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q-Pseudoconvexity and Regularity at the Boundary for Solutions of the $\bar{\partial}$-problem

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Abstract. For a domain Ω of $\mathbb{C}^N$ we introduce a fairly general and intrinsic condition of weak q-pseudoconvexity, and prove, in Theorem 4, solvability of the $\bar{\partial}$-complex for forms with $C^\infty(\bar{\Omega})$-coefficients in degree $\geqslant q+1$.

All domains whose boundary have a constant number of negative Levi eigenvalues are easily recognized to fulfill our condition of q-pseudoconvexity; thus we regain the result of Michel (with a simplified proof).

Our method deeply relies on the L^2-estimates by Hörmander (with some variants). The main point of our proof is that our estimates (both in weightened-L^2 and in Sobolev norms) are sufficiently accurate to permit us to exploit the technique by Dufresnoy for regularity up to the boundary.

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Let Ω be a domain of $\mathbb{C}^N$, z_o a point of $M := \partial\Omega$, U a neighborhood of z_o. We consider an orthonormal basis of $(1,0)$–forms $\omega_1, \dots, \omega_N$ on U, and the dual basis $\partial_{\omega_1}, \dots, \partial_{\omega_N}$ of $(1,0)$–derivatives. We assume that M is C^2, take a defining function ρ for Ω (thus $\Omega = \{\rho < 0\}$) and denote by $\big(\rho_{ij}(z)\big)$ the matrix of the Hermitian form $\bar{\partial}\partial\rho(z)$ in the basis $\{\omega_i\}$. We assume that, for a suitable choice of $\{\omega_i\}$ with C^2–coefficients and with $\omega_N = \partial\rho$, and for an integer q with $1 \leqslant q < N$, we have

$$\Big(\rho_{ij}(z)\Big)_{ij \leqslant q} \leqslant 0, \quad \Big(\rho_{ij}(z)\Big)_{q+1 \leqslant ij \leqslant N-1} \geqslant 0, \quad \Big(\rho_{ij}(z)\Big)_{i \leqslant q, q+1 \leqslant j \leqslant N-1} = 0$$
$$\forall z \in M \cap U. \tag{1}$$

Remark 1. Put $\mathcal{M}(z) = span\{\partial_{\omega_1}, \dots, \partial_{\omega_q}\}$, then $\mathcal{M}$ is a C^2 *majorant of the negative eigenspace* $\mathcal{M}_M^-$ of $\bar{\partial}\partial\rho|_{\partial\rho^\perp}$. Here, as in the following, $\partial\rho^\perp$ is the complex hyperplane of $\mathbb{C}^N$ orthogonal to $\partial\rho$. We shall also use in the following the notation $\mathcal{M}_M^0$ and $\mathcal{M}_M^+$ for the null and positive eigenspace respectively.

Note also that (1) is independent of the choice of the 'defining' function ρ.

Denote by $s_M^\pm(z)$ the numbers of respectively positive and negative eigenvalues of the form $\bar{\partial}\partial\rho(z)_{\partial\rho^\perp(z)}$ and consider the condition

$$s_M^-(z) \equiv q \, \forall z \in M \cap U. \tag{2}$$

LEMMA 2. *Let Ω be C^4. Then (2) is equivalent, in a suitable C^2 basis $\{\omega_i\}$, to (1) with the additional requirement: $(\rho_{ij}(z))_{ij \leqslant q} < 0$ (instead of $\leqslant 0$).*

Proof. Let $\mu_1(z) \leqslant \mu_2(z) \leqslant \ldots \leqslant \mu_{N-1}(z)$ be the eigenvalues of $(\rho_{ij}(z))|_{\partial\rho(z)^\perp}$. It is clear that

$$\mu_q(z) < 0, \quad \mu_{q+1}(z) \geqslant 0 \ \forall z \in U.$$

Thus the eigenspace of the first q (resp. second $N-1-q$) eigenvectors depend C^2 on z and coincide with $\mathcal{M}_M^-$ (resp. $\mathcal{M}_M^0 \cup \mathcal{M}_M^+$). □

For ordered multi-indices $J = (j_1 < \ldots < j_k)$ of a given length $|J| = k$, we shall consider vectors $w = (w_J)$. For any permutation σ we shall also put $w_{\sigma(J)} := \operatorname{segn}(\sigma) w_J$.

PROPOSITION 3. *Assume (1). Then for a suitable ρ and with $\phi(z) = -log(-\rho)(z) + \lambda'|z|^2$ (λ' real positive), we get an exhaustion function of Ω at z_o such that for suitable λ' and for any $k \geqslant q+1$:*

$$\sum_{|K|=k-1}{}' \sum_{ij=1,\ldots,N}{}' \phi_{ij}(z) w_{iK} \bar{w}_{jK} - \sum_{|J|=k}{}' \sum_{i \leqslant q} \phi_{ii}(z)|w_J|^2 \geqslant \lambda |w|^2 \ \forall z \in \Omega \cap U,$$
$$\forall w \in \mathbb{C}^N \tag{3}$$

(*with a new $\lambda > 0$ and where $\sum'$ indicates the sum restricted to ordered indices*).

Proof. We begin by solving this initial problem. In condition (3) the Levi form is evaluated at points of Ω, whereas in the assumption (1) it is evaluated at $\partial\Omega$. To fill this gap we represent $\partial\Omega$ as a graph $x_N = h$ and consider the projection $\Omega \to \partial\Omega$, $z \mapsto z^*$ along the x_N-axis. For $\rho = x_N - h$ we clearly have:

$$\partial\rho^\perp(z) = \partial\rho^\perp(z^*), \quad \bar{\partial}\partial\rho(z) = \bar{\partial}\partial\rho(z^*).$$

For this reason, (1) is in fact fulfilled also in Ω (even though in this form it is no more intrinsic and depends on our particular choice of the defining function ρ). Thus we shall forget z in the following and always suppose it ranges through Ω.

We shall also use the notation $\omega' = (\omega_1, \ldots, \omega_{N-1})$, $\omega_N = \partial\rho$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and $\mu_1 \leqslant \mu_2 \leqslant \ldots$ be the eigenvalues of $\bar{\partial}\partial\phi$ and $\bar{\partial}\partial\rho|_{\partial\rho^\perp}$ respectively. Since $\bar{\partial}\partial\phi = |\rho|^{-1}\bar{\partial}\partial\rho + |\rho|^{-2}\bar{\omega}_N \wedge \omega_N + \lambda'\bar{\omega} \wedge \omega$, then $|\rho|^{-1}\mu_i + \lambda'$ are the eigenvalues of $\bar{\partial}\partial\phi|_{\partial\rho^\perp}$. Also it is clear that:

$$\begin{aligned} &\sum_{|K|=k-1}{}' \sum_{ij=1,\ldots,N}{}' \phi_{ij}(z) w_{iK} \bar{w}_{jK} \geqslant \left(\sum_{i=1,\ldots,k} \lambda_i \right) |w|^2, \\ &\sum_{|J|=k}{}' \sum_{i \leqslant q} \phi_{ii}(z)|w_J|^2 = \left((-\rho)^{-1} \left(\sum_{i \leqslant q} \mu_i \right) + \lambda' q \right) |w|^2. \end{aligned} \tag{4}$$

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We claim that for a suitable $c > 0$,

$$\sum_{i=1,\dots,k} \lambda_i - \rho^{-1} \sum_{i=1,\dots,q} \mu_i - \lambda' q \geqslant ((k-q)\lambda' - kc) =: \lambda. \tag{5}$$

(where in turn λ is positive for suitable λ'). In fact:

$$\begin{aligned} \bar\partial\partial\phi &= (-\rho)^{-1}\bar\partial\partial\rho + \rho^{-2}\bar\omega_N \wedge \omega_N + \lambda'\bar\omega \wedge \omega \\ &= (-\rho)^{-1}\bar\partial'\partial'\rho + \left[\rho^{-2}\bar\omega_N \wedge \omega_N + 2(-\rho)^{-1}\Re\mathrm{e}\bar\partial'\partial_{\omega_N}\rho + c|\omega'|^2\right] \\ &\quad - c\bar\omega' \wedge \omega' + \lambda'\bar\omega \wedge \omega. \end{aligned} \tag{6}$$

Now for suitable large c we can make the term between brackets '[·]' in the second line of (6) to be positive. It follows:

$$\bar\partial\partial\phi \geqslant (-\rho)^{-1}\bar\partial'\partial'\rho - c\bar\omega' \wedge \omega' + \lambda'\bar\omega \wedge \omega. \tag{7}$$

Let $\{N_k\}$ describe the family of complex k-dimensional planes in $\mathbb{C}^N$. We have:

$$\begin{aligned} \sum_{i=1,\dots,k} \lambda_i &= \inf_{N_k} \mathrm{trace}(\bar\partial\partial\phi|_{N_k}) \\ &\geqslant \inf_{N_k} \mathrm{trace}\big(((-\rho)^{-1}\bar\partial'\partial'\rho - c\bar\omega \wedge \omega + \lambda'\bar\omega \wedge \omega)|_{N_k}\big) \\ &\geqslant (k\lambda' - kc) + (-\rho)^{-1} \sum_{i=1,\dots,k} \mu_i. \end{aligned} \tag{8}$$

(where the central inequality is due to (7)). From (8) and (1) our claim (5) immediately follows. (5) and (4) imply in turn (3). The proof is complete. □

We shall consider forms $f = \sum'_J f_J \bar\omega_J$ (resp. $u = \sum'_K u_K \bar\omega_K$) of type $(0, k)$ (resp. $(0, k-1)$). (Since all forms shall be understood to be antiholomorphic we shall only mention in the following their degree k instead of their type $(0, k)$.)

THEOREM 4. *Assume that in a C^2 basis of ω_i's, (1) is fulfilled. Then there is a fundamental system of neighborhoods $\{U\}$ of z_o such that if $k(=$ degree$(f)) \geqslant q+1$ and $\bar\partial f = 0$ in $\overline{\Omega \cap U}$, then the equation*

$$\bar\partial u = f \quad \text{is solvable in } C^\infty(\overline{\Omega \cap U'}) \text{ for any } U'CCU. \tag{9}$$

The proof will be given in many steps. For a real positive function ϕ and for an integer $k \geqslant 0$, we define $L^2_\phi(\Omega)^k$ to be the space of k-antiholomorphic forms $f = \sum'_{|J|=k} f_J \bar\omega_J$ with $||f_J||_\phi (:= \left(\int_\Omega \mathrm{e}^{-\phi}|f_J|^2 \mathrm{dV}\right)^{\frac{1}{2}}) < +\infty$ (dV = the Lebesgue measure on $\mathbb{C}^N$, $\{\omega_i\}$ = a basis over $\mathbb{C}^N$). Here, as always, $\sum'$ indicates the sum over ordered indices. We let $\bar\partial$ act as a complex:

$$L^2_\phi(\Omega)^{k-1} \xrightarrow{\bar\partial} L^2_\phi(\Omega)^k \xrightarrow{\bar\partial} L^2_\phi(\Omega)^{k+1}. \tag{10}$$

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We denote by $\bar\partial^*$ (resp. δ_{ω_i}) the adjoint of $\bar\partial$ (resp. $-\partial_{\bar\omega_i}$) in the $L^2_\phi(\Omega)$-norm. We have

$$\begin{aligned}&\sum_{|K|=k-1}' \sum_{ij=1,\dots,N}' \int_\Omega e^{-\phi}\left(\delta_{\omega_i} f_{iK}\overline{\delta_{\omega_j} f_{jK}} - \partial_{\bar\omega_j} f_{iK}\overline{\partial_{\bar\omega_i} f_{jK}}\right)\mathrm{dV}+\\ &\quad+\sum_{|J|=k}' \sum_{i=1,\dots,N} \int_\Omega e^{-\phi}|\partial_{\bar\omega_i} f_J|^2\mathrm{dV} = ||\bar\partial^* f + R(f)||^2_\phi + ||\bar\partial f + R(f)||^2_\phi \\ &\quad \forall f \in \mathrm{C}^\infty_c(\Omega)^k,\end{aligned} \tag{11}$$

where $R(f)$ is an error where no f_J is differentiated and which involves the derivatives of the coefficients of the ω_i's. Let (I) be the left side of (11). We then get

$$(I) \leqslant 2(||\bar\partial^* f||^2_\phi + ||\bar\partial f||^2_\phi) + \sigma_1^2 ||f||^2_\phi \;\; \forall f \in \mathrm{C}^\infty_c(\Omega)^k, \tag{12}$$

where σ_1 denotes terms which can be estimated by the sup-norm of the first derivatives of the ω_i's over the support of f. (In the following we shall also use the notation σ_2 for constants which can be estimated by the second derivatives.) If we introduce now a new $\psi \geqslant 0$, and replace (10) by:

$$L^2_{\phi-2\psi}(\Omega)^{k-1} \xrightarrow{\bar\partial} L^2_{\phi-\psi}(\Omega)^k \xrightarrow{\bar\partial} L^2_\phi(\Omega)^{k+1}, \tag{13}$$

we get:

$$\begin{aligned}(I) &= ||e^\psi \bar\partial^* f + R(f) + \partial\psi\cdot f||^2_\phi + ||\bar\partial f + R(f)||^2_\phi \\ &\leqslant 2(||\bar\partial^* f||^2_{\phi-2\psi} + ||\bar\partial f||^2_\phi) + \sigma_1^2||f||^2_\phi + 2|||\partial\psi|f||^2_\phi \quad \forall f \in \mathrm{C}^\infty_c(\Omega),\end{aligned} \tag{14}$$

where $\partial\psi\cdot f := \sum_K' \sum_i \partial_{\omega_\bullet}\psi f_{iK}$. The main ingredient of the proof of Th. 4 is contained in the following

PROPOSITION 5. *For any orthonormal* C^2*-basis* $\{\omega_i\}$, *and with* (ϕ_{ij}) *denoting the matrix of* $\bar\partial\partial\phi$ *in such basis, we have*

$$\begin{aligned}&\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_\Omega \mathrm{e}^{-\phi}\phi_{ij} f_{iK}\bar f_{jK}\mathrm{dV} - \sum_{|J|=k}' \int_\Omega \mathrm{e}^{-\phi}\phi_{ii}|f_J|^2\mathrm{dV}\\ &\quad\leqslant 2(||\bar\partial^* f||^2_{\phi-2\psi} + ||\bar\partial f||^2_\phi + |||\partial\psi|f||^2_\phi) + (\sigma_1^2+\sigma_2)||f||^2_\phi \;\; \forall f \in \mathrm{C}^\infty_c(\Omega)^k.\end{aligned} \tag{15}$$

Proof. We recall that

$$\begin{aligned}&\delta_{\omega_i} = -\partial^*_{\bar\omega_i},\\ &\delta_{\omega_i}\partial_{\bar\omega_j} - \partial_{\bar\omega_j}\delta_{\omega_i} = \partial_{\bar\omega_j}\partial_{\omega_i}\phi + \sum_h c^h_{ji}\partial_{\omega_h} - \sum_h \bar c^h_{ij}\partial_{\bar\omega_h}\\ &\qquad\qquad = \phi_{ji} + \sum_h c^h_{ji}\delta_{\omega_h} - \sum_h \bar c^h_{ij}\partial_{\bar\omega_h},\end{aligned} \tag{16}$$

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where the terms c_{ji}^h involve the antiholomorphic derivatives of the coefficients of the ω_i's. We apply (16) to the terms in the first sums of (I) with $i \neq j$ or $i = j \geqslant q+1$. The remaining terms added to the second sum give

$$\sum_{|K|=k-1}' \sum_{i \leqslant q} ||\delta_{\omega_i} f_{iK}||_\phi^2 + \sum_{|J|=k}' \sum_{i \geqslant q+1 or\ i \notin J} ||\partial_{\bar{\omega}_i} f_J||_\phi^2. \tag{17}$$

We also apply (16) to the terms in the second sums in (17) with $i \leqslant q, i \notin J$. Thus (17) becomes:

$$\sum_{|J|=k}' \sum_{i \leqslant q} ||\delta_{\omega_i} f_J||_\phi^2 + \sum_{|J|=k}' \sum_{i \geqslant q+1} ||\partial_{\bar{\omega}_i} f_J||_\phi^2 - \sum_{|J|=k}' \sum_{i \leqslant q i \notin J} \int_\Omega \mathrm{e}^{-\phi} \phi_{ii} |f_J|^2 \mathrm{dV}.$$

Thus we get

$$\begin{aligned} &\left(\sum_{|K|=k-1}' \sum_{ij=1,\ldots,N} \int_\Omega \mathrm{e}^{-\phi} \phi_{ij} f_{iK} \bar{f}_{jK} \mathrm{dV} - \sum_{|J|=k}' \sum_{i \leqslant q} \int_\Omega \mathrm{e}^{-\phi} \phi_{ii} |f_J|^2 \mathrm{dV} \right) + \\ &\quad + \left(\sum_{|J|=k}' \sum_{i \leqslant q} ||\delta_{\omega_i} f_J||_\phi^2 + \sum_{|J|=k}' \sum_{i \geqslant q+1} ||\partial_{\bar{\omega}_i} f_J||_\phi^2 \right) \\ &\quad \leqslant 2\left(||\bar{\partial}^* f||_{\phi-2\psi}^2 + ||\bar{\partial} f||_\phi^2 + |||\partial\psi| f||_\phi^2 \right) + \sigma_1^2 ||f||_\phi^2 + \\ &\quad + \left(\sum_K' \sum_{hij} \left| \int_\Omega \mathrm{e}^{-\phi} c_{ji}^h \delta_{\omega_h}(f_{iK}) \bar{f}_{jK} \mathrm{dV} \right| + \sum_K' \sum_{hij} \left| \int_\Omega \mathrm{e}^{-\phi} \bar{c}_{ij}^h \partial_{\bar{\omega}_h}(f_{iK}) \bar{f}_{jK} \mathrm{dV} \right| \right). \end{aligned} \tag{18}$$

Let us denote by A, B, C, D, the four lines in (18). To get a good estimation for D we remark that:

$$\int_\Omega \mathrm{e}^{-\phi} c_{ji}^h \delta_{\omega_h} f_{iK} \bar{f}_{jK} \mathrm{dV} = -\int_\Omega \mathrm{e}^{-\phi} c_{ji}^h f_{iK} \overline{\partial_{\bar{\omega}_h} f_{jK}} \mathrm{dV} - \int_\Omega \mathrm{e}^{-\phi} \partial_{\omega_h}(c_{ji}^h) f_{iK} \bar{f}_{jK} \mathrm{dV}. \tag{19}$$

It follows:

$$D \leqslant \sigma_1 ||f||_\phi \left(\frac{B}{2} \right)^{\frac{1}{2}} + \sigma_2 ||f||_\phi^2 \leqslant \frac{B}{2} + (\sigma_1^2 + \sigma_2) ||f||_\phi^2. \tag{20}$$

Then the conclusion follows. □

Let us denote by $D_{\bar{\partial}^*}$ and $D_{\bar{\partial}}$ the domains of $\bar{\partial}^*$ and $\bar{\partial}$ respectively defined by (13).

PROPOSITION 6. *Let Ω be bounded and endowed with an exhaustion function which satisfies* (3) *($\forall z \in \Omega$) in a suitable basis of ω_i over $\bar{\Omega}$. Then if $k \geqslant q+1$ and for a new ϕ*

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and a suitable ψ, we have:

$$||f||^2_{\phi-\psi} \leqslant ||\bar{\partial}^* f||^2_{\phi-2\psi} + ||\bar{\partial} f||^2_{\phi} \quad \forall f \in D_{\bar{\partial}} \cap D_{\bar{\partial}^*}. \tag{21}$$

Moreover for any compact subset $K \subset\subset \Omega$, we may choose $\psi|_K \equiv 0$ and $\phi|_K \equiv (2 + \sigma_1^2 + \sigma_2)|z|^2$.

Proof. We choose ψ according to [5, Lemma 4.1.3]; (in particular, $\forall K$, we can choose $\psi|_K \equiv 0$). This ensures density of C_c^∞ into L^2-forms.

We then take an exhaustion function ϕ for Ω which satisfies (3) $\forall z \in \Omega$. We go back to (15) of Proposition 5; this holds now for L^2 instead of C_c^∞ forms. Moreover, in the present situation the left side is larger than $\lambda ||f||^2_\phi$ for some constant $\lambda > 0$ independent of K. Let $c \geqslant \phi|_K$; we replace the above ϕ by $\chi(\phi) + (2 + \sigma_1^2 + \sigma_2)|z|^2$, where χ is a positive convex function of a real argument t which satisfies:

$$\begin{aligned} &\chi(t) \equiv 0, && \text{for } t \leqslant c, \\ &\dot{\chi}(t) \geqslant \sup_{\{z:\phi(z) \leqslant t\}} \frac{2(|\partial\psi|^2 + e^\psi)}{\lambda}, && \text{for } t \geqslant c. \end{aligned} \tag{22}$$

Under this choice of ϕ and ψ, (21) clearly follows. □

With the conclusions of Proposition 6 at our disposal, the rest of the proof of Theorem 4 can be carried out along classical lines. First we need to translate the basic estimate (21) into two results on existence and regularity of solutions of the system $(\bar{\partial}, \bar{\partial}^*)$. For their proof we give [5, Lemma 4.41 and Th. 4.2.5] as general reference and [13, Prop. 2.1 and Prop. 2.2] for a specific proof. We shall denote by $m = m(z)$ the (strictly plurisubharmonic) function $m = (2 + \sigma_1^2 + \sigma_2)|z|^2$. We shall denote by $\bar{\partial}$ (resp. $\bar{\partial}^*$) the $\bar{\partial}$-complex (resp. its adjoint) over $L^2_m(\Omega)$-forms.

PROPOSITION 7. *Let Ω be bounded, assume (3) $\forall z \in \Omega$ in a C^2 basis of ω_i, and let $k \geqslant q + 1$. Then for any $f \in L^2_m(\Omega)^k$ with $\bar{\partial} f = 0$ there exists $u \in L^2_m(\Omega)^{k-1}$ such that*

$$(\bar{\partial} u = f,\ \bar{\partial}^* u = 0), \quad ||u||^2_m \leqslant ||f||^2_m. \tag{23}$$

Let $||\cdot||_{(s)}$ denote the norm of the Sobolev space $W^s(\Omega)$ of index s. Let $\Omega_\varepsilon = \{z \in \Omega | dist(z, \partial\Omega) > \varepsilon\}$.

PROPOSITION 8. *Let Ω be bounded, suppose (3) be satisfied $\forall z \in \Omega$ in a suitable basis of ω_i, and let $k \geqslant q + 1$. Then for any $f \in C^\infty(\Omega)^k$ with $\bar{\partial} f = 0$ there is $u \in C^\infty(\Omega_\varepsilon)$ such that for any $s > 0$ and for suitable $S_s > 0$ (independent of f):*

$$(\bar{\partial} u = f,\ \bar{\partial}^* u = 0), \quad ||u||_{(s+1)} \leqslant \frac{S_s}{\varepsilon^{s+1}} ||f||_{(s)}, \tag{24}$$

where the norm of f and u are over Ω and Ω_ε respectively.

End of Proof of Theorem 4 (cf. Dufresnoy [2]). We choose a decreasing sequence of domains $\ldots, \Omega_\nu \supset \Omega_{\nu+1} \ldots \supset \Omega$ which inherit from Ω the property (1) (and hence, if they are small enough, (3)), and require that for η with $0 < \eta < \frac{1}{2}$ we have $\eta^{2^{v+1}} < dist(\partial\Omega_v, \partial\Omega) < (\eta^{2^v}/2)$. For instance, if Ω is defined in a neighbourhood of z_o by $x_N - h < 0$, we can define Ω_v by $x_n - h < (\eta^{2^v}/2)$. Let U (resp. U_v) be the sphere with center z_o and radius σ (resp. $\sigma + (\eta^{2^v}/2)$. we consider the functions

$$\phi_v = -\log\left(-x_N + h + \frac{\eta^{2^v}}{2}\right) + \lambda|z|^2 - \log\left(-|z - z_o|^2 + \left(\sigma + \frac{\eta^{2^v}}{2}\right)^2\right).$$

Clearly the functions ϕ_v verify (3) for a smooth basis of ω_i's on $\overline{\Omega_v \cap U_v}$. Let f be a smooth form in $\overline{\Omega \cap U_{v_o}}$ for v_o large. To solve the equation $\bar\partial u = f$ in $\bar\Omega \cap U$, we first extend f to $\tilde f$ in Ω_v, $v \geqslant v_o$ such that $\tilde f$ is still C^∞ and

$$||\bar\partial\tilde f|_{\Omega_v} \cap U_v||_{(s)} \leqslant C_{S\,s}\eta^{2^v S} \tag{25}$$

for any S and for suitable $C_{S\,s}$. (This is clearly possible because $\bar\partial\tilde f| \equiv 0$ on Ω.) On account of Proposition 8, we take solutions h_v over $\Omega_v \cap U_v$ of

$$\begin{cases} \bar\partial h_v = \bar\partial\tilde f \\ ||h_v|_{\Omega_{v+1}\cap U_{v+1}}||_{(s+1)} \leqslant S_s(\eta^{2^{v+1}})^{-(s+1)}||\bar\partial\tilde f||_{(s)}, \end{cases} \tag{26}$$

a solution α_1 of $\bar\partial\alpha_1 = \tilde f - h_1$, and finally solutions α_{v+1} of

$$\begin{cases} \bar\partial\alpha_{v+1} = h_v - h_{v+1} \\ ||\alpha_{v+1}||_{(s+2)} \leqslant S_{s+1}(\eta^{2^{v+2}})^{-(s+2)}||h_v - h_{v+1}||_{(s+1)} \leqslant C_{S,s}(\frac{1}{2})^v, \end{cases}$$

(for S and v large). This is clearly possible by recalling (25), (26). It follows that the series $\sum_v \alpha_v$ converges in $C^\infty(\bar\Omega \cap \bar U)$ and solves $\bar\partial(\sum_v \alpha_v) = \tilde f - \lim_v h_v = \tilde f$. This completes the proof of Theorem 4. □

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