THE SUMMABILITY OF FORMAL SOLUTIONS OF FUNCTIONAL EQUATIONS

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In 1923 Nörlund [1] considered the difference equation

$$u(x+\omega)-u(x) = \theta(x) \tag{1}$$

and showed that the formal solution of (1)

$$u(x) = c- \sum_0^\infty \theta(x+n\omega) \tag{2}$$

obtained by iteration, although in general divergent, is in fact Abel summable to a solution of (1). He writes the arbitary constant c as

$$\int_a^\infty \theta(s)ds,$$

and thus the "principal solution" of the difference equation is

$$u(x) = \lim_{\lambda\to 0+} \left\{\int_a^\infty \theta(s)e^{-\lambda s}ds - \sum_0^\infty \theta(x+n\omega)e^{-\lambda(x+n\omega)}\right\}.$$

Nörlund showed further that the series (2) is summable to a solution by a large class of methods.

Bellman [2] solved the analogous problem for Fredholm integral equations of the second kind with both continuous and L^2 kernels. The equation

$$u(x) = f(x)+\lambda \int K(x, s)u(s)ds \tag{3}$$

has as a formal solution the familiar Neumann series and it is shown in [2] that if λ is not a characteristic value and if $\mathscr{S}$ is a regular summability method which satisfies the following condition

G: "$\mathscr{S}$ sums the geometric series $\sum_0^\infty z^n$ $(z \neq 1)$ to $1/(1-z)$ in some set of points $D = D(\mathscr{S})$ of the complex plane",

then $\mathscr{S}$ sums the Neumann series to a solution of (3) for all $\lambda \in \Lambda(D)$, a set determined by D.

Bellman concludes with the remark "· · · it would seem that for those linear functional equations where a solution in the large is known 'a priori',

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a summability theory of the formal solution . . . may be framed". In this note it is shown that for a large class of abstract functional equations this conjecture is valid.

Let X be an arbitrary Banach space and T a bounded linear transformation of X into itself. We call an open set U *T-admissible* if

(i) $U \supset \sigma(T)$, the spectrum of T; and

(ii) U has a finite number of components and the boundary of U consists of a finite number of disjoint rectifiable Jordan curves.

The general linear functional equation will be written as

$$(\lambda I - T)u = f \tag{4}$$

and we assume throughout that $f \neq 0$. Now, as is well known [3], in any Banach Algebra $\boldsymbol{A}$ with $T \in \boldsymbol{A}$ and $|\lambda| > \|T\|$, the resolvent $R(\lambda, T)$ has the expansion

$$\lambda^{-1}(I + \lambda^{-1}T + \lambda^{-2}T^2 + \cdots) \tag{5}$$

convergent in the norm topology, in this case, the uniform topology of operators. We are thus led to the prototype of formal solutions of (4)

$$u = \lambda^{-1} \textstyle\sum_0^\infty \lambda^{-n} T^n f. \tag{6}$$

Suppose now that $\mathscr{S} = (c_{n,m})_{n,m=0}^\infty$ is a regular summability matrix satisfying G (the following results will still hold if instead $\mathscr{S}$ is a sequence-to-function transformation) and for all $z \in D$ consider the transformed partial sums of the geometric series

$$\begin{aligned} t_m(z) &= \textstyle\sum_0^\infty c_{n,m} \sum_0^n z^\nu, \qquad m = 0, 1, \cdots \\ &= (1-z)^{-1} \textstyle\sum_0^\infty c_{n,m}(1 - z^{n+1}), \end{aligned} \tag{7}$$

then for m sufficiently large, say $m > m_0$, each of the series (7) converge and hence $t_m(z)$ is analytic in D. Now by regularity [4], we have

$$\textstyle\sum_0^\infty c_{n,m} \to 1 \quad \text{as} \quad m \to \infty,$$

and as $\mathscr{S}$ satisfies condition G, $(1-z)^{-1} \sum_0^\infty c_{n,m} z^{n+1} \to 0$ for all $z \in D$. Further

$$|t_m(z) - (1-z)^{-1}| \leqq |1-z|^{-1}\{|\textstyle\sum_0^\infty c_{n,m} - 1| + |\sum_0^\infty c_{n,m} z^{n+1}|\}$$

and by the above, given any $\varepsilon > 0$ there exists an $m_0 = m_0(\varepsilon)$ such that

$$|\textstyle\sum_0^\infty c_{n,m} - 1| < \varepsilon \qquad \text{for all } m > m_0.$$

If now z is contained in some compact subset V of D, there is a constant $M > 0$ such that $|1-z|^{-1} < M$, and further, given any $\eta > 0$ there exists an $m_1 = m_1(\eta)$ such that $|\sum_0^\infty c_{n,m} z^{n+1}| < \eta$ for all $m > m_1$. Hence we have that $t_m(z) \to 1/(1-z)$ uniformly on any compact subset of D.

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Now for all z in some T-admissible set U

$$\lambda^{-1}t_m(z/\lambda) \to 1/(\lambda - z)$$

uniformly in z and for all $\lambda \in \Lambda(D) = \{w \mid U \subseteqq wV\}$. So finally because of the T-admissibility of U we may apply the operational calculus for bounded linear operators [3] — and assert that for all $\lambda \in \Lambda(D)$

$$\lambda^{-1}t_m(T/\lambda) \to R(\lambda, T)$$

in the uniform operator topology. Therefore the series (5) is summable ($\mathscr{S}$) to the resolvent of T and hence, for all $f \in X$, (6) is summable to a solution of equation (4) — the desired result. Hence we have proven the

THEOREM 1. *Suppose $\mathscr{S}$ is a regular summability method satisfying G, then for any T-admissible set U and any compact set $V \subseteqq D$, the formal series* (6) *is summable to a solution of* (4) *for all* $\lambda \in \{w \mid U \subseteqq wV\}$.

In the case when X is a Hilbert Space and the convergence of (6) is considered in the weak topology, we see that the regularity of $\mathscr{S}$ in the above theorem is not a necessary condition. For example take T to be self-adjoint and degenerate (that is, the range of T is a finite-dimensional subspace of X) and the result is a direct consequence of the Spectral Decomposition Theorem for such operators.

Note also that with $r(T)$ denoting the spectral radius of T, Theorem 1 implies that the series (6) may be continued analytically across the boundary $|\lambda| = r(T)$ in such a way that the extension is a solution of (4). However [3], $R(\lambda, T)$ cannot be extended analytically into $\sigma(T)$ and so the method $\mathscr{S}$ will lead to a representation of $R(\lambda, T)$ valid in a certain region of the resolvent set $\rho(T)$. For example, if $\mathscr{S}$ represents Borel Summability we arrive at the familiar representation

$$R(\lambda, T) = \int_0^\infty e^{-\lambda x} \exp(xT) dx$$

valid in the Borel polygon of some T-admissible set U.[1]

It may happen — with a given T — that for certain $u \in X$ $R(\lambda, T)u$ is extendable into $\sigma(T)$ and hence deeper results might be expected by considering the problem in the strong topology of operators. Work in this direction is currently in progress.

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[1] A variety of methods satisfying condition G are to be found on page 190 of Hardy's book [4].

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References

[1] N. E. Nörlund, 'Mémoire sur le calcul aux différences finies'. *Acta Math.* 44 (1923), 71–211.
[2] R. Bellman, 'The summability of formal solutions of linear integral equations'. *Duke Math. J.* 17 (1950), 53–55.
[3] E. Hille & R. S. Phillips, *Functional analysis and semi-groups* (New York, 1958).
[4] G. H. Hardy, *Divergent series* (O.U.P. 1949).

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