Areal coordinate proof of Theorems 2 and 3

The area of triangle $DEF$ is worked out using areal coordinates by evaluating the $3 \times 3$ determinant whose entries in rows 1, 2, 3 are the coordinates of $D, E, F$ respectively, and then the result is $[DEF]/[ABC]$, the ratios of the areas of triangles $DEF$ and the fundamental triangle $ABC$. Using a computer algebra package, we obtain

$$[DEF] = [ABC] \frac{a^2 b^2 + j(a^2 + b^2 - c^2 + j)}{a^2 + b^2 + j}.$$

Repeating the calculation using the coordinates of $D', E', F'$ we find that $[D'E'F']$ has the same area as $[DEF]$. It is also interesting to work out the sides of these triangles. In areal coordinates, if a displacement $ST = (x, y, z)$, a displacement being the difference in the normalised coordinates of $T$ and $S$, then $ST^2 = -a^2yz - b^2zx - c^2xy$, see Bradley [2]. Using this formula we obtain $DE^2 = F'D'^2 = pa^2$, $EF^2 = D'E'^2 = pb^2$, $FD^2 = E'F'^2 = pc^2$, where

$$p = \frac{a^2(b^2 + j) + j(b^2 - c^2 + j)}{(a^2 + b^2 + j)^2}.$$

This proves that triangles $DEF$ and $D'E'F'$ are in fact congruent and that $DEF$ is similar to $BCA$ and $D'E'F'$ is similar to $CAB$ (order of letters being indicative).

Finally, if the Brocard points of $ABC$ are $\Omega\left(\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}\right)$ and $\Omega'\left(\frac{1}{c^2}, \frac{1}{a^2}, \frac{1}{b^2}\right)$, then $\Omega$ is one of the Brocard points of $DEF$ and $\Omega'$ is the other Brocard point of $D'E'F'$. The proof of these results is left for the reader.

References


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95.17 The golden ratio in conic sections

Inspired by Domenico [1, p. 261], I identify two examples of the golden ratio which are presented below.

**Example 1:** Suppose that $P_1P_2$ is a focal chord of a parabola with focus $S(a, 0)$. If $a, SP_1, SP_2$ form a geometric progression, then $\cot^2 \theta$ equals the golden ratio, where $\theta$ is the angle between the tangent at $P_2$ and the axis of the parabola.
Proof: Let the equation of the parabola be \( y^2 = 4ax \) and the coordinates of \( P_1 \) and \( P_2 \) be \((at_1^2, 2at_1)\) and \((at_2^2, 2at_2)\) respectively. Then
\[
t_1t_2 = -1. \tag{1}
\]
Let the lengths of the segments \( SP_1 \) and \( SP_2 \) be \( l_1 \) and \( l_2 \) respectively.
\[
l_1 = a(t_1^2 + 1) \tag{2}
\]
and
\[
l_2 = a(t_2^2 + 1). \tag{3}
\]
But \( a, l_1, l_2 \) are in G.P.
\[
l_1^2 = al_2
\]
and we have
\[
\cot \theta = t_2.
\]
From (2) and (3), we get
\[
a^2(t_1^2 + 1)^2 = a^2(t_2^2 + 1).
\]
From (1), we have
\[
\left( \frac{1}{t_2^2} + 1 \right)^2 = t_2^2 + 1
\]
or
\[
\frac{t_2^2 + 1}{t_2^2} = 1 \quad \text{since} \quad t_2^2 + 1 > 0
\]
or
\[
u^2 - u - 1 = 0, \quad \text{where} \quad u = t_2^2 = \cot^2 \theta.
\]
\[
u = \left( \frac{1 \pm \sqrt{1 + 4}}{2} \right) = \frac{1 \pm \sqrt{5}}{2}.
\]
But being a perfect square, \( u \) cannot be negative.
\[
u = \frac{\sqrt{5} + 1}{2}, \text{ which is the golden ratio.}
\]

Example 2: Suppose that a hyperbola has semi-major and semi-minor axes of length \( a \) and \( b \) respectively. If the radius of the director circle of the hyperbola is the geometric mean of \( a \) and \( b \), then \( \frac{a}{b} \) equals the golden ratio.

Proof: Since the equation of the hyperbola is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), the equation of the director circle is \( x^2 + y^2 = a^2 - b^2 \). Hence its radius is \( \sqrt{a^2 - b^2} \).

According to the given condition
\[
a^2 - b^2 = ab
\]
or \( \left( \frac{a}{b} \right)^2 - 1 = \frac{a}{b} \).

\[
\therefore \quad v^2 - v - 1 = 0, \text{ where } v = \frac{a}{b}.
\]

Hence \( v \) equals golden ratio.

Reference


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95.18 Normals to the Euler line

In both Cartesian and areal coordinates, the parallelism of lines is simply expressed by means of a null determinant but perpendicularity is rather more problematical in the latter framework.

In this note we will first use areal coordinates to obtain the set \( S \) of normals to the Euler line of the triangle of reference \( ABC \). We will then consider the set \( T \) of circles whose centres lie on the Euler line (with radical axes of pairs of circles in \( T \) thus belonging to \( S \)).

The normals

Denoting the triangle area by \( \Delta \), we first find the length of the Euler line segment \( GH \) (for centroid \( G \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \) and orthocentre \( H \left[ \alpha\beta, \gamma\alpha, \alpha\beta \gamma \right] \) in actual areal coordinates).

Thus, with angle cotangents \( \alpha, \beta, \gamma \) (\( \beta\gamma + \gamma\alpha + \alpha\beta = 1, \alpha + \beta + \gamma = \Sigma, \alpha\beta\gamma = \Pi \)), the expression

\[
\frac{GH^2}{2\Delta} = \alpha \left( \frac{1}{3} - \beta\gamma \right)^2 + \beta \left( \frac{1}{3} - \gamma\alpha \right)^2 + \gamma \left( \frac{1}{3} - \alpha\beta \right)^2 = \frac{(\Sigma - 9\Pi)}{9}
\]

follows easily.

Then by considering the point \( P \left[ 0, \frac{(\Sigma - \gamma)}{(\beta - \gamma)}, \frac{(\Sigma - \beta)}{(\gamma - \beta)} \right] \) on \( BC \), we will show that \( GP \) is perpendicular to the Euler line.

Now

\[
\frac{GP^2}{2\Delta} = \alpha \left( \frac{1}{3} - 0 \right)^2 + \beta \left( \frac{1}{3} - \frac{(\Sigma - \gamma)}{(\beta - \gamma)} \right)^2 + \gamma \left( \frac{1}{3} - \frac{(\Sigma - \beta)}{(\gamma - \beta)} \right)^2
\]

leading to \( \frac{GP^2}{2\Delta} = \frac{(\Sigma - 9\Pi)}{9(\beta - \gamma)^2} \) after some algebra.