Finally, $E = (E_{ij})_{i,j=1}^{n}$ is the $n \times n$ matrix of bilateral exposures. More specifically, E_{ij} is the value of all liabilities of the institution j to i (in other words, the exposure of i to j) at the date of computation. Therefore, E is the adjacency matrix of the direct graph, which demonstrates the set of edges as well as the weight E_{ij} of the edge from i to j. If $E_{ij} = 0$, then there is no (direct) edge from i to j. It follows that the diagonal elements of the E are equal to 0, since banks cannot lend to themselves.

Additionally, if we sum up the ith column of the matrix E, then we obtain the total interbank liabilities of the bank i. Therefore, the *interbank liabilities vector* can be defined as $\mathbf{L} = (L_i)_{i=1}^{n} = \left(\sum_{j=1}^{n} E_{ji}\right)_{i=1}^{n}$. Similarly, if we sum up the ith raw of the matrix, then we obtain the total interbank assets of the bank i. Therefore, $\mathbf{a} = (a_i)_{i=1}^{n} = \left(\sum_{j=1}^{n} E_{ij}\right)_{i=1}^{n}$ is the *interbank assets vector*.

We now describe the fundamental mechanism of default contagion adopted in this study. We assume that a subset of banks denoted as A defaults due to their inability to meet legal obligations, such as servicing loans. While real-world cases involve only a few banks in set A, our abstract model allows for any subset of X to be A. We are primarily concerned with studying the propagation of default contagion through the interbank market, rather than the specific causes leading to A's default.

The initial shock sets off a chain reaction of defaults: banks in subset A cannot meet their obligations to creditors in $X \setminus A$; hence, any creditor $j \in X \setminus A$ faces losses equal to its exposures $\sum_{i \in A} E_{ji}$. At this point, we make some further assumptions, which are realistic, especially in short run, and they have been used in other studies to investigate the worst-case scenarios. First, we assume that there is no recovery rate and, therefore, the creditor loses all its interbank assets held against the defaulted institutions A (see Chinazzi et al., 2015; Gai & Kapadia, 2010). Second, we assume that the loss of the creditor j is imputed directly to the capital of this bank (see Cont et al., 2010). Consequently, the capital level of the bank j (and thus its ability to absorb any losses) becomes $c_j - \sum_{i \in A} E_{ji}$. In the case where the new capital is non positive, that is, if the losses exceed the initial capital c_j, then the bank j becomes insolvent. Our third assumption is that insolvent banks are also considered as defaulted (see Leventides et al., 2019). This is not always the case, since the bank j may be able to find some liquidity and fulfill its legal obligations. However, this is extremely difficult as the creditworthiness of the bank has been irreparably affected and j cannot be funded through short-term debt. Therefore, to analyze the worst-case scenarios, we consider the insolvent institutions as defaulted.

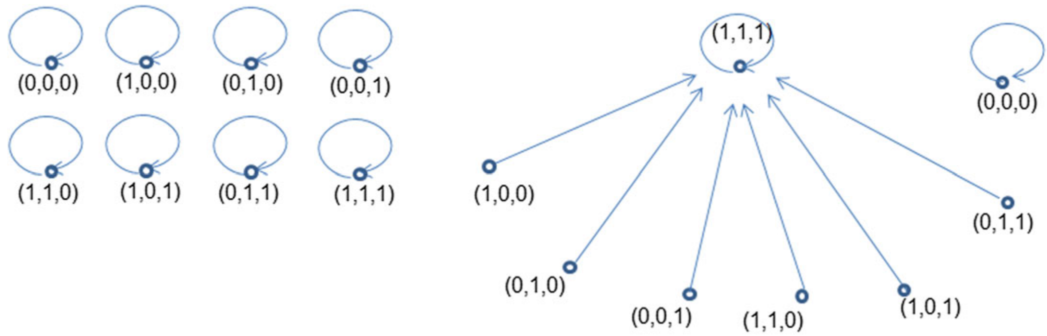

Figure 1. The contagion graph for a network consisting of three banks. The left is the graph in the case of zero contagion, whereas the right one depicts the case of total contagion.

In summary, an initial shock causing the failure of banks can lead to a cascade of defaults as the shock spreads through the interbank network. In scenarios with limited time, bolstering bank capital emerges as a reliable strategy for fortifying the interbank market against contagion.

2.2 Contagion map and contagion graph

The phenomenon of default contagion through an interbank network can be modeled with the so-called *contagion map* $T\colon 2^X \to 2^X$ (see Leventides et al., 2022), where 2^X is the powerset of X (the collection of all subsets of X). Given $A \subseteq X$, the map T returns the set $T(A) \subseteq X$ which contains A and also all the creditors that bankrupt due to the failure of the banks of the set A. By the description of the mechanism of default contagion, the map T is formally defined as $T(A) = \{j \in X \mid j \in A \text{ or } \sum_{i\in A} E_{ji} \geq c_j\}$.

It should be noted that $T(A)$ contains the banks of the set A as well as the banks that fail at the first stage after the initial shock. The set $T(A)$ may contain new defaulted institutions which, in turn, create losses to their creditors and a new round of failures occurs. Hence, at the second stage, the set of defaulted institutions is $T(T(A))$ and the propagation may continue in the same manner. The transmission of the initial shock ends when we reach an equilibrium set of T, that is, a set V with the property that $T(V) = V$.

The contagion map T produces a dynamical system on the powerset 2^X. The trajectories $\{A, T(A), T^2(A), \ldots, T^m(A)\}$ (where m has the property that $T^{m+1}(A) = T^m(A)$) of this dynamical system describe the propagation of default through the interbank market for all possible initial shocks A. Furthermore, the powerset 2^X acquires the structure of a directed graph which has (directed) edges from A to $T(A)$ for all $A \in 2^X$. This is called the *contagion graph,* and it is different from the initial triplet (X, C, E) (which is also modeled as a directed weighted graph).

For the contagion map T (and consequently for the contagion graph), there are two extreme cases denoted by T_0 and T_1. The first one is defined by $T_0(A) = A$, for all $A \in 2^X$, that is, in this case there is no contagion at all. The corresponding contagion graph has 2^n edges of the form (A, A), for all $A \in 2^X$. The other extreme case is the total contagion defined by $T_1(A) = X$ for all non-empty sets A. Hence, any initial shock promptly results in the complete collapse of the entire market. The contagion graph consists of the $2^n - 1$ directed edges (A, X) for all non-empty sets A. Figure 1 illustrates the contagion graph in these two extreme cases for a network consisting of three banks.

In general, the contagion graph contains several connected components. Each component has a single maximal element and various minimal elements. The maximal element is an equilibrium point of the map T and, therefore, it describes a situation where the contagion ends.

Because handling subsets is not convenient for calculations, an equivalent formulation of the contagion map would be desirable. This can be obtained by identifying as usual a subset A of X with the indicator vector $\mathbf{1_A}$ whose entries are zero except those corresponding to the elements of A which are equal to 1. In terms of the interbank market, 1 means that the corresponding bank defaults, while 0 implies non-default. Hence, the powerset 2^X can be identified with the set $\mathbb{Z}_2^n = \{0, 1\}^n$ of all vectors with entries 0 or 1. As a consequence, the contagion map can be seen as a map from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$ which associates every vector $\mathbf{1_A}$ with the vector $1_{T(A)}$. In this setting, it can be proved that the contagion map is given by $T(\mathbf{1_A}) = g(C - E_0 \cdot \mathbf{1_A})$, where E_0 is the $n \times n$ matrix which follows from E by replacing the zero diagonal with the vector C of capitals and $g\colon \mathbb{R}^n \to \mathbb{Z}_2^n$ is the function:

$$g(\mathbf{x})(i) = \begin{cases} 1, & \mathbf{x}(i) \leq 0; \\ 0, & \mathbf{x}(i) > 0. \end{cases}$$

for any vector $\mathbf{x} \in \mathbb{R}^n$ ($\mathbf{x}(i)$ denotes the ith coordinate of $\mathbf{x}$).

2.3 The bankruptcy sets

The *bankruptcy set* of the bank $i \in X$ (see Leventides et al., 2020), denoted by U_i, is defined as the collection of all $A \subseteq X \setminus \{i\}$ with the following property: if the banks belonging to A default, then i becomes insolvent. By the description of the mechanism of contagion it follows that:

$$A \in U_i \Leftrightarrow i \in T(A) \Leftrightarrow \sum_{j \in A} E_{ij} \geq c_i.$$

Observe that, if $A \in U_i$ and B is any set with $A \subseteq B$, then B also belongs to the bankruptcy set U_i. Hence, U_i has the structure of an upper set with respect to the order of inclusion.

2.4 Quantitative analysis of the network

If the bank i is prone to financial contagion, then it is expected to be seen in the set $T(A) \setminus A$ for many $A \in 2^X$ not containing i. This justifies the definition of the *contagion vector* as:

$$\mathbf{V} = \sum_{A \in 2^X} (\mathbf{1_{T(A)}} - \mathbf{1_A}) \in \mathbb{Z}^n.$$

Consequently, the smaller coordinates of the above vector correspond to the more robust banks of the network, while the bigger coordinates indicate weaker institutions. For instance, in the extreme case T_0 of zero contagion, the corresponding vector $\mathbf{V_0}$ is zero. On the other hand, in the case of total contagion T_1, the coordinates of the contagion vector have the maximum possible value, that is, $\mathbf{V_1} = (2^{n-1} - 1) \cdot \mathbf{1}$ (where $\mathbf{1}$ is the vector with all coordinates equal to 1).

Furthermore, from the contagion vector the indicator I_1 can be defined as:

$$I_1(T) = \frac{\langle \mathbf{V}, \mathbf{1} \rangle}{\langle \mathbf{V_1}, \mathbf{1} \rangle} = \frac{\langle \mathbf{V}, \mathbf{1} \rangle}{n \cdot (2^{n-1} - 1)},$$

where $\mathbf{V_1} = (2^{n-1} - 1)\mathbf{1}$ is the vector in the case of total contagion. The above quantity takes values between 0 and 1 (in the cases of zero and total contagion, respectively). Roughly speaking, it measures the stability of the network as a proportion of the corresponding network in total contagion. The smaller the indicator I_1, the more stable the network and vice versa.

Additionally, two other indicators can be defined. The indicator I_2 is related to the capitals that are destroyed and the indicator I_3 to the loans that are not paid back (again as a proportion of the

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network in total contagion). More specifically,

$$I_2(T) = \frac{\langle \mathbf{V}, C\rangle}{\langle \mathbf{V_1}, C\rangle} = \frac{\langle \mathbf{V}, C\rangle}{(2^{n-1}-1)\,\langle \mathbf{1_X}, C\rangle}, \quad I_3(T) = \frac{\langle \mathbf{V}, \mathbf{L}\rangle}{\langle \mathbf{V_1}, \mathbf{L}\rangle} = \frac{\langle \mathbf{V}, \mathbf{L}\rangle}{(2^{n-1}-1)\,\langle \mathbf{1_X}, \mathbf{L}\rangle}.$$

The quantities take also values between 0 and 1, and smaller values indicate more stable networks.

Finally, the indicator $m = \frac{f}{2^n}$ can be defined, where f denotes the number of equilibrium sets (or fixed points) of the contagion map T. These are the sets $A \subseteq X$ with the property that $T(A) = A$. Their importance stems from the fact that they are the points where the propagation of an economic crisis eventually stops. There are at least two fixed points (the sets $\emptyset$ and X). Consequently, the indicator m takes values between $\frac{1}{2^{n-1}}$ and 1. The higher the index, the more stable the network, since there are a lot of fixed points.

3. The liabilities ordering on the set $2^{X\setminus\{i\}}$

When any bank i of the network increases its capital, it is expected that i will become more robust and the whole network more resistant. However, the capital of i must exceed a certain threshold to produce tangible results. More generally, not all capital increases have the same effect. There must be various thresholds, and each time the capital exceeds one of them better results should be observed. The present section captures this intuition in a more rigorous context by introducing a suitable ordering.

Henceforth, let $i \in X$ be a bank of the network. The sets $A \subseteq X \setminus \{i\}$ which are "poisonous" for i (in the sense that the failure of the banks of A would make the bank i be insolvent) are those with the property: $\sum_{j\in A} E_{ij} \geq c_i$. The quantity $\sum_{j\in A} E_{ij}$ (representing the total liabilities of the banks of A toward i) plays a major role in the analysis of default contagion, and it will be denoted by r_A. Since $E_{ij} \geq 0$ for any (i, j), it follows that $r_A \leq r_B$ whenever $A \subseteq B \subseteq X \setminus \{i\}$.

We now introduce a binary relation on the set $2^{X\setminus\{i\}}$ (the collection of all subsets of $X \setminus \{i\}$) as follows: for any $A, B \in 2^{X\setminus\{i\}}$,

$$A <_L B \Longleftrightarrow r_A < r_B.$$

As usual, for any $A, B \in 2^{X\setminus\{i\}}$ we also define

$$A \leq_L B \Longleftrightarrow A = B \text{ or } A <_L B \Longleftrightarrow A = B \text{ or } r_A < r_B.$$

Theorem 3.1. *The relation $\leq_L$ is a partial order on the powerset $2^{X\setminus\{i\}}$.*

Proof. By its definition, it follows that the binary relation $\leq_L$ is reflexive, antisymmetric, and transitive. Hence, it is a partial order. □

The order $\leq_L$ is called the *liabilities ordering* on $2^{X\setminus\{i\}}$. Clearly, this set has also a natural partial order induced by inclusion. Since $r_A \leq r_B$, for $A \subseteq B$, the next proposition follows immediately.

Proposition 3.2. *For any $A, B \in 2^{X\setminus\{i\}}$, if $A \subseteq B$, then $A \leq_L B$. The converse is not necessary true.*

Remark 3.3. The liabilities ordering $\leq_L$ is not a linear order on the set $2^{X\setminus\{i\}}$. Indeed, we may have $A \neq B$ and yet $r_A = r_B$, in which case the sets A, B are $\leq_L$-noncomparable. However, for any value r there are only finitely many sets $A \in 2^{X\setminus\{i\}}$ such that $r_A = r$. Consequently, it is possible to turn $\leq_L$ into a linear order by considering (for any value r) a linear ordering of the finitely many sets A such that $r_A = r$. There is no unique way to do this. Nevertheless, the exact definition of the linear order is not essential for our purposes. For this reason, we refer to $\leq_L$ as a linear order on the set $2^{X\setminus\{i\}}$.

The liabilities ordering classifies the elements of $2^{X\setminus\{i\}}$ in a linear order. Recall that the cardinality of $2^{X\setminus\{i\}}$ is equal to 2^{n-1}. Therefore, the sets belonging to $2^{X\setminus\{i\}}$ can be enumerated as $\{A_j\}_{j=1}^{2^{n-1}}$, such that $A_1 = \emptyset$, $A_{2^{n-1}} = X \setminus \{i\}$ and

$$A_1 \leq_L A_2 \leq_L \ldots \leq_L A_j \leq_L A_{j+1} \leq_L \ldots \leq_L A_{2^{n-1}},$$

that is,

$$0 = r_{A_1} \leq r_{A_2} \leq \ldots \leq r_{A_j} \leq r_{A_{j+1}} \leq \ldots \leq r_{A_{2^{n-1}}} = \sum_{j=1, j\neq i}^{n} E_{ij}.$$

The intervals $\{(r_{A_j}, r_{A_{j+1}}]\}_{j=1}^{2^{n-1}-1}$ (except those which may be empty) and $(r_{A_{2^{n-1}}}, \infty)$ define a partition of the set $(0, \infty)$ of positive real numbers. The next proposition shows that the numbers $(r_{A_j})_{j=1}^{2^{n-1}}$ quantify the initial intuition as discussed at the beginning of this section. Hence, they demonstrate the levels that the capital c_i of the bank i needs to go through for network improvements to occur.

Theorem 3.4. *If the capital c_i assumes values only within the interval $(r_{A_k}, r_{A_{k+1}}]$, then the indicators I_1, I_3, m remain constant.*

Proof. The indicators I_1, m depend actually on the contagion map T, while I_3 depends on T and the liabilities vector $\mathbf{L}$. Since, $\mathbf{L}$ does not change, it suffices to prove that T remains constant when c_i assumes values within the interval $(r_{A_k}, r_{A_{k+1}}]$. The map T, in turn, depends on the exposures matrix and the capital vector and, more precisely, it is determined by the relation of $\sum_{j\in A} E_{\ell j}$ with c_ℓ for all $1 \leq \ell \leq n$ and $A \subseteq X$. Because the matrix E and the capitals of the other banks remain constant and c_i does not cross the thresholds r_{A_k} and $r_{A_{k+1}}$, it follows that T remains the same. □

In particular, when the capital levels cross some threshold, then any further increase does not contribute to the stability of the interbank network.

Corollary 3.5. *In the case where $c_i > r_{A_{2^{n-1}}} = \sum_{j\neq i} E_{ij}$, then any further increase to the capital of the bank i does not contribute to the value of the indicators.*

Remark 3.6. The indicator I_2 is particularly sensitive to variations in the capital vector C. Consequently, even a slight augmentation in the capital of the bank i can induce alterations in the I_2 value. However, if the capital c_i falls within the range of $(r_{A_k}, r_{A_{k+1}}]$, the contagion map T and the contagion graph remain unaltered. This signifies that, despite capital increments, the bank i remains subject to identical risk exposures. Thus, it is reasonable to assert that while the I_2 value may fluctuate, it doesn't lead to an enhancement in network stability.

Finally, the liabilities ordering determines the bankruptcy set U_i as the next proposition demonstrates.

Proposition 3.7. *Assume that the capital c_i falls within the interval $(r_{A_k}, r_{A_{k+1}}]$. The bankruptcy set U_i of the bank i coincides with the final segment of the linearly ordered space $(2^{X\setminus\{i\}}, \leq_L)$ consisting of the sets $A_{k+1}, \ldots, A_{2^{n-1}}$, that is,*

$$U_i = \left\{A_{k+1}, \ldots, A_{2^{n-1}}\right\}.$$

Proof. The bankruptcy set U_i consists of the sets $A \in 2^{X\setminus\{i\}}$ such that $\sum_{j\in A} E_{ij} \geq c_i$, that is, $r_A > c_i$. These sets are exactly the sets $A_{k+1}, \ldots, A_{2^{n-1}}$. □

4. Analysis of the sensitivity of the network in the capital raise of a bank

It is reasonable to anticipate that capital increases in different banks will have varying impacts on the network's stability. The capital increase of bank i might exert a more significant influence on the network compared to bank j. In essence, the interbank market might exhibit varying degrees of

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sensitivity to different banks. This section introduces an index designed to quantify this sensitivity concerning the capital increase of each individual bank.

Let i be any bank of the network and let $A_1 \leq_L A_2 \leq_L \ldots \leq_L A_{2^n-1}$ be the liabilities ordering of the corresponding set $2^{X\setminus\{i\}}$. Assume also that the capital c_i falls within the interval $(r_{A_k}, r_{A_{k+1}}]$. The value $r_{A_{k+1}}$ signifies the minimum capital level that needs to be exceeded (at least by one unit) for an improvement in network stability to become evident.

If I is any of the indicators I_1, I_2, I_3, the next quotient can be considered:

$$\frac{\Delta I}{\Delta c} = \frac{I_a - I_\tau}{r_{A_{k+1}} + 1 - c_i}, \tag{4.1}$$

where I_a is the initial value of the indicator and I_τ is its value when the capital c_i of the bank i increases to $c_i^* = r_{A_{k+1}} + 1$. In the case of the indicator m, however, the increase in the capital of i implies that the stability of the network improves; thus, the value of the indicator is bigger. Consequently, the corresponding quotient should be $\frac{\Delta m}{\Delta c} = \frac{m_\tau - m_a}{r_{A_{k+1}} + 1 - c_i}$.

The quotient of Equation (4.1) quantifies the extent to which network stability improves in relation to the bank's minimum capital increase. Consequently, it provides a measure for the sensitivity of the network to the capital increase of the bank i: the bigger it is the more sensitive if the network. For this reason, it is called the *sensitivity index to the bank's capital increase in terms of the indicator I*. Furthermore, these numbers can also serve as an index for assessing the importance of each bank i to the entire network.

5. Indicators of the stability of the network as functions on the vector of capitals

Let I be any of the indicators for the stability of the network described in Section 2. Since we have assumed that all the other features of the interbank market remain constant and we are allowed to modify the capital levels, the indicator I becomes actually a function of the vector $C = (c_1, c_2, \ldots, c_n)$ of capitals:

$$\mathbb{R}_+^n \to \mathbb{R}$$

$$(c_1^*, c_2^*, \ldots c_n^*) \mapsto I(c_1^*, c_2^*, \ldots c_n^*).$$

Since the computation of the indicators is based on the contagion graph, either directly or through the contagion vector, the above function is actually decomposed as a composition of two functions. The first one is a function with domain $\mathbb{R}_+^n$, it takes values in the set of all contagion graphs and it assigns to each capital vector $\tilde{C} = (c_1^*, c_2^*, \ldots c_n^*)$ the contagion graph which corresponds to the network $(X, \tilde{C}, E)$. The second one is the function that assigns to each contagion graph the corresponding value of the indicator I.

$$\begin{array}{ccc} \tilde{C} = (c_1^*, c_2^*, \ldots c_n^*) & \longrightarrow & \text{contagion graph} \\ & \searrow & \downarrow \\ & & I(c_1^*, c_2^*, \ldots c_n^*) \end{array}$$

We next argue that the above function takes only finitely many values. Indeed, let i be any bank of the network and let $A_1 \leq_L A_2 \leq_L \ldots \leq_L A_{2^n-1}$ be the liabilities linear ordering of $2^{X\setminus\{i\}}$. Assume also that $r_{A_{k_i-1}} < c_i \leq r_{A_{k_i}}$. By Section 3, we know that $r_{A_{k_i}}, r_{A_{k_i+1}}, \ldots r_{A_{2^n-1}}$ are the thresholds that the capital of the i bank should cross so that to observe some improvement in the stability and robustness of the network. It has also been justified that if c_i^* takes any value in the interval $(r_{A_j}, r_{A_{j+1}}]$, then the contagion graph remains the same and thus the stability of the network does

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not change. In other words,

$$I(c_1^*, c_2^*, \ldots, c_i^* \ldots, c_n^*) = I(c_1^*, c_2^*, \ldots, \hat{c}_i^* \ldots, c_n^*),$$

for any $c_i^*, \hat{c}_i^* \in (r_{A_j}, r_{A_{j+1}}]$. Consequently, if we change only the ith coordinate of the vector of capitals, then the indicator $I(c_1^*, c_2^*, \ldots c_n^*)$ can assume at most $2^n - k_i$ different values.

The above remain valid for any bank $i \in X = \{1, 2, \ldots, n\}$. It follows that there are in total $\prod_{i=1}^n (2^n - k_i)$ possible combinations of bank capital raise which may affect the value of the indicator I. Consequently, the indicator I takes finitely many values (at most $\prod_{i=1}^n (2^n - k_i)$). Each value corresponds to some of the $\prod_{i=1}^n (2^n - k_i)$ different scenarios for bank capital raise. Therefore, we have proved the following Proposition.

Proposition 5.1. *There are $\prod_{i=1}^n (2^n - k_i)$ different scenarios for the raise of the capital levels of the banks belonging to network. Therefore, each indicator can take only finitely many values ($\prod_{i=1}^n (2^n - k_i)$ at most).*

As a result, although c_i, $i = 1, \ldots, n$ are continuous variables, we are only interested in finitely many discrete changes of them.

6. The lattice structure on the set of contagion graphs

Suppose that all characteristic features of the interbank network remain constant and only capital levels are increased. By the previous section, it is clear that there is a finite number of scenarios that can occur. These scenarios can be parametrized by finitely many points of $\mathbb{R}^n$, namely the points belonging to the lattice $\mathcal{L} = \prod_{i=1}^n S_i$, where S_i is a finite subset of $\mathbb{N}$, namely $S_i = \{0, 1, 2, \ldots, 2^n - k_i\}$. For instance, the point $(1, 0, \ldots, 0)$ corresponds to the scenario where all banks keep their capital levels constant except the first one which increases its capital so that to cross the threshold $r_{A_{k_1}}$. Similarly, $(2, 0, \ldots, 0)$ means that the first bank increases its capital so that to surpass the second threshold, etc. Of course, in the case of the scenario $(0, 0, \ldots, 0)$ no change is applied to the capitals of the banks.

Each scenario $d = (d_1, d_2, \ldots, d_n) \in \mathcal{L}$ generates a triple (X, C_d, E), where C_d stands for the capital vector of the banks in the scenario d. This triple, in turn, creates a contagion graph, which we denote by G_d. (Note that $G_{(0,\ldots,0)}$ coincides with the initial contagion graph.) Therefore, we have a set whose elements are the contagion graphs for all possible scenarios:

$$\mathfrak{G} = \{G_d \mid d \in \mathcal{L}\}$$

and our purpose is to study the structure of this set.

To this end, we define on $\mathfrak{G}$ the relation $\leq_{\mathfrak{G}}$ as follows:

$$G_d \leq_{\mathfrak{G}} G_{d'} \iff \mathrm{Fix}(G_d) \subseteq \mathrm{Fix}(G_{d'}),$$

where $\mathrm{Fix}(G_d)$ stands for the set of fixed points of the graph.

Proposition 6.1. *The map*

$$\mathcal{L} \to \mathfrak{G}$$

$$d \mapsto G_d$$

is order-preserving.

Proof. Let $d \leq d'$ be elements of $\mathcal{L}$. We denote by (X, C_d, E), where $C_d = (c_1^*, c_2^*, \ldots, c_n^*)$, and T_d the interbank network and the contagion map corresponding to the case of scenario d. Similarly, $(X, C_{d'}, E)$, with $C_{d'} = ({c_1}', {c_2}', \ldots, {c_n}')$ and $T_{d'}$ are the network and the contagion map for the scenario d'.

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Since the order in $\mathcal{L}$ is defined pointwise, $d \leq d'$ implies that $d_i \leq d_i'$ for any $i = 1, 2, \ldots, n$. Therefore, $c_i^* \leq c_i'$ for any $i = 1, 2, \ldots, n$. The proof now is intuitively obvious, since in the case of scenario d' the banks possess more capitals and, hence, they are more resilient to default contagion.

More formally, let A be a fixed point of the contagion map T_d. This implies that $T_d(A) = A$. Consequently, for any $j \notin A$, one has $\sum_{i \in A} E_{ji} < c_j^*$. It follows that $\sum_{i \in A} E_{ji} < c_j'$ and, hence, $j \notin T_{d'}(A)$. Therefore, $T_{d'}(A) \subseteq A$. Since, the other inclusion holds true, we have that $T_{d'}(A) = A$, that is, A remains a fixed point for the contagion graph $G_{d'}$. □

Corollary 6.2. *The set $\mathfrak{G}$ of all possible contagion graphs with the relation $\leq_{\mathfrak{G}}$, which is defined in terms of the set of fixed points, becomes a lattice. In particular, this set inherits the lattice structure of $\mathcal{L}$.*

7. A systematic approach to networks reengineering

In this section, we present a systematic way for reengineering interbank networks. The ultimate purpose is to increase the robustness of the network and to strengthen its resilience against systemic risk. As we have commented earlier, the balance sheet size of the banks belonging to network contain several factors that could be changed (e.g. capitals, liabilities, assets, deposits, etc.). However, in short-time scenarios, the increase of capital levels appears to be the most reliable and effective method. Therefore, we adopt the assumption that the other factors remain constant and we examine in which ways the capitals should be altered so that to achieve lower levels of vulnerability.

To assess the robustness of an interbank network, we may utilize several indicators (see Section 2). These indicators take values between 0 and 1. The closer to 1 they are, the more robust the network is and vice versa.

We now consider two kinds of problems related to networks reengineering. The first one is figuring out the optimal way for achieving a specific goal of robustness (we use some of the indicators as a measure of robustness) and at the same time to keep the capital increase as low as possible. The second problem, which could be thought of as the dual to the first one, is to consider a fixed total capital raise and find how this should be allocated among banks so that the optimal increase of some indicator would be achieved. Our approach to each problem comprises of three steps that are briefly described below.

7.1 Problem I: Increase the robustness with the minimum cost

In this approach, we seek to reach some predefined improvement of the network given that the capital raise should be the less possible. Therefore, we have the following three steps.

Step 1: Measurements and network evaluation. The first step involves the evaluation of the interbank network at a particular stage. The supervisors of the network examine the balance sheet size of each bank. Capital levels, liabilities, and assets are measured. The contagion graph can also be constructed showing the development of a crisis through the interbank market for all possible scenarios of initial default. Finally, the indicators are calculated and the network is evaluated for its robustness and resilience to systemic risk.

Step 2: Setting the goal. The supervisors and policymakers may observe that the indicators are not satisfactory and some criteria are not met. This implies that the network is weak or it has some weak points. Consequently, they may decide that specific proactive actions should be taken. In this work, we only examine the capital raise of each bank.

At this stage, a specific goal should be set. For instance, the supervisors should agree on a specific value $a \in (0, 1)$ for the indicator I that has to be met.

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Step 3: Capital raise. After the goal has been set, the best method for achieving it has to be found. This translates into the following problem that has to be solved. Find the capitals $(c_1^*, c_2^*, \ldots, c_n^*)$ such that

$$I(c_1^*, c_2^*, \ldots, c_n^*) \leq a \quad (\text{or } m(c_1^*, c_2^*, \ldots, c_n^*) \geq a \text{ for the indicator } m) \tag{7.1}$$

$$(c_1^*, c_2^*, \ldots, c_n^*) \geq (c_1, c_2, \ldots, c_n) \quad \text{in the pointwise sense} \tag{7.2}$$

$$\text{minimize} \sum_{j=1}^{n} c_i^*. \tag{7.3}$$

7.2 Problem II: Increase the robustness with respect to a specific capital raise

In this approach, we seek to find the optimal improvement of the network given that a total capital raise has been decided for the set of banks. We have again three steps.

Step 1: Measurements and network evaluation. The network should be evaluated as in the previous problem.

Step 2: Setting the goal. A total capital raise should be agreed for the set of banks. This means that we will have new vector of capitals $C = (c_1^*, c_2^*, \ldots, c_n^*)$, such that $c_i \leq c_i^*$ for every $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} c_i^* = A$, where A is a constant representing the new total capital of the interbank market.

Step 3: Capital raise. Finally, the following optimization problem has to be solved. Find the capitals $(c_1^*, c_2^*, \ldots, c_n^*)$ such that

$$c_i \leq c_i^* \text{ for every } i = 1, 2, \ldots, n, \tag{7.4}$$

$$\sum_{i=1}^{n} c_i^* = A, \tag{7.5}$$

$$\text{minimize the indicator } I \quad (\text{or maximize the indicator } m). \tag{7.6}$$

7.3 Algorithm

At this stage, we present an algorithm for addressing the above problems. The algorithm contains the next steps.

Step 1: The first step is to measure the characteristic features of the interbank market at a particular stage and to depict the contagion graph showing the progression of a crisis for all possible initial defaults. Additionally, the indicators should be calculated which provide a measure of the network robustness.

Step 2: For each bank i, we place in ascending order the potential losses that may be created from other banks. This contains the loan levels $r_A = \sum_{j \in A} E_{ij}$ for all $A \in 2^{X \setminus \{i\}}$. A chain of losses is created as follows:

$$\begin{array}{c} r_{A_1} \\ r_{A_2} \\ \vdots \\ r_{A_{2^{n-1}}} \end{array}$$

Then the capital level c_i is placed within this chain of losses in order to evaluate the minimum raise in capital levels that has to be made for the network to be resilient in case of a systematic shock:

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$$c_i \quad \rightarrow \quad \begin{matrix} r_{A_1} \\ r_{A_2} \\ \vdots \\ \\ \vdots \\ r_{A_{2n-1}} \end{matrix}$$

The condition that suffices higher network robustness is that each time the capital levels c_i of each bank should surpass by 1 the elements of the chain. To decide upon the optimal strategy, that will provide such a robustness, the minimum capital raise each time for c_i should be as least as the value of the next chain element in order and a bit more. In other words, if $r_{A_j} < c_i \leq r_{A_{j+1}}$ then the new capital level $c_i^* = r_{A_{j+1}} + \epsilon$ for ϵ relatively small.

Step 3: The next step is the new Boolean graph of the network to be created and the new values of the indicators that measure network's resilience or robustness to be computed.

Step 4: The previous steps create actually a map which assigns to every possible combination of (discrete) capital raise the contagion graph of the network and the values of the indicators after the capital raise has been realized. Therefore, a pivot table can be created with all possible scenarios and indicators, in which the supervisors could allocate the optimal solution according to their needs.

8. Applications

To illustrate the ideas and techniques of the previous sections and depict the aforementioned approach, the following example has been created. In this example, the indicator that measures network's robustness is the number of fixed points in the contagion graph.

Let an interbank network with $n = 3$ banks where the capital vector and the exposures matrix are given as follows:

$$C = (10, 8, 4) \quad E = \begin{bmatrix} 0 & 7 & 4 \\ 9 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}.$$

The interbank network is depicted in Figure 2.

Step 1: The initial Boolean graph of the previous network is formulated and the number of fixed points is determined. The contagion graph is shown in Figure 3 and the total number of fixed points is 4.

The contagion vector is computed easily and it is $V = (1, 2, 1)$. Hence, the values of the indicators can be derived. Let us use the indicator I_2 which is related to the capitals that are destroyed. One has

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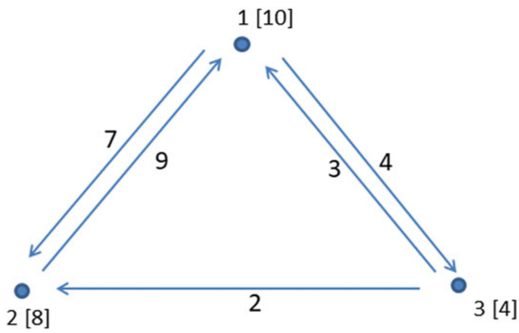

Figure 2. The interbank network of the example consisting of three banks. The numbers in brackets correspond to the capitals of the banks. The weights on the edges of the graph correspond to bilateral exposures.

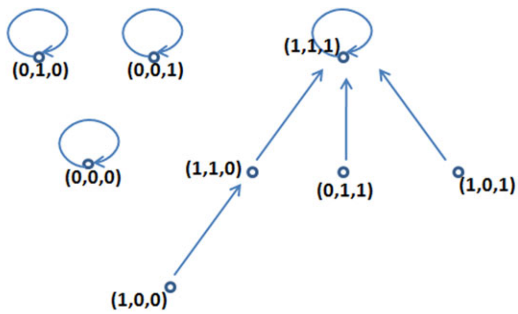

Figure 3. The contagion graph corresponding to the interbank network of the example.

$$I_2 = \frac{\langle V, C\rangle}{(2^2 - 1)\cdot(c_1 + c_2 + c_3)} \approx 0,45.$$

Step 2: The chain of losses for each bank is created and the corresponding capital levels are compared.

Bank 1		Bank 2		Bank 3	
	4	8 $\rightarrow$			2
	7		9		3
10 $\rightarrow$					5
	11			5 $\rightarrow$	

Step 3: The new contagion graphs are created taking into account each time the necessary capital raise for each bank. Several scenarios have to be addressed. In this particular example, there are $2 \cdot 2 \cdot 2 = 8$ different scenarios.

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Step 4: The pivot table of this example is the following:

A/N scenario	Initial capital	Scenario	Capital raise	Fixed points	Indicator I_2	Sensitivity = $\frac{\text{Raise of fixed points}}{\text{Capital raise}+1}$	Sensitivity of I_2 $\frac{\lvert I_{2\tau}-I_{2a}\rvert}{\text{Capital raise}+1}$
0	22	No capital raise	–	4	0.45	–	–
1	22	$c_2^* = 9$	1	6	0.19	$2/2 = 1$	0.13
2	22	$c_1^* = 11$	1	5	0.28	$1/2 = 0.5$	0.085
3	22	$c_3^* = 5$	1	5	0.38	$1/2 = 0.5$	0.035
4	22	$c_1^* = 11, c_2^* = 9$	2	7	0.05	$3/3 = 1$	0.133
5	22	$c_1^* = 11, c_3^* = 5$	2	6	0.21	$2/3 \approx 0.67$	0.08
6	22	$c_2^* = 9, c_3^* = 5$	2	7	0.13	$3/3 = 1$	0.107
7	22	$c_1^* = 11, c_2^* = 9, c_3^* = 5$	3	8	0	$4/4 = 1$	0.112

As we can see, the initial value of the indicator I_2 is 0.45 which is rather large. If we wish to achieve $I_2 \leq 30\%$ with the minimum capital raise, then there are two choices represented by scenarios: 1 and 2. However, the table shows that scenario 1, that is, the capital raise of the second bank, gives the best results. Similarly, in order to achieve $I_2 \leq 20\%$, then we should again follow the first scenario. However, for $I_2 \leq 15\%$, the possible scenarios are 4 and 6, with the fourth scenario giving the best result.

On the other hand, if total capital raise equal to 1 has been decided, the optimal scenario is the first one, while for capital raise equal to 2 the best choice is the fourth scenario.

Finally, the table (as well as the above discussion) reveals that the network is very sensitive to the capital raise of the second bank. This can be seen by the last column as well as by the fact that the bank 2 participates in the two best scenarios (which are the first and fourth ones).

8.1 The Lattice structure on the set of contagion graphs

For the interbank network of the example, there are totally $2^3 = 8$ contagion graphs corresponding to all possible combinations of capital raise of the banks. We denote this graphs by G_δ where $\delta = (\delta_1, \delta_2, \delta_3) \in \{0, 1\}^3$. Thus, $\delta_i = 1$ means that the capital of the bank i increases. According to the results of Section 6, if $\delta \leq \delta'$ (in the pointwise sense) then the set $\mathrm{Fix}(G_\delta)$ of fixed points of G_δ is contained in the set $\mathrm{Fix}(G_\delta{}')$.

The set $\mathfrak{G}$ of all contagion graphs inherits the partial order:

$$G_\delta \leq_{\mathfrak{G}} G_{\delta'} \Leftrightarrow \mathrm{Fix}(G_\delta) \subseteq \mathrm{Fix}(G_{\delta'}).$$

This set becomes a lattice isomorphic to the lattice $\mathcal{L} = \{0, 1\}^3$. This lattice is depicted in Figure 4.

9. An optimization model for reengineering interbank networks

The problem of reengineering interbank networks, as described in Section 7, is a problem that concerns how to conduct and coordinate the operations (i.e. distribution of new capitals) within an organization (i.e. an interbank market). Additionally, it is a case where conflicts of interest among

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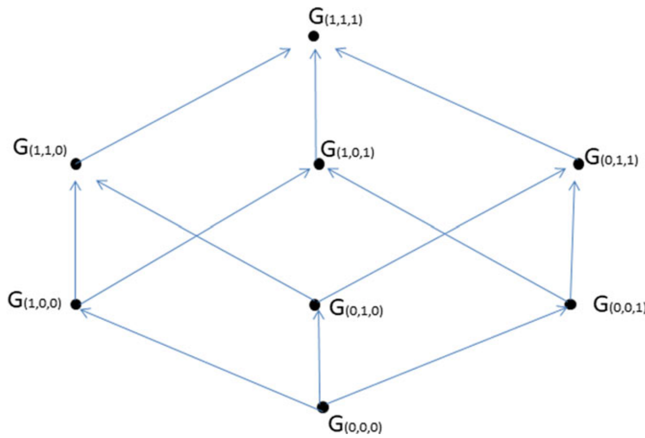

Figure 4. The lattice of contagion graphs.

the components of the interbank network may arise. However, policymakers and regulators of the interbank network are interested in a best possible course of action, which is the ideal for the market as a whole.

Consequently, the problem of reengineering interbank networks can be clearly considered as a decision-making problem. In fact, the problem of analyzing interbank payments has already been studied in the context of Operational Research since 1998 (see Guntzer et al., 1998). Subsequent approaches include the studies Elsinger et al. (2006), Shafransky & Doudkin (2006), Cohen-Cole et al. (2015), Chen et al. (2016), and Torri et al. (2018). Most of these studies model interbank payments by means of networks and employ heuristics to determine the best possible course of action, which is ideal for the market as a whole.

The purpose of the present section is to formulate the problem as a decision-making problem in terms of operations research and, then, to apply methods of (nonlinear) programming to obtain the optimal allocation of new capital in the system. As a result, more realistic applications (than the one described in Section 8) can also be addressed.

9.1 Formulation in terms of Operational Research

The mathematical model describing the interbank market has been extensively analyzed in previous sections. Hence, we can now proceed with the formulation in terms of Operational Research. To be more specific, we consider the problem described in Section 7.2. Therefore, we assume that a predefined capital raise has been agreed, and the objective is to find out how this new capital should be delivered between the banks of the market. The problem has the parameters shown in Table 1.

Furthermore, the variables of the problem are as follows:

$$IC_i : \text{ new (increased) capitals at bank i}$$

$$T_{s,i,j} = \begin{cases} 1, & \text{if bank i defaults because of bank j at stage s of the process;} \\ 0, & \text{otherwise.} \end{cases}$$

The objective of the proposed model is to determine optimal values for the new capitals IC_i such that the contagion effects are minimized. The model is formulated as follows:

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Table 1. Parameters of the problem

N	Number of banks in the interbank network
E_{ij}	The liability of j toward i
C_i	Capital of the ith bank
R	Available new capitals for supporting the system

$$\text{minimize} \quad Z = \frac{\sum_i [(C_i + IC_i) \cdot \sum_j T_{N,i,j}]}{\sum_i (C_i + IC_i)},$$

subject to:

$$T_{s,i,j} = 1 \qquad \text{for all } s, i, j \text{ with } i = j \tag{9.1}$$

$$T_{1,i,j} \geq \frac{E_{ij} - (C_i + IC_i)}{\sum_{k,l} E_{kl}} + \epsilon \qquad \text{for all } i, j \text{ with } i \neq j \tag{9.2}$$

$$T_{1,i,j} \leq 1 + \frac{E_{ij} - (C_i + IC_i)}{\sum_{k,l} E_{kl}} \qquad \text{for all } i, j \text{ with } i \neq j \tag{9.3}$$

$$T_{s,i,j} \geq \frac{\sum_k T_{(s-1),k,j} \cdot E_{ik} - (C_i + IC_i)}{\sum_{k,l} E_{kl}} + \epsilon \qquad \text{for all } s, i, j \text{ with } s > 1 \text{ and } i \neq j \tag{9.4}$$

$$T_{s,i,j} \leq 1 + \frac{\sum_k T_{(s-1),k,j} \cdot E_{ik} - (C_i + IC_i)}{\sum_{k,l} E_{kl}} \qquad \text{for all } s, i, j \text{ with } s > 1 \text{ and } i \neq j \tag{9.5}$$

$$\sum_i IC_i \leq R \tag{9.6}$$

$$IC_i \geq 0 \qquad \text{for all } i \tag{9.7}$$

$$T_{s,i,j} \in \{0, 1\} \qquad \text{for all } s, i, j \tag{9.8}$$

The objective function expresses the index of lost capitals, initial and new ones, due to the failure of banks and the possible contagion effects throughout the whole network.

Constraints (9.1) ensure that at each stage s of the default process, the elements along the main diagonal of the matrix $T_{s,i,j}$ are equal to 1, reflecting the default of bank i. Constraints (9.2) stipulate that at the initial stage of the default process, when the exposure of bank i to bank j exceeds the capitals available at bank i, then the variable $T_{1,i,j}$ is equal to 1, indicating that bank i also defaults as a result of the failure of bank j. Note that $0 < \epsilon \ll 1$ is a small positive number. More specifically, when $E_{ij} \geq C_i + IC_i$, then $T_{1,i,j} > 0$, which forces $T_{1,i,j} = 1$. On the other hand, constraints (9.3) ensure that when the total capitals of bank i are sufficient to cover its loans to j, namely when $E_{ij} < C_i + IC_i$, then $T_{1,i,j} \leq f < 1$, which forces $T_{1,i,j} = 0$. Similarly, constraints (9.4) and (9.5) determine the values of variables $T_{s,i,j}$ for $i \neq j$ at each stage s of the process, taking into account all banks that may have defaulted at stage $s - 1$. Hence, constraints (9.2) to (9.5) describe the contagion effects throughout the network due to possible defaults. Constraint (9.6) expresses the restriction in the availability of new capitals to support the network. Finally, constraints (9.7) and (9.8) express the nature of the variables.

The model is a MINLP model due to the objective function. It was implemented within the AIMMS modeling platform which is essentially a programming environment for developing and solving optimization problems under constraints. The model is solved using the standard Outer Approximation Algorithm (OAA), introduced by Duran and Grossman (1986), which is available within AIMMS (2023).

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9.2 Computational experiments

The model was tested on a series of randomly generated instances with 20 or 80 banks in the system. The values of the loans E_{ij} were generated such that the total loans issued by each bank sum up to twice its initial capitals with a standard deviation of 10%. The number of loans issued by each bank was also randomly selected such that the density of the whole table is around 10%. Finally, the values of a randomly generated set of loans were artificially increased to ensure that contagion effects are created in the system.

Concerning the available new capitals for supporting the system (parameter R), we experimented with different values ranging from 10% down to 3% of the value D = SL-SC, where SL is the sum of loans issued by all the banks in the system and SC the sum of their initial capitals.

We applied the OA Algorithm in its default settings, with a maximum of 10 iterations. At this stage, we did not experiment with different settings of the OAA parameters such as different multi-start strategies or penalty functions, since our objective was to investigate whether the proposed approach can be used to model contagion effects in realistic bank networks. In all instances, the OA Algorithm was able to converge to solutions that kept the contagion effects under control given the restricted availability of new capitals.

Indicatively, in Table B1 of Appendix B we present the solutions given by the algorithm in a randomly generated instance with 80 banks. Columns (2) and (3) of the table show the initial capital and the total loans issued by each bank, respectively. Columns (4)–(6) show the allocation of new capitals in the system to alleviate contagion under the three scenarios mentioned earlier.

These results indicate that the banks that should be supported by the increased capital are primarily the ones having relatively small initial capitals and whose total loans are more than twice the level of these initial capitals. On the other hand, banks with sufficiently large initial capitals do not seem to be requiring assistance even when they have large total loans that are distributed over many other banks. These results make sense from a practical point of view and further support the validity of the model and its applicability in realistic situations.

10. Conclusions

In Leventides et al. (2020) and Leventides et al. (2022), a framework for the study of default contagion in an interbank market has been established. In this framework, the interbank market is represented by a directed weighted graph (X, C, E) and the contagion of a financial crisis is studied with tools from pure and applied mathematics, such as dynamical systems, Boolean networks, and linear operators. Through this investigation, several indicators have been developed whose purpose is to assess the robustness of the interbank network and its resilience to default contagion and systemic risk. These indicators combine many characteristic features of the network, for instance, capital levels, bilateral exposures, interconnectedness, leverages, fixed points, etc.

The purpose of the present work, which is a natural continuation of the previous research, is to use the aforementioned framework in order to investigate methods for reengineering interbank networks. The aim is to enhance the stability of the network and its ability to absorb losses or shocks that may occur. In this paper, we assume that the capital levels of the banks are raised, while the bilateral exposures of the banks and other features remain constant. Then, we first develop a theoretical framework in order to (a) study how the capital raise affects the stability of the network and its resilience to systemic shocks and (b) investigate how the reengineering of the network should be designed to achieve the best possible results. Future works may also investigate reengineering of the networks in terms of bilateral exposures while the capital levels remain constant, or a combination of capitals and exposures may also be considered.

Based on this theoretical framework, we also develop a MINLP model whose objective is to optimally determine the allocation of the new capitals such that the contagion effects are minimized. We tested the model on several artificially generated instances involving up to 80 banks,

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which is a reasonable number in realistic applications. The results indicate that the overall framework may indeed be utilized to analyze the stability of real interbank networks and assist the relevant policymakers and regulators to determine optimal policies.

This work may be extended in several ways. First, the model may be viewed in a dynamic setting, where bank defaults may occur in different time periods, and decisions to allocate new capitals are also taken dynamically, according to the evolution of the default process. Second, it may be worth investigating how the solution method of the MINLP model may be improved to obtain optimal solution times in shorter computation times or in larger instances. Finally, the optimization model may be extended in a multi-objective setting to include the simultaneous analysis of more than one performance indicator that describe different aspects of the stability of the network.

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Data availability statement. The authors confirm that the data supporting the findings of this study are available within the article.

References

Acemoglu, D., Ozdaglar, A., & Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. *American Economic Review*, *105*(2), 564–608.

AIMMS. (2023). Prescriptive Analytics Platform. www.aimms.com [1].

Allen, F., & Gale, D. (2000). Financial contagion. *Journal of Political Economy*, *108*, 1–33.

Amini, H., Cont, R., & Minca, A. (2016). Resilience to contagion in financial networks. *Mathematical Finance*, *26*, 329–365.

Anand, K., Craig, B., & Von Peter, G. (2015). Filling in the blanks: Network structure and interbank contagion. *Quantitative Finance*, *15*(4), 625–636.

Bardoscia, M., Battiston, S., Caccioli, F., & Caldarelli, G. (2017). Pathways towards instability in financial networks. *Nature Communications*, *8*, Article14416. https://doi.org/10.1038/ncomms14416

Barucca, P., Bardoscia, M., Caccioli, F., D'Errico, M., Visentin, G., Battiston, S., & Caldarelli, G. (2016, June 14). *Network valuation in financial systems*. Available at SSRN: https://ssrn.com/abstract=2795583 or 10.2139/ssrn.2795583

Battiston, S., Puliga, M., Kaushik, R., Tasca, P., & Caldarelli, G. (2012). Debtrank: Too central to fail? Financial networks, the fed and systemic risk. *Scientific Reports*, *2*, Article541. https://doi.org/10.1038/srep00541

Caccioli, F., Catanach, T. A., & Farmer, J. D. (2012). Heterogeneity, correlations and financial contagion. *Advances in Complex Systems*, *15*(supp02), 1250058.

Caccioli, F., Shrestha, M., Moore, C., & Farmer, J. (2014). Stability analysis of financial contagion due to overlapping portfolios. *Journal of Banking & Finance*, *46*(C), 233–245.

Chen, N., Liu, X., & Yao, D. D. (2016). An optimization view of financial systemic risk modeling: Network effect and market liquidity effect. *Operations Research*, *64*(5), 1089–1108.

Chinazzi, M., Pegoraro, S., & Fagiolo, G. (2015). *Defuse the bomb: Rewiring interbank networks*. Pisa, Italy: LEM Working Paper Series, No. 2015/16, Scuola Superiore Sant'Anna, Laboratory of Economics and Management (LEM).

Cifuentes, R., Shin, H. S., & Ferrucci, G. (2005). Liquidity risk and contagion. *Journal of the European Economic Association*, *3*(2-3), 556–566.

Cimini, G., Squartini, T., Garlaschelli, D., & Gabrielli, A. (2015). Systemic risk analysis on reconstructed economic and financial networks. *Scientific Reports*, *5*, Article15758. https://doi.org/10.1038/srep15758

Cohen-Cole, E., Patacchini, E., & Zenou, Y. (2015). Static and dynamic networks in interbank markets. *Network Science*, *3*(1), 98–123.

Cont, R., Moussa, A., & Santos, E. B. (2010). *Network structure and systemic risk in banking systems*. Available at SSRN: https://ssrn.com/abstract=1733528 or 10.2139/ssrn.1733528

De Souza, S. R. S., Silva, T. C., Tabak, B. M., & Guerra, S. M. (2016). Evaluating systemic risk using bank default probabilities in financial networks. *Journal of Economic Dynamics and Control*, *66*, 54–75.

Duran, M.A., & Grossmann, I.E. (1986). An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Mathematical Programming*, 36, 307–339.

Elliott, M., Golub, B., & Jackson, M. O. (2014). Financial networks and contagion. *American Economic Review*, *104*(10), 3115–3153.
Elsinger, H., Lehar, A., & Summer, M. (2006). Risk assessment for banking systems. *Management Science*, *52*(9), 1301–1314.
Gai, P., & Kapadia, S. (2010). Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *466*, 2401–2423.
untzer, G. M., Jungnickel, D., & Leclerc, M. (1998). Efficient algorithms for the clearing of interbank payments. *European Journal of Operational Research*, *106*(1), 212–219.
Iori, G., Jafarey, S., & Padilla, F. (2006). Systemic risk on the interbank market. *Journal of Economic Behavior & Organization*, *61*, 525–542.
Ladley, D. (2013). Contagion and risk-sharing on the inter-bank market. *Journal of Economic Dynamics and Control*, *37*, 1384–1400.
Leventides, J., Livada, M., & Poulios, C. (2020). The dynamics of interbank networks. In: M. T. Rassias, & A. Raigorodskii (Eds.), *Discrete mathematics and applications*, Springer optimization and its applications (Vol. *165*, pp. 369–395). Cham: Springer.
Leventides, J., Loukaki, K., & Papavassiliou, V. G. (2019). Simulating financial contagion dynamics in random interbank networks. *Journal of Economic Behavior & Organization*, *158*, 500–525.
Leventides, J., Poulios, C., & Camouzis, E. (2022). A mathematical approach to interbank financial contagion dynamics. *IMA Journal of Mathematical Control and Information*, *39*(2), 409–442.
Mistrulli, P. E. (2011). Assessing financial contagion in the interbank market: Maximum entropy versus observed interbank lending patterns. *Journal of Banking & Finance*, *35*(5), 1114–1127.
Shafransky, Y., & Doudkin, A. (2006). An optimization algorithm for the clearing of interbank payments. *European Journal of Operational Research*, *171*(3), 743–749.
Torri, G., Giacometti, R., & Paterlini, S. (2018). Robust and sparse banking network estimation. *European Journal of Operational Research*, *270*(1), 51–65.
Upper, C. (2011). Simulation methods to assess the danger of contagion in interbank markets. *Journal of Financial Stability*, *7*(3), 111–125.
Wishnick, D. A. (2021). Reengineering financial market infrastructure. *Minnesota Law Review*, *105*(5), 2379–2442.

Appendix A: Contagion graphs of the scenarios

In this appendix, we present the contagion graphs of the several scenarios examined in the example of Section 8.

Scenario 1: $c_2^* = 9$ (Figure A1).

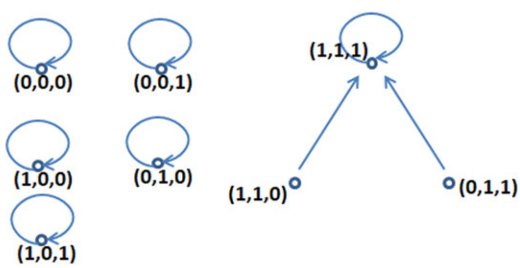

Figure A1. The contagion graph $G_{(0,1,0)}$ corresponding to the scenario $c_2^* = 9$.

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Scenario 2: $c_1^* = 11$ (Figure A2).

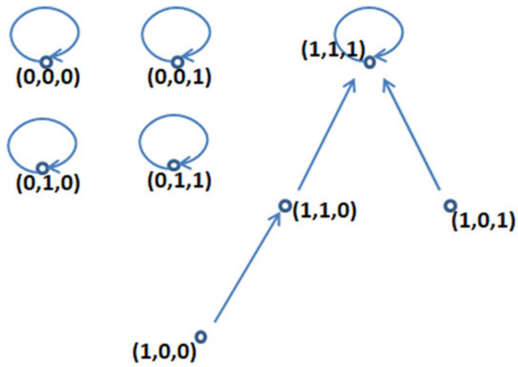

Figure A2. The contagion graph $G_{(1,0,0)}$ corresponding to the scenario $c_1^* = 11$.

Scenario 3: $c_3^* = 5$ (Figure A3).

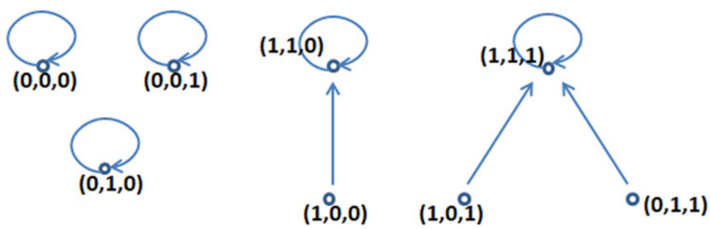

Figure A3. The contagion graph $G_{(0,0,1)}$ corresponding to the scenario $c_3^* = 5$.

Scenario 4: $c_1^* = 11$, $c_2^* = 9$ (Figure A4).

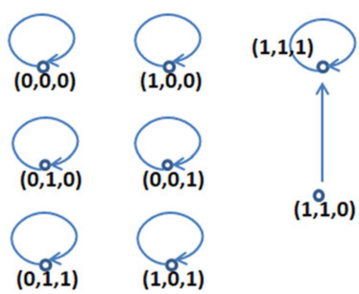

Figure A4. The contagion graph $G_{(1,1,0)}$ corresponding to the scenario $c_1^* = 11$, $c_2^* = 9$.

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Scenario 5: $c_1^* = 11$, $c_3^* = 5$ (Figure A5).

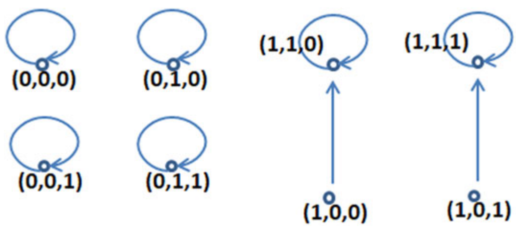

Figure A5. The contagion graph $G_{(1,0,1)}$ corresponding to the scenario $c_1^* = 11$, $c_3^* = 5$.

Scenario 6: $c_2^* = 9$, $c_3^* = 5$ (Figure A6).

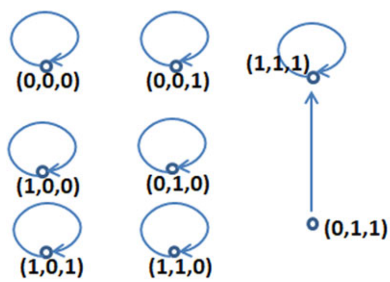

Figure A6. The contagion graph $G_{(0,1,1)}$ corresponding to the scenario $c_2^* = 9$, $c_3^* = 5$.

Appendix B: Table of solutions in an interbank market with 80 banks

Table B1. Solutions for an interbank network with 80 banks

Bank (1)	Initial capital (2)	Loans (3)	Scenario 1 (10% of D) (4)	Scenario 2 (5% of D) (5)	Scenario 1 (3% of D) (6)
1	43	38			0.0200
2	12	70	8.4649	8.4649	8.4860
3	10	20			0.0200
4	25	20			0.0200
5	37	48			0.0200
6	12	20			0.0200
8	13	48	7.4649	7.4649	
9	45	94			0.0200

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Table B1. Continued.

Bank (1)	Initial capital (2)	Loans (3)	Scenario 1 (10% of D) (4)	Scenario 2 (5% of D) (5)	Scenario 1 (3% of D) (6)
10	15	22			0.0200
11	35	198	15.4649	15.4649	15.4849
12	40	76			0.0200
13	48	120			0.0200
14	54	104			0.0200
15	20	22			0.0200
16	24	24			0.0200
17	65	128			0.0200
18	34	82			0.0200
19	73	122			0.0200
20	51	58			0.0200
21	25	58			0.0200
22	24	40			0.0200
23	30	54			0.0200
24	47	92			0.0200
25	13	12			0.0200
26	17	20			0.0200
27	10	6			0.0200
28	50	80			0.0200
29	12	98	8.4649	8.4649	8.4860
30	15	96	5.4649	5.4649	5.4856
31	62	164			0.0200
32	23	36			0.0200
33	29	101	6.4649	6.4649	6.4857
34	12	14			0.0200
35	42	52			0.0200
36	14	14			0.0200
37	25	44			0.0200
38	10	26	10.4649	10.4649	
39	29	36			0.0200
40	40	82			0.0200

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Table B1. Continued.

Bank (1)	Initial capital (2)	Loans (3)	Scenario 1 (10% of D) (4)	Scenario 2 (5% of D) (5)	Scenario 1 (3% of D) (6)
41	20	66	0.4649	0.4649	0.4849
42	45	74			0.0200
43	14	20			0.0200
44	10	36	5.4649	5.4649	
45	12	40	8.4649	8.4649	
46	15	20			0.0200
47	10	38	10.4649	10.4649	2.4649
48	19	20			0.0200
49	10	16			0.0200
50	37	94			0.0200
51	27	54			0.0200
52	26	28			0.0200
53	11	12			0.0200
54	54	142			0.0200
55	48	80			0.0200
56	30	66			0.0200
57	31	66			0.0200
58	70	150			0.0200
59	19	20			0.0200
60	47	60			0.0200
61	34	72			0.0200
62	39	76			0.0200
63	50	70			0.0200
64	21	40			0.0200
65	40	56			0.0200
66	40	54			0.0200
67	12	10			0.0200
68	24	78	12.4649	12.4649	12.4865
69	42	136			0.0200
71	20	48			0.0200
72	29	40			0.0200

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Table B1. Continued.

Bank (1)	Initial capital (2)	Loans (3)	Scenario 1 (10% of D) (4)	Scenario 2 (5% of D) (5)	Scenario 1 (3% of D) (6)
73	15	62	5.4649	5.4649	5.4856
74	37	46			0.0200
75	46	40			0.0200
76	20	36			0.0200
77	63	16			0.0200
78	37	72			0.0200
79	36	48			0.0200
80	23	60			0.0200

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