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# ON AN ELLIPTIC EQUATION OF $p$-KIRCHHOFF TYPE VIA VARIATIONAL METHODS

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This paper is concerned with the existence of positive solutions to the class of nonlocal boundary value problems of the $p$-Kirchhoff type

$$-\left[M\left(\int_\Omega |\nabla u|^p\, dx\right)\right]^{p-1}\Delta_p u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

and

$$-\left[M\left(\int_\Omega |\nabla u|^p\, dx\right)\right]^{p-1}\Delta_p u = f(x,u) + \lambda|u|^{s-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N$, $s \geqslant p^* = (pN)/(N-p)$ and $M$ and $f$ are continuous functions.

## 1. Introduction

The purpose of this article is to investigate the existence of positive solutions to the class of nonlocal boundary value problems of the $p$-Kirchhoff type

$$(P) \qquad \begin{cases} -\left[M\left(\|u\|^p\right)\right]^{p-1}\Delta_p u = f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

and

$$(P)_\lambda \qquad \begin{cases} -\left[M\left(\|u\|^p\right)\right]^{p-1}\Delta_p u = f(x,u) + \lambda|u|^{s-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where, through this work, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions that satisfy conditions which will be stated later, $\Delta_p u$ is the $p$-Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right), \ 1 < p < N,$$

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and $\|.\|$ is the usual norm in $W_0^{1,p}(\Omega)$ given by

$$\|u\|^p = \int_\Omega |\nabla u|^p.$$

The main goal of this paper is establishing conditions on $M$ and $f$ under which problem $(P)$ and $(P)_\lambda$ possess positive solutions.

Problem $(P)$ and $(P)_\lambda$ are called nonlocal because of the presence of the term $M(\|u\|^p)$ which implies that the equations in $(P)$ and $(P)_\lambda$ are no longer pointwise identities. This provokes some mathematical difficulties which makes the study of such a problem particulary interesting. This problme has a physical motivation. The operator $\left[M(\|u\|^p)\right]^{p-1}\Delta_p u$, with $p = 2$, appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\text{(1.1)} \qquad \begin{cases} u_{tt} - M(\|u\|^2)\Delta u = f(x,u) \text{ in } \Omega \times (0,T) \\ u = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \quad , \quad u_t(x,0) = u_1(x). \end{cases}$$

Hence, problem $(P)$ and $(P)_\lambda$, in case $p = 2$, are the stationary counterpart of the above evolution equation.

Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\text{(1.2)} \qquad \rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff [**8**]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (1.2) have the following meanings: $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density and $P_0$ is the initial tension.

Problem (1.1) began to attract the attention of several researchers mainly after the work of Lions [**9**], where a functional analysis approach was proposed to attack it.

The reader may consult [**1, 2, 4, 10, 13**] and the references therein, for more informations on $(P)$ and $(P)_\lambda$, in case $p = 2$.

Motivated by papers [**2, 5**] and by some ideas developed in [**3, 6**], we prove the existence of positive solutions to $(P)$ and $(P)_\lambda$. However, in this work, we use a different approach to those explored in [**2, 5, 3**], because here we are working with the $p$-Laplacian operator. Some estimates for this type of operator can not be obtained using the same kind of ideas explored for the case p=2. For example, results involving uniform a priori estimate of the Gidas and Spruck type [**7**] does not hold for the $p$-Laplacian. To overcome these difficulties, we use comparison between minimax levels of energy.

This paper is organised as follows: in Section 2, we show the existence of positive solutions for the equation $(P)$. In Section 3, we show the existence of positive solutions for the equation $(P)_\lambda$.

## 2. The Subcritical Case

In this section we assume that $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and satisfies the subcritical growth conditions

$$(f_1) \qquad |f(x,t)| \leqslant C|t|^{q-1},$$

for all $x \in \Omega$ and for all $t \in \mathbb{R}$, where $p < q < p^* = (pN)/(N-p)$.

We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the problem $(P)$ if it satisfies

$$\left[M\left(\|u\|^p\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi - \int_\Omega f(x,u)\phi = 0$$

for all $\phi \in W_0^{1,p}(\Omega)$.

We shall look for solutions of $(P)$ by finding critical points of the $C^1$-functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\widehat{M}\left(\|u\|^p\right) - \int_\Omega F(x,u)dx$$

where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1} ds$ and $F(x,t) = \int_0^t f(x,s)ds$.

Note that

$$I'(u)\phi = \left[M\left(\|u\|^p\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi - \int_\Omega f(x,u)\phi,$$

for all $\phi \in W^{1,p}(\Omega)$.

In order to use critical point theory we first derive results related to the Palais–Smale compactness condition.

We say that a sequence $(u_n)$ is a Palais–Smale sequence for the functional $I$ if

$$I(u_n) \to c \text{ and } \|I'(u_n)\| \to 0 \text{ in } \left(W_0^{1,p}(\Omega)\right)'$$

If every Palais–Smale sequence of $I$ has a strongly convergent subsequence, then one says that $I$ satisfies the Palais–Smale condition ((PS) for short).

Through this paper, we assume that $M$ is a continuous function and satisfies:

$$(M_1) \qquad M(t) \geqslant m_0 > 0 \text{ for all } t \in \mathbb{R}^+.$$

We have the following lemma:

**Lemma 2.1.** *Assume that conditions $(f_1)$ and $(M_1)$ hold. Then, any bounded Palais–Smale sequence of $I$ has a strongly convergent subsequence.*

PROOF: Let $(u_n)$ be a bounded Palais–Smale sequence for $I$. Thus, passing to a subsequence if necessary, we have

$$\|u_n\|^p \to t_0 \tag{2.1}$$

and there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. From $(f_1)$, the Lebesgue Dominated Convergence Theorem and the Sobolev Imbedding, we see that

$$\int_\Omega f(x,u_n)u \to \int_\Omega f(x,u)u \text{ and } \int_\Omega f(x,u_n)u_n \to \int_\Omega f(x,u)u.$$

Let us now consider the sequence

$$P_n = I'(u_n)u_n + \int_\Omega f(x,u_n)u_n - I'(u_n)u - \int_\Omega f(x,u_n)u.$$

We have that $P_n \to 0$ and

$$P_n = \left[M\left(\|u_n\|^p\right)\right]^{p-1}\|u_n\|^p - \left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u.$$

Moreover, from (2.1), we get $M\left(\|u_n\|^p\right) \to M(t_0)$ and from the weak convergence, we have

$$-\left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_n + \left[M\left(\|u_n\|\right)\right]^{p-1}\|u\|^p = o_n(1).$$

Hence,

$$\begin{aligned} o_n(1) + P_n = {} & \left[M\left(\|u_n\|^p\right)\right]^{p-1}\|u_n\|^p - \left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u \\ & - \left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_n + \left[M\left(\|u_n\|^p\right)\right]^{p-1}\|u\|^p. \end{aligned}$$

Consequently,

$$o_n(1) + P_n = \left[M\left(\|u_n\|^p\right)\right]^{p-1}\int_\Omega \left\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\right\rangle.$$

Using the standard inequality in $\mathbb{R}^N$ given by

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x-y\right\rangle \geqslant C_p|x-y|^p \quad if \quad p \geqslant 2$$

or

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x-y\right\rangle \geqslant \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}} \quad if \quad 2 > p > 1,$$

where $\langle .,. \rangle$ is the Euclidian inner product in $\mathbb{R}^N$ (see appendix or [**12**]) and from $(M_1)$, we obtain

$$o_n(1) + P_n \geqslant m_0^{p-1}C_p\int_\Omega |\nabla u_n - \nabla u|^p.$$

Thus, we conclude that $\|u_n - u\| \to 0$ in $W_0^{1,p}(\Omega)$. ∎

Now, let us show a basic existence result as a motivation to our main theorem. Here, we use the version due to Willem for the Mountain Pass Theorem (see [**15**, p. 12]).

**LEMMA 2.2.** *Let $X$ be a Banach space, $I \in C^1(X,\mathbb{R})$ with $I(0)=0$. Suppose that:*

($H_1$) *There exists $\alpha, r > 0$ such that $I(u) \geqslant \alpha > 0$ for all $u \in X$ with $\|u\| = r$*

($H_2$) *There exists $e \in X$ such that $\|e\| > r$ and $I(e) < 0$.*

*Then there exists a sequence $(u_n) \subset X$ such that*

$$I(u_n) \longrightarrow c \text{ and } I'(u_n) \longrightarrow 0 \text{ in } X'$$

*where*

$$0 < c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} I\big(\gamma(t)\big)$$

*and*

$$\Gamma = \Big\{\gamma \in C\big([0,1], X\big) \ : \gamma(0) = 0, \gamma(1) = e\Big\}.$$

**THEOREM 2.3.** *Assume that ($f_1$) and ($M_1$) hold. Furthermore, let us suppose that*

$$(f_2) \qquad 0 < \mu F(x,t) \leqslant f(x,t)t \text{ for all } t > 0.$$

*for some $\mu \in \mathbb{R}$ with $p < \mu < q$. Then, if*

$$(2.2) \qquad \widehat{M}(t) \geqslant \big[M(t)\big]^{p-1} t \text{ for all } t \geqslant 0,$$

*problem ($P$) has a positive solution.*

PROOF: Note that $I(0) = 0$ and using the condition (2.2), we obtain

$$\begin{aligned} I(u) &= \frac{1}{p}\widehat{M}(\|u\|^p) - \int_\Omega F(x,u) \\ &\geqslant \frac{1}{p}\Big[M\big(\|u\|^p\big)\Big]^{p-1}\|u\|^p - \int_\Omega F(x,u). \end{aligned}$$

From ($f_1$) and ($M_1$), there exists $r, \alpha > 0$ such that $I(u) \geqslant \alpha > 0$, for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = r$. From ($f_2$), there exists $u \in W_0^{1,p}(\Omega)$ such that $I(u) < 0$. By Lemma 2.2, we find a Palais–Smale sequence $(u_n) \subset W_0^{1,p}(\Omega)$, for the functional $I$. We claim that such a sequence $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Indeed, using ($f_2$) and (2.2) again, we have

$$\begin{aligned} C + \|u_n\| &\geqslant I(u_n) - \frac{1}{\mu} I'(u_n)u_n \\ &= \frac{1}{p}\widehat{M}(\|u_n\|^p) - \frac{1}{\mu}\Big[M\big(\|u_n\|^p\big)\Big]^{p-1}\|u_n\|^p + \int_\Omega \Big[\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\Big] \\ &\geqslant \Big(\frac{1}{p} - \frac{1}{\mu}\Big)\Big[M\big(\|u_n\|^p\big)\Big]^{p-1}\|u_n\|^p \geqslant C\|u_n\|^p \end{aligned}$$

with $C > 0$. Hence, $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ and from lemma 2.1, there exists $u \in W_0^{1,p}(\Omega)$ such that $I(u) = c_M > 0$, where

$$c_M = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big) \text{ and } \Gamma = \Big\{\gamma \in C\big([0,1]\big) : \gamma(0) = 0 \text{ and } I\big(\gamma(1)\big) < 0\Big\}.$$

Using $u^-$ as a test function, the condition $(f_1)$ and the maximum principle, we obtain $u > 0$. Thus, we conclude the proof of Theorem 2.3. ∎

REMARK 2.4. As we can see in [2], to the original meaning of $M$, in the Kirchhoff equation

$$u_{tt} - M\big(\|u\|^2\big)\Delta u = f(x,u),$$

it should be an increasing function. Then

$$\widehat{M}(u) < \int_0^u M(u)ds = M(u)u\ , \quad for \ \ all \ \ u > 0$$

and therefore, condition (2.2) cannot be satisfied.

In what follows, we consider the existence of positive solutions of $(P)$ where $M$ may be increasing. To this end, we first suppose that $M$ is bounded. More precisely, we assume that there exists $m_1 \geqslant m_0$ and $t_0 > 0$ such that

$$M(t) = m_1 \text{ for all } t \geqslant t_0. \tag{2.3}$$

THEOREM 2.5. *Suppose that $f$ satisfies $(f_1)$ and $(f_2)$. Assume, in addition, that $M$ is a function satisfying $(M_1)$ and (2.3) with*

$$\Big(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu}\Big) > 0. \tag{2.4}$$

*Then, problem $(P)$ has a positive solution.*

PROOF: We argue as in Theorem 2.3 to show the functional $I$ has a nonzero critical point. From $(M_1)$ and (2.3), we see that

$$\widehat{M}(t) \geqslant m_0^{p-1}t \text{ for all } t \geqslant 0 \tag{2.5}$$

and

$$\widehat{M}(t) \leqslant m_1^{p-1}t + m_2 \text{ for all } t \geqslant t_0 \tag{2.6}$$

where

$$m_2 = \left|\int_0^{t_0} \big[M(s)\big]^{p-1} ds - m_1^{p-1}t_0\right|.$$

Using standard arguments, we infer that $I$ satisfies

$$I(u) \geqslant C\|u\|^p - C\|u\|^q$$

for all $u \in W_0^{1,p}(\Omega)$, where here and elsewhere we may use the same letter $C$ to indicate (possibly different) positive constants. If $\phi \geqslant 0$ is a nonzero function, we get from $(f_2)$ and (2.6) that

$$I(t\phi) \leqslant \frac{t^p m_1^{p-1}}{p}\|\phi\|^p - t^\mu C\|\phi\|^\mu + C < 0 \quad (t > 0 \text{ large}).$$

Thus, from Lemma 2.2, there exists a $(PS)_c$ sequence $(u_n) \subset W_0^{1,p}(\Omega)$ for $I$. We claim that $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Indeed, assume, by contradiction, that, up to a subsequence, $\|u_n\| \to +\infty$. Thus $M(\|u_n\|) = m_1$, if $n$ is large enough, and by (2.5) and $(f_2)$, we have

$$\begin{aligned} C + \|u_n\| &\geqslant I(u_n) - \frac{1}{\mu} I'(u_n)u_n \\ &\geqslant \Big(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu}\Big)\|u_n\|^p. \end{aligned}$$

Using (2.4), we conclude that $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, which contradicts $\|u_n\| \to +\infty$. ∎

Our goal is to extend Theorem 2.5 to a large class of $M$, including the increasing linear functions. This is done by using truncation arguments and a priori estimates obtained via relations between minimax levels $c_M$ and $c_0$, related to functional $I_0$ associated to the problem

$$(P_0) \qquad \begin{cases} -\Delta_p u = \dfrac{1}{m_1^{p-1}} f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

that is,

$$I_0(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{m_1^{p-1}}\int_\Omega F(x,u).$$

Next we prove a lemma that establishes a relation between the $W_0^{1,p}(\Omega)$ norm of the solutions of problem $(P)$ and $M(\|u\|^p)$.

**Lemma 2.6.** *Let $u$ be the solution of $(P)$ obtained in Theorem 2.5. Then, there exist $\widetilde{C} > 0$ and $\theta > 0$ independent on $M$, such that*

$$\|u\| \leqslant \widetilde{C} \text{ and } \|u\|^p \leqslant \big[M(\|u\|^p)\big]^{1-p}\,\theta.$$

Proof: If $\|u\| < t_0$, we choose $\widetilde{C} = t_0$. If $\|u\|^p \geqslant t_0$, we have $M(\|u\|^p) = m_1$ and

$$c_M = I(u) - \frac{1}{\mu} I'(u)u \geqslant \Big(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu}\Big)\|u\|^p + \int_\Omega \Big[\frac{1}{\mu} f(x,u)u - F(x,u)\Big].$$

By $(f_2)$ again

$$(2.7) \qquad c_M \geqslant I(u) - \frac{1}{\mu} I'(u)u \geqslant \Big(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu}\Big)\|u\|^p.$$

Moreover, by (2.3) and (2.6), for all $u \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} I(u) &= \frac{1}{p}\widehat{M}(\|u\|^p) - \int_\Omega F(x,u) \leqslant \frac{1}{p} m_1^{p-1}\|u\|^p - \int_\Omega F(x,u) + m_2 \\ &= m_1^{p-1}\Big[I_0(u) + \frac{m_2}{m_1^{p-1}}\Big]. \end{aligned}$$

Thus, we conclude that

$$c_M \leqslant m_1^{p-1} c_0 + \frac{m_2}{m_1^{p-1}}. \tag{2.8}$$

By (2.7) and (2.8), we get

$$\|u\|^p \leqslant \left(\frac{p\mu}{m_0^{p-1}\mu - m_1^{p-1}p}\right)\left(m_1^{p-1}c_0 + \frac{m_2}{m_1^{p-1}}\right) = \widetilde{C}^p.$$

By $(f_1)$

$$\left[M\left(\|u\|^p\right)\right]^{p-1}\|u\|^p = \int_\Omega f(x,u)u \leqslant C\widetilde{C}^q.$$

Hence

$$\|u\|^p \leqslant \left[M\left(\|u\|^p\right)\right]^{1-p}\theta,$$

where $\theta = C\widetilde{C}^q$. ∎

**Theorem 2.7.** *Suppose that $f$ satisfies $(f_1)$and $(f_2)$. Assume, in addition, that $M$ satisfies $(M_1)$ and there exists $k > 0$ such that*

$$[M(k)]^{p-1} < \mu\frac{m_0^{p-1}}{p} \tag{$M_2$}$$

*and*

$$\left[M(k)\right]^{1-p} \leqslant \frac{k}{\theta}, \tag{$M_3$}$$

*where $\theta$ was given in Lemma 2.5. Then, problem $(P)$ has a positive solution.*

Proof: Let us define the truncated function

$$M_k(t) = \begin{cases} M(t) & \text{if } t \leqslant k \\ M(k) & \text{if } t > k. \end{cases}$$

Then, the assumption $(M_2)$ imply that $M_k$ satisfies (2.4) with $m_1 = M(k)$. We can apply Theorem 2.5 to obtain a solution $u_k > 0$ of the truncated problem

$$\begin{cases} -\left[M_k\left(\|u\|^p\right)\right]^{p-1}\Delta_p u = f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

From Lemma 2.6, we know that

$$\|u_k\|^p \leqslant \left[M_k\left(\|u\|^p\right)\right]^{1-p}\theta.$$

This implies that if $\|u_k\|^p > k$, so

$$k < \left[M(k)\right]^{1-p}\theta,$$

which contradicts $(M_3)$. Therefore, $\|u_k\|^p \leqslant k$, which shows that $u_k$ is, in fact, a positive solution of the (nontruncated) problem $(P)$. ∎

## 3. Case critical/supercritical

First of all, we have to note that because $f(x,t)+\lambda|t|^{s-2}t$ has a supercritical growth we can not use directly the variational techniques, by virtue of the lack of compactness of the Sobolev immersions. So, we construct a suitable truncation of $f(x,t)+\lambda|t|^{s-2}t$ in order to use variational methods or, more precisely, the Mountain Pass Theorem. This truncation was used by Rabinowitz [**14**] (see [**3, 6**]).

Let $K>0$ be a real number, whose precise value will be fixed later, and consider the function $g_K:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ given by

$$g_K(x,t)=\begin{cases}0 \quad \text{if} \quad t<0\\ f(x,t)+\lambda t^{s-1} \quad \text{if} \quad 0\leqslant t\leqslant K\\ f(x,t)+\lambda K^{s-q}t^{q-1} \quad \text{if} \quad t\geqslant K.\end{cases}$$

We study the truncated problem associated to $g_K$

$$(T)_\lambda \qquad \begin{cases}-\left[M\left(\|u\|^p\right)\right]^{p-1}\Delta_p u=g_K(x,u) \text{ in } \Omega\\ u=0 \text{ on } \partial\Omega.\end{cases}$$

Such a function enjoys the following conditions:

$$(g_{K,1}) \qquad \left|g_K(x,t)\right|\leqslant (C+\lambda K^{s-q})t^{q-1}$$

for all $x\in\Omega$, for all $t\in\mathbb{R}$, where $C>0$ and $p<q<p^*=(pN)/(N-p)$ and

$$(g_{K,2}) \qquad 0<\mu G_K(x,t)\leqslant g_K(x,t)t,$$

for all $x\in\Omega$, for all $t>0$ and where $G_K(x,t)=\displaystyle\int_0^t g_K(x,\xi)d\xi$. Assuming $(M_1)-(M_2)-(M_3)$, by Theorem 2.7, we have a positive solution $u_\lambda$ of $(T)_\lambda$, such that $I_\lambda(u_\lambda)=c_\lambda$, where $c_\lambda$ is the Mountain Pass level associated to the functional

$$I_\lambda(u_\lambda)=\frac{1}{p}\widehat{M}(\|u_\lambda\|^2)-\int_\Omega G_K(x,u_\lambda)$$

which is related to the problem $(T)_\lambda$, where $\widehat{M}(t)=\displaystyle\int_0^t M(s)ds$. Furthermore,

$$(3.1) \qquad I_\lambda(u_\lambda)-\frac{1}{\mu}I'_\lambda(u_\lambda)u_\lambda\geqslant\left(\frac{m_0^{p-1}}{p}-\frac{M(k)^{p-1}}{\mu}\right)\|u_\lambda\|^p + \int_\Omega\left[\frac{1}{\mu}g_K(x,u_\lambda)u_\lambda-G_K(x,u_\lambda)\right].$$

To prove the main result of this section, we need the following estimate.

**Lemma 3.1.** *If $u_\lambda$ is a solution (positive) of problem $(T)_\lambda$, then $\|u_\lambda\|\leqslant\overline{C}$ for all $\lambda\geqslant 0$, where $\overline{C}>0$ is a constant does not depend on $\lambda$.*

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PROOF: Since $G_K(x,t) \geqslant F(x,t)$ for all $x \in \Omega$ and for all $t \geqslant 0$, one has $c_\lambda \leqslant c_M$, where $c_M$ is the Mountain Pass level related to the functional $I$. Furthermore

$$c_M \geqslant c_\lambda = I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{\mu} I'_\lambda(u_\lambda)u_\lambda$$

and from (3.1)

$$c_M \geqslant \Big(\frac{m_0^{p-1}}{p} - \frac{M(k)^{p-1}}{\mu}\Big)\|u_\lambda\|^p + \int_\Omega \Big[\frac{1}{\mu} g_K(x,u_\lambda)u_\lambda - G_K(x,u_\lambda)\Big].$$

From $(g_{K,2})$, we get

$$\|u_\lambda\| \leqslant \Big(\frac{p\mu^{p-1}c_M}{\mu m_0^{p-1} - pM(k)^{p-1}}\Big)^{1/p} := \overline{C},$$

for all $\lambda \geqslant 0$. ∎

Next, we are going to use the Moser iteration method [**11**](see [**3, 6**]).

**THEOREM 3.2.** *Let us suppose that the function $M$ satisfies $(M_1)-(M_2)-(M_3)$ and $f$ satisfies $(f_1)$-$(f_2)$. Then there exists $\lambda_0 > 0$ such that problem $(P)_\lambda$ possesses a positive solution for each $\lambda \in [0, \lambda_0]$.*

PROOF: Let $u_\lambda$ be a solution of problem $(T)_\lambda$. We shall show that there is $K_0$ such that for all $K > K_0$ there exists a corresponding $\lambda_0$ for which

$$|u_\lambda|_{L^\infty(\Omega)} \leqslant K \text{ for all } \lambda \in [0, \lambda_0].$$

This is the case one has $g_K(x,u_\lambda) = f(x,u_\lambda) + \lambda|u_\lambda|^{s-1}$ and so $u_\lambda$ is a solution of problem $(P_\lambda)$, for all $\lambda \in [0, \lambda_0]$.

For the sake of simplicity, we shall use the following notation:

$$u_\lambda := u$$

For $L > 0$, let us define the following functions

$$u_L = \begin{cases} u & \text{if} \quad u \leqslant L \\ L & \text{if} \quad u > L, \end{cases}$$

$$z_L = u_L^{p(\beta-1)}u \quad \text{and} \quad w_L = uu_L^{\beta-1}$$

where $\beta > 1$ will be fixed later. Let us use $z_L$ as a test function, that is,

$$\Big[M\big(\|u\|^p\big)\Big]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla z_L = \int_\Omega g_K(x,u)z_L$$

which implies

$$M\big(\|u\|^p\big)^{p-1} \int_\Omega u_L^{p(\beta-1)}|\nabla u|^p = -p(\beta-1)M\big(\|u\|^p\big)^{p-1} \int_\Omega u_L^{p\beta-3}u|\nabla u|^{p-2}\nabla u \nabla u_L + \int_\Omega g_K(x,u)uu_L^{p(\beta-1)}.$$

From the definition of $u_L$, we have

$$p(\beta-1)M\big(\|u\|^p\big)^{p-1}\int_\Omega u_L^{p\beta-3}u|\nabla u|^{p-2}\nabla u\nabla u_L \geqslant 0$$

and using $(g_{K,1})$ and $(M_1)$, we have

$$\int_\Omega u_L^{p(\beta-1)}|\nabla u|^p \leqslant (C+\lambda K^{s-q})\frac{1}{m_0^{p-1}}\int_\Omega u^q u_L^{p(\beta-1)},$$

that is,

$$\int_\Omega u_L^{p(\beta-1)}|\nabla u|^p \leqslant C_{\lambda,K}\int_\Omega u^q u_L^{p(\beta-1)}, \tag{3.2}$$

where $C_{\lambda,K} = \big(C+\lambda K^{s-q}\big)1/m_0^{p-1}$.

On the other hand, from the continuous Sobolev immersions, one gets

$$|w_L|_{p^*}^p \leqslant C_1\int_\Omega |\nabla w_L|^p = C_1\int_\Omega \big|\nabla(uu_L^{\beta-1})\big|^p.$$

Consequently

$$|w_L|_{p^*}^p \leqslant C_1\int_\Omega u_L^{p(\beta-1)}|\nabla u|^p + C_1(\beta-1)^p\int_\Omega u_L^{p(\beta-2)}u^p|\nabla u_L|^p$$

which gives

$$|w_L|_{p^*}^p \leqslant C_2\beta^p\int_\Omega u_L^{p(\beta-1)}|\nabla u|^p. \tag{3.3}$$

From (3.2) and (3.3), we get

$$|w_L|_{p^*}^p \leqslant C_2\beta^p C_{\lambda,K}\int_\Omega u^q u_L^{p(\beta-1)}$$

and hence,

$$|w_L|_{p^*}^p \leqslant C_2\beta^p C_{\lambda,K}\int_\Omega u^{q-p}(uu_L^{\beta-1})^p = C_2\beta^p C_{\lambda,K}\int_\Omega u^{q-p}w_L^p.$$

We now use Hölder inequality, with exponents $p^*/[q-p]$ and $p^*/[p^*-(q-p)]$, to obtain

$$|w_L|_{p^*}^p \leqslant C_2\beta^p C_{\lambda,K}\left(\int_\Omega u^{p^*}\right)^{(q-p)/p^*}\left(\int_\Omega w_L^{pp^*/\left[p^*-(q-p)\right]}\right)^{[p^*-(q-p)]/p^*}, \tag{3.4}$$

where $p < (pp^*)/\big(p^*-(q-p)\big) < p^*$. Considering the continuous Sobolev immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $p-1 \leqslant q \leqslant p^*$, we obtain

$$|w_L|_{p^*}^p \leqslant C_2'\beta^p C_{\lambda,K}\|u\|^{q-p}|w_L|_{\alpha^*}^p.$$

where $\alpha^* = pp^*/(p^* - (q-p))$.

Using Lemma 3.1

$$|w_L|_{p^*}^p \leqslant C_3\beta^p C_{\lambda,K}\overline{C}^{q-p}|w_L|_{\alpha^*}^p \tag{3.5}$$

Since $w_L = uu_L^{\beta-1} \leqslant u^\beta$ and supposing that $u^\beta \in L^{\alpha^*}(\Omega)$, we have from (3.5) that

$$\left(\int_\Omega |\, uu_L^{\beta-1}\,|^{p^*}\right)^{p/p^*} \leqslant C_4\beta^p C_{\lambda,K}\left(\int_\Omega u^{\beta\alpha^*}\right)^{p/\alpha^*} < +\infty.$$

We now apply Fatou's lemma to the variable $L$ to obtain

$$|u|_{\beta p^*}^{p\beta} \leqslant C_4 C_{\lambda,K}\beta^p|u|_{\beta\alpha^*}^{p\beta}.$$

and so

$$|u|_{\beta p^*} \leqslant (C_4C_{\lambda,K})^{1/\beta p}\beta^{1/\beta}|u|_{\beta\alpha^*}. \tag{3.6}$$

Furthermore, by considering $\chi = p^*/\alpha^*$, we have $p^* = \chi\alpha^*$ and $\beta\chi\alpha^* = \beta p^*$, for all $\beta > 1$ so that $u^\beta \in L^{\alpha^*}(\Omega)$. Let us consider two cases:

First Case. First we consider $\beta = p^*/\alpha^*$ and note that

$$u^\beta \in L^{\alpha^*}(\Omega).$$

Hence, from the Sobolev immersions, Lemma 3.1 and relation (3.6), we get

$$|u|_{(p^*)^2/\alpha^*} \leqslant (C_4C_{\lambda,K})^{1/p\beta}\beta^{1/\beta}\overline{C}C_5.$$

and so

$$|u|_{\chi^2\alpha^*} \leqslant C_6(C_4C_{\lambda,K})^{1/\chi p}\chi^{1/\chi} \tag{3.7}$$

Second Case. We now consider $\beta = (p^*/\alpha^*)^2$, and note again that

$$u^\beta \in L^{\alpha^*}(\Omega).$$

From inequality in (3.6) we obtain,

$$|u|_{(p^*)^3/(\alpha^*)^2} \leqslant C_6(C_4C_{\lambda,K})^{1/\beta p}\beta^{1/\beta}|u|_{(p^*)^2/\alpha^*},$$

which implies

$$|u|_{\chi^3\alpha^*} \leqslant C_6(C_4C_{\lambda,K})^{1/\chi^2 p}(\chi^2)^{1/\chi^2}|u|_{\chi^2\alpha^*}$$

or,

$$|u|_{\chi^3\alpha^*} \leqslant C_7(C_4C_{\lambda,K})^{1/\chi^2 p+1/\chi p}(\chi^2)^{p/\chi^2+1/\chi}\overline{C}.$$

An iterative process leads to

$$|u|_{\chi^{(m+1)}\alpha^*} \leqslant C_8(C_4C_{\lambda,K})^{\sum_{i=1}^m \frac{\chi^{-i}}{p}}\chi^{\sum_{i=1}^m i\chi^{-i}}\overline{C}.$$

Taking limit as $m \to \infty$, we obtain

$$|u|_{L^\infty(\Omega)} \leqslant C_8(C_4C_{\lambda,K})^{\sigma_1}\chi^{\sigma_2}\overline{C}.$$

where $\sigma_1 = \sum_{i=1}^{\infty}(\chi^{-i})/p$ and $\sigma_2 = \sum_{i=1}^{\infty} i\chi^{-i}$. In order to choose $\lambda_0$, we consider the inequality

$$C_8(C_4C_{\lambda,K})^{\sigma^1}\chi^{\sigma^2}\overline{C} = C_8\Big[C_4(C+\lambda K^{s-q})\frac{1}{m_0^{p-1}}\Big]^{\sigma_1}\chi^{\sigma_2}\overline{C} \leqslant K,$$

from which

$$(C+\lambda K^{s-q})^{\sigma_1} \leqslant \frac{Km_0^{(p-1)\sigma_1}}{C_4^{\sigma_1}\chi^{\sigma_2}\overline{C}C_8}.$$

Choosing $\lambda_0$ to satisfy the inequality

$$\lambda_0 \leqslant \Big[\frac{K^{1/\sigma_1}m_0^{p-1}}{C_4\chi^{\sigma_2/\sigma_1}C_8^{1/\sigma_1}\overline{C}^{1/\sigma_1}} - C\Big]\frac{1}{K^{s-q}}$$

and fixing $K$ such that

$$\Big[\frac{K^{1/\sigma_1}m_0^{p-1}}{C_8^{1/\sigma^1}C_4\chi^{\sigma_2/\sigma_1}\overline{C}^{1/\sigma_1}} - C\Big] > 0,$$

we obtain

$$|u_\lambda|_{L^\infty(\Omega)} \leqslant K \quad \forall\, \lambda \in [0,\lambda_0].$$

and the proof of the theorem is over. ∎

## Appendix

**Lemma 3.3.** *Let $x$, $y \in \mathbf{R}^N$ and let $\langle .,. \rangle$ be the standard inner product in $\mathbf{R}^N$. Then*

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x-y\big\rangle \geqslant C_p|x-y|^p \text{ if } p \geqslant 2$$

*or*

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x-y\big\rangle \geqslant \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}} \text{ if } 2 > p > 1.$$

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PROOF: By homogeneity we can assume that $|x| = 1$ and $|y| \leqslant 1$. Moreover by choosing a convenient basis in $\mathbb{R}^N$ we can assume

$$x = (1,0,0,\dots,0) \;\;,\;\; y = (y_1, y_2, 0, \dots, 0) \text{ and} \sqrt{y_1^2 + y_2^2} \leqslant 1.$$

(i) Case $2 > p > 1$. It is clear that the inequality is equivalent to the next one

$$\left\{\left(1 - \frac{y_1}{(y_1^2+y_2^2)^{(2-p)/2}}\right)(1-y_1) + \frac{y_2^2}{(y_1^2+y_2^2)^{(2-p)/2}}\right\}\frac{(1-\sqrt{y_1^2+y_2^2})^{2-p}}{(1-y_1)^2+y_2^2} \geqslant C$$

But

$$1 - \frac{y_1}{(\sqrt{y_1^2+y_2^2})^{2-p}} \geqslant 1 - \frac{y_1}{|y_1|^{2-p}} \geqslant (p-1)(1-y_1) \text{ if } 0 \leqslant y_1 \leqslant 1$$

or

$$1 - \frac{y_1}{(\sqrt{y_1^2+y_2^2})^{2-p}} \geqslant 1 - y_1 \geqslant (p-1)(1-y_1) \text{ if } y_1 \leqslant 0,$$

then

$$(p-1)\left\{(1-y_1)^2 + y_2^2\right\}\frac{(1+y_1+y_2)^{(2-p)/2}}{(1-y_1)^2+y_2} \geqslant p-1.$$

(ii) Case $p \geqslant 2$. The inequality is equivalent to prove

$$\frac{[1 - y_1(y_1^2+y_2^2)^{(p-2)/2}](1-y_1) + y_2^2(y_1^2+y_2^2)^{(p-2)/2}}{((1-y_1)^2+y_2^2)^{p/2}} \geqslant C.$$

Denoting $t = |y|/|x|$ and $s = \langle x, y\rangle / (|x||y|)$ then, we must show that the function

$$f(t,s) = \frac{1 - (t^{p-1}+t)s + t^p}{(1-2ts+t^2)^{p/2}}$$

is bounded from below. Direct calculation shows that fixed t, $\dfrac{\partial f}{\partial s} = 0$ if

$$1 - (t^{p-1}+t)s + t^p = \frac{t^{p-2}+1}{p}(1-2ts+t^2),$$

we have

$$f(t,s) = \frac{t^{p-2}+1}{p(1-2ts+t^2)^{(p-2)/2}} \geqslant \frac{1}{p}\min_{0\leqslant t\leqslant 1}\frac{t^{p-2}+1}{(t+1)^{p-2}} \geqslant \frac{1}{2p},$$

which concludes the proof of lemma. ∎

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