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REMARKS CONCERNING THE 2-HILBERT CLASS FIELD OF IMAGINARY QUADRATIC NUMBER FIELDS

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Let k be an imaginary quadratic number field and let k_1 be the 2-Hilbert class field of k. If $C_{k,2}$, the 2-Sylow subgroup of the ideal class group of k, is elementary and $|C_{k,2}| \geqslant 8$, we show that $C_{k_1,2}$ is not cyclic. If $C_{k,2}$ is isomorphic to $Z/2Z \times Z/4Z$ and $C_{k_1,2}$ is elementary we show that k has finite 2-class field tower of length at most 2.

1. INTRODUCTION

Let k be a number field and let $C_{k,2}$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group, C_k, (in the wider sense) of k. We let k_1 denote the 2-Hilbert class field of k; that is the maximal unramified (including the infinite primes) Abelian field extension of k which has degree a power of 2. It is well known that if $C_{k,2}$ is elementary of rank 2, that is isomorphic to $Z/2Z \times Z/2Z$, which we shall denote by (2, 2), then $C_{k_1,2}$ is cyclic (see Kisilevsky [6], Gorenstein [4], Taussky [8]). However, in the case where k is an imaginary quadratic number field and $C_{k,2}$ is elementary of rank greater than 2, we show that $C_{k_1,2}$ is not cyclic.

We next examine the case where k is an imaginary quadratic number field, $C_{k,2}$ is isomorphic to $Z/2Z \times Z/4Z$ (denoted by (2, 4)), and $C_{k_1,2}$ is elementary. From our earlier paper (see Snyder and Benjamin [7]), and the group tables in Hall and Senior [5] we are able to show that $C_{k_1,2}$ does not have rank 3, which implies that k has finite 2-class field tower of length at most 2 (see Blackburn [1, 2]). In the process of examining various groups G of order 64 with $G/G' \cong (2, 4)$ we have uncovered some errors in Hall and Senior [5] which we describe.

2. RESULTS WHEN $C_{k,2}$ IS ELEMENTARY

We begin by stating a result that characterises the Galois group of the second Hilbert 2-class field, k_2, over k for k a number field and $C_{k,2} \cong (2, 2)$. We let Q_m, D_m, S_m be the quaternion, dihedral, and semidihedral groups of order 2^m. It is well known that (see Gorenstein [4]):

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LEMMA 1. *If G is a 2-group of order 2^m, $m > 2$, such that $G/G' \cong (2, 2)$, then G is isomorphic to D_m, Q_m or S_m. In particular G' is cyclic.*

We now define $G = \mathrm{Gal}(k_2/k)$. Then from class field theory we know that $G' = \mathrm{Gal}(k_2/k_1) \cong C_{k_1,2}$ and $G/G' \cong \mathrm{Gal}(k_1/k) \cong C_{k,2}$ where G' is the commutator subgroup of G (see Gorenstein [4]). We can therefore conclude that if k is a number field with $C_{k,2} \cong (2, 2)$ then $C_{k_1,2}$ is cyclic.

At this point, assuming k to be an imaginary quadratic number field, we are able to formulate an interesting contrast when $C_{k,2}$ is elementary of rank greater than 2. We let L be an unramified quadratic extension of k with $H = \mathrm{Gal}(k_2/L)$ and let j be the homomorphism from $C_k \to C_L$ induced by the extension of ideals from k to L. The ideal classes in the kernel of j, that is those ideal classes which become principal in L, are said to "capitulate" in L. It is well known that if k is an imaginary quadratic number field then $|\ker j| = 2$ if $N_{L/k}(E_L) = \{1, -1\}$ and $|\ker j| = 4$ if $N_{L/k}(E_L) = \{1\}$ where E_L denotes the group of units in L and $N_{L/k}(E_L)$ denotes the norm from L to k of E_L. We know that the order of any ideal class in $\ker j$ divides $[L : k] = 2$ (see for example Furtwangler [3]) and $\ker j$ is therefore contained in $C_{k,2}$. The Artin map ϕ induces the following commutative diagram:

$$\begin{array}{ccc} C_k & \xrightarrow{\ \phi\ } & G/G' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle t} \\ C_L & \xleftarrow{\ \phi\ } & H/H' \end{array}$$

the rows are isomorphisms and $t\colon G/G' \to H/H'$ is the group tranfer map (Verlagerung) which has the following simple characterisation when H is of index 2 in G. Let $G = H \cup zH$; then $t(hG') = hzhz^{-1}H'$ and $t(zhG') = (zh)^2 H'$. Thus $\ker j$ is determined by $\ker t$. We are now able to prove the following theorem.

THEOREM 1. *Let k be an imaginary quadratic number field with $C_{k,2}$ elementary and $|C_{k,2}| \geqslant 8$. Then $C_{k_1,2}$ is not cyclic.*

PROOF: Let $G = \mathrm{Gal}(k_2/k)$; therefore $G' = \mathrm{Gal}(k_1/k) \cong C_{k_1,2}$ and $G/G' = \mathrm{Gal}(k_1/k) \cong C_{k,2}$. Assume G' is cyclic; therefore there exists $x, y \in G$ such that $G' = \langle \alpha \rangle$ with $xyx^{-1}y^{-1} = \alpha$. Since G has rank $\geqslant 3$ (by the Burnside Basis Theorem) there exists a maximal subgroup H containing x and y, and thus $H' = G' = \langle \alpha \rangle$. Since G has rank $m \geqslant 3$ we let $G/G' = \langle \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_m \rangle$ and examine the two cases for the transfer map $t\colon G.G' \to H/H'$:

CASE 1. If $\alpha_i \notin H$, $t(\overline{\alpha}_i) = \alpha_i^2 \langle \alpha \rangle$.

CASE 2. If $\alpha_i \in H$, $t(\overline{\alpha}_i) = \alpha_i \alpha_j \alpha_i \alpha_j^{-1} \langle \alpha \rangle$, $\alpha_j \notin H$.

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Since $\alpha_i^2 \in \langle\alpha\rangle$ then in Case 1 $t(\overline{\alpha}_i) = I\langle\alpha\rangle$. Since $\alpha_i\alpha_j\alpha_i\alpha_j^{-1} = \alpha_i^2(\alpha_i^{-1}\alpha_j\alpha_i\alpha_j^{-1}) \in \langle\alpha\rangle$ then in Case 2 we also have $t(\overline{\alpha}_i) = I\langle\alpha\rangle$. Thus the kernel of t is G/G'. From the commutative diagram above we conclude that all ideals in $C_{k,2}$ capitulate in the corresponding unramified quadratic extension to H. However $|C_{k,2}| \geqslant 8$ and since k is an imaginary quadratic number field at most 4 ideals in $C_{k,2}$ can capitulate. Consequently we have a contradiction and therefore $C_{k_1,2}$ is not cyclic. ∎

3. Results when $C_{k,2} \cong (2, 4)$

We next examine the case where k is an imaginary quadratic number field, $C_{k,2} \cong (2, 4)$, and $C_{k_1,2}$ is elementary. Let $G = \mathrm{Gal}(k_2/k)$; since $G/G' \cong C_{k,2} \cong (2, 4)$, $G/G' = \langle\overline{a}, \overline{b}\rangle$ for some $a, b \in G$ such that $\overline{a}^2 = I = \overline{b}^4$. We note that G has three maximal subgroups given by $H_1 = \langle b, G'\rangle$, $H_2 = \langle ab, G'\rangle$, $H_3 = \langle a, b^2, G'\rangle$. Let K_i be the subfield of k_2 fixed by H_i, $i = 1, 2, 3$. Then K_1, K_2, K_3 comprise all of the unramified quadratic extensions of k. We let $j_i\colon C_k \to C_{K_i}$ be the homomorphism induced by extension of ideals from k to K_i. We use the following result implicit in our earlier work (see Snyder and Benjamin [7]).

Lemma 2. *Let k be an imaginary quadratic number field with $C_{k,2} \cong (2, 2^m)$. If $C_{k_1,2}$ is not cyclic, then $|\ker j_1| = 2$ or $|\ker j_2| = 2$.*

We now define the 2-class field tower of k as the sequence $k \leqslant k_1 \leqslant k_2 \ldots k_i \leqslant k_{i+1} \ldots$ where k_{i+1} is the 2-Hilbert class field of k_i. If $k_i = k_{i+1}$ and i is minimal we say that k has finite 2-class field tower of length i; otherwise k has infinite 2-class field tower. We now state and prove the following theorem.

Theorem 2. *Let k be an imaginary quadratic number field with $C_{k,2} \cong (2, 4)$ and $C_{k_1,2}$ elementary. Then k has finite 2-class field tower of length at most 2.*

Proof: By Blackburn [1] we know that if $G/G' \cong (2, 4)$ and k has 2-class field tower of length greater than 2, then G' must have rank $\leqslant 3$. By Blackburn [2] if G has rank 2 and the rank of G' is less than or equal to 2 then G' is abelian. Therefore since $C_{k,2} \cong (2, 4)$, $C_{k_1,2}$ has rank equal to 3. By Hall and Senior [5] there are two non-isomorphic groups G with $G/G \cong (2, 4)$ and $G' \cong (2, 2, 2)$ ($(2, 2, 2)$ denotes $Z/2Z \times Z/2Z \times Z/2Z$). These groups can be described as follows:

$$G_1 = \langle q, s\rangle\colon x^2 = y^2 = z^2 = 1,\ zrzr = x,\ ysys^{-1} = x,\ qrqr = y,$$
$$zszs^{-1} = y,\ qsqs^{-1} = z,\ q^2 = 1,\ r^2 = 1$$
$$G_2 = \langle q, s\rangle\colon x^2 = y^2 = z^2 = 1,\ zrzr = x,\ ysys^{-1} = x,\ qrq^{-1}r = y,$$
$$zszs^{-1} = y,\ qsq^{-1}s^{-1} = z,\ q^2 = x,\ r^2 = 1,\ s^2 = r.$$

In each case $G/G' = \langle\overline{q}, \overline{s}\rangle$, $\overline{q}^2 = 1$, $\overline{s}^4 = 1$, $G' = \langle x, y, z\rangle$ and the three maximal subgroups are $H_1 = \langle s, G'\rangle$, $H_2 = \langle qs, G'\rangle$, $H_3 = \langle q, s^2, G'\rangle$. We note that $H_i' =$

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$\langle x, y\rangle$ for $i = 1, 2, 3$ and for each group G we find the kernels of the transfer maps $t_i\colon G/G' \to H_i/H_i'$ are as follows: $\ker t_1 = \langle \overline{q}, \overline{s}^2\rangle$, $\ker t_2 = \langle \overline{q}, \overline{s}^2\rangle$, $\ker t_3 = \langle \overline{s}^2\rangle$. We observe that t_1 and t_2 have 4 ideals that capitulate, and t_3 has 2 ideals that capitulate. From Lemma 2 we see that this capitulation cannot occur for k imaginary and therefore if $|C_{k_1,2}| = 8$, $C_{k_1,2}$ is not elementary. We conclude that k has 2-class field tower of length at most 2.

In the process of attempting to refine our earlier results for k imaginary, $C_{k,2} \cong (2, 4)$ and $C_{k_1,2}$ cyclic, we observed three groups in Hall and Senior [5] that are in error.

These groups are listed as groups 64/140, 64/141 and 64/143 in Hall and Senior, and can be described as follows:

GROUP 64/140. $x^2 = 1$, $y^2 = x$, $z^2 = y^{-1}$, $q^2 = z^{-1}$, $yry^{-1}r^{-1} = x$, $zrz^1r^{-1} = y$, $qrq^{-1}r^{-1} = z$, $r^2 = w$, $w^2 = 1$.

GROUP 64/141. $x^2 = 1$, $y^2 = x$, $z^2 = y^{-1}$, $q^2 = z^3$, $r^2 = w$, $yry^{-1}r^{-1} = x$, $zrz^{-1}r^{-1} = y$, $qrq^{-1}r^{-1} = z$, $w^2 = 1$.

GROUP 64/143. $x^2 = 1$, $y^2 = x$, $z^2 = y^{-1}$, $x = w^2$, $q^2 = z^{-1}$, $r^2 = w$, $yry^{-1}r^{-1} = x$, $zrz^{-1}r^{-1} = y$, $qrq^{-1}r^{-1} = z$.

We note that in all three groups G_i, $i = 1, 2, 3$, we have $G_i = \langle q, r\rangle$, $G_i' = \langle z\rangle$, $G_i/G_i' = \langle \overline{q}, \overline{r}\rangle$, and $Z_i = \langle x, w\rangle$ where Z_i is the centre of G_i. We also note that H_3, the non-cyclic maximal subgroup as a factor group of G', is always abelian since in all three groups $H_3 = \langle q, r^2, z\rangle = \langle q, w\rangle$. However, in all three groups G_i there is a problem with the transfer map $t_i\colon G_i/G_i' \to H_3/H_3'$. The transfer map t_i can be described for all three groups G_i in the following way: $t_i(\overline{I}) = t_i(\overline{z}) = zrzr^{-1} = yrzzr^{-1} = yry^{-1}r^{-1} = x$. Thus the transfer map takes the identity element to an element that is not the identity, which of course cannot happen. ∎

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