EXCLUSIVE SEMIGROUPS

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0. Introduction

First, we introduce the following condition (C1) for semigroups S. Let n, k be fixed positive integers such that $n > k$.

(C1) For any given sequence $x_1, x_2, \cdots, x_n$ of n elements of S, there exist k elements $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$, where $1 \leqq i_1 < i_2 \cdots < i_k \leqq n$, such that

$$x_1 x_2 \cdots x_n = x_1 \cdots \hat{x}_{i_1} \cdots \hat{x}_{i_2} \cdots \hat{x}_{i_k} \cdots x_n,$$

where

$$x_1 \cdots \hat{x}_{i_1} \cdots \hat{x}_{i_2} \cdots \hat{x}_{i_k} \cdots x_n$$

means

$$x_1 \cdots x_{i_1 - 1} x_{i_1 + 1} \cdots x_{i_2 - 1} x_{i_2 + 1} \cdots x_{i_k - 1} x_{i_k + 1} \cdots x_n.$$

For example, if $n = 2$, $k = 1$ then the condition (C1) is "$x_1 x_2 = \hat{x}_1 x_2$ or $x_1 \hat{x}_2$ (that is, $x_1 x_2 = x_1$ or x_2) for any given x_1, x_2". The structure of semigroups satisfying the condition "$x_1 x_2 = x_1$ or x_2 for any given x_1, x_2" was completely determined by the author [8].

If $n = 3$ and $k = 2$, then the condition (C1) means "$x_1 x_2 x_3 = \hat{x}_1 \hat{x}_2 x_3$, $\hat{x}_1 x_2 \hat{x}_3$ or $x_1 \hat{x}_2 \hat{x}_3$ (that is, $x_1 x_2 x_3 = x_1$, x_2 or x_3) for any given x_1, x_2, x_3" Hewitt and Zuckerman [3] described explicitly the structure of semigroups satisfying the condition "$x_1 x_2 x_3 = x_1$, x_2 or x_3 for any given x_1, x_2, x_3".

Now, let us consider the case $n = 3$, $k = 1$. Then, the condition (C1) means the following:

(C2) $\qquad x_1 x_2 x_3 = x_1 x_2$, $x_2 x_3$ or $x_1 x_3$ for any given x_1, x_2, x_3.

The problem of describing all the semigroups satisfying the condition (C2) is just the problem proposed by Schein in the Semigroup Forum, Vol. 1, p. 91. We shall call a semigroup satisfying condition (C2) an *exclusive semigroup*. The structure of commutative exclusive semigroups has been completely deter-

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mined by Tamura [6], and recently the author has heard that O'Carroll and Schein [5] have completely solved the problem of constructing all exclusive bands. In this paper, we deal with exclusive semigroups which are not necessarily commutative and not necessarily bands. The paper is divided into four sections. In the first two sections, the structure of exclusive semigroups whose non-zero idempotents are primitive will be clarified. Next, we shall investigate a certain class of exclusive semigroups called "exclusive homobands". In the final section we shall deal with medial exclusive homobands and show how we can construct them.

1. Exclusive (R)-semigroups

Let S be a semigroup with idempotents, for example, an exclusive semigroup. We shall say that S is an *(R)-semigroup* if it satisfies the following:

(1.1) *For any idempotents e and f, $ef = fe$ implies $e = f$.*

If S is an (R)-semigroup with zero 0, then S has no idempotent except 0. Hence, in this case, S is a unipotent semigroup.[1] On the other hand, if S has no zero element then S is clearly a primitive semigroup without zero.[2] Thus, any exclusive (R)-semigroup must be either a *unipotent exclusive semigroup* or a *primitive exclusive semigroup without zero.*

By using this result, we have

THEOREM 1.1. *Let S be an exclusive semigroup, and E the set of idempotents of S. Then, the following conditions are equivalent:*

(1) *For any $e, f \in E$, $ef = fe$ implies $e = f$.*

(2) *E is a rectangular band.*

(3) *S is a unipotent exclusive semigroup or a primitive exclusive semigroup without zero.*

PROOF. (1) $\Rightarrow$ (2): Let e, f be idempotents of S. Then, $efeefe = (ef)e(fe) = efe$. Hence, $efe \in E$. Now, $(efe)e = e(efe) = efe$. Therefore, $efe = e$. This means that E is a rectangular band.

(2) $\Rightarrow$ (3): Suppose that S has a zero element 0. Then for any $e \in E$, $e0e = 0$. Since E is a rectangular band, we have also $e0e = e$. Hence, $e = 0$. Therefore, in this case, S is unipotent. Next, it is obvious that S is primitive if S has no zero element.

(3) $\Rightarrow$ (1): If S is a unipotent exclusive semigroup, then S clearly satisfies (1). Therefore, assume that S is a primitive exclusive semigroup without zero. By the

(1) A semigroup is said to be unipotent if it has just one idempotent.

(2) Let S be a semigroup with idempotents, and E the set of idempotents of S. Then a non-zero element $e \in E$ is called primitive if $ef = fe = f$, $f \in E$, implies $f = 0$ or $e = f$. If every non-zero element in E is primitive, then S is called a primitive semigroup.

same method as in (1) $\Rightarrow$ (2), we can prove that efe is an idempotent for any $e, f \in E$. Let g, h be elements of E such that $gh = hg$. Then since $(ghg)g = g(ghg) = ghg$ and since S is primitive, we have $ghg = g$. Hence $gh = hg = g$. Since S is primitive, $gh = hg = g$ implies $g = h$.

Now, let S be an exclusive semigroup and E the set of idempotents of S. For each $e \in E$, let

$$S_e = \{x \in S: x^2 = e\}.$$

Then

THEOREM 1.2. (1) *S_e is a unipotent exclusive subsemigroup of S, and* (2) *$S = \sum \{S_e: e \in E\}$ (the disjoint sum of all S_e). Hence, S is a disjoint sum of unipotent exclusive semigroups.*

PROOF. Let x, y be elements of S_e. Then $x^2 = y^2 = e$. We shall show that $xy \in S_e$. Since S is exclusive,

$$(xy)^2 = xyxy = xyx,\ yxy \text{ or } xy.$$

If $(xy)^2 = xyx$, then

$$xyxy = (xyx)y = xyxy^2 = xyx^3 = xyx^2 = xyy^2 = xy^3 = xy^2 = x^3 = x^2 = e.$$

Therefore, $xy \in S_e$. Similarly if $(xy)^2 = yxy$, then we have $xy \in S_e$. Assume that $(xy)^2 \neq xyx,\ yxy$. Then, $(xy)^2 = xy$ and hence xy is an idempotent. Put $xy = f$. Then, $fx = xyx = f$, e or yx. If $fx = e$, then $fxy = ey$ and hence

$$f = ey = y^3 = y^2 = e.$$

Therefore $(xy)^2 = e$, and we have $xy \in S_e$. If $fx = f$, then

$$f = fxy = fy = xyy = xe = x^3 = x^2 = e.$$

Therefore, $(xy)^2 = e$ and hence $xy \in S_e$. Finally let $fx = yx$. Then $xyx = yx$, and hence $x^2yx = xyx$. Therefore,

$$xyx = eyx = y^3x = ex = x^3 = e.$$

Thus we have $fx = e$, and by the result above $xy \in S_e$. Since $xy \in S_e$ for any $x, y \in S_e$, S_e is a subsemigroup. The second part (2) is obvious.

In particular, if S is an exclusive (R)-semigroup then

THEOREM 1.3. (1) *E is an ideal of S.*

(2) *S_e is a unipotent exclusive semigroup. In particular, S_e is a null semigroup if there exists $f \in E$ such that $ef \neq e, f$.*

(3) $S = \sum \{S_e: e \in E\}$.

(4) *$S_f S_h = fh$ if f, $h \in E$ and $fh \neq f$, h.*

PROOF. First, we prove part (4). Let x and y be elements of S_f and S_h, res-

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pectively. Since $fyy = fy$ or y^2 and since $fh \neq f$, h, we have $fh = fy$. Hence,

$$fh = fy = x^2y = x^2 \text{ or } xy.$$

Since $fh \neq f$, we have $fh = xy$. Thus, part (4) is proved. Part (2), except for the latter half, is obvious by Theorem 1.2. Let $e, f \in E$ be such that $ef \neq e, f$, and let $x, y \in S_e$. Then, $xfy = xf, fy$ or xy. Since, by part (4), $xfy = efe$, $xf = ef$ and $fy = fe$, we have $xfy \neq xf$ and $xfy \neq fy$. Hence $xfy = xy$, and hence

$$e = efe = xfy = xy.$$

Therefore, S_e is a null semigroup. Part (3) is obvious.

Next, we prove part (1). By Theorem 1.1, E is a rectangular band. Let $x \in S$ and $e \in E$, and put $x^2 = f$. We shall show that xe is an idempotent. First, we have

$$xexe = \{xe \text{ or } xex\} = xe,\ ex \text{ or } x^2.$$

If $xexe = x^2$, then $xexe = xe(xexe)xe = xex^3e = xefe = xe$.

If $xexe = ex$, then $xexe = xe(xexe)xe = xex^2e = xefe = xe$. Hence, xe is an idempotent. Similarly, we can prove that ex is an idempotent. Therefore, E is an ideal of S.

A band is said to be *purely rectangular* if it is rectangular but is neither a left zero semigroup nor a right zero semigroup. Under this definition if S is an exclusive semigroup in which the set E of idempotents is a purely rectangular band, then

THEOREM 1.4. (1) *Each S_e is a null semigroup.* (2) *$S_f S_h = fh$ for all $f, h \in E$.*
Accordingly, an exclusive (R)-semigroup whose idempotents E form a purely rectangular band is an inflation of the band E.

PROOF. First, we prove part (1). Since E is a purely rectangular band, for $e \in E$ there exists $f \in E$ such that $ef \neq e, f$. Therefore, by Theorem 1.3, S_e is a null semigroup. Next, part (2) is proved as follows. Let a, b be elements of S_f, S_h respectively. We need only show that $ab = fh$. If $fh \neq f$, h, then by Theorem 1.3 we obtain $ab = fh$. If $f = h$, then by part (1) we have $ab = f = fh$. Hence, we assume that $f \neq h$ and $fh = f$ or h. Suppose that $fh = f$. In this case, $hf = h$. Since E is purely rectangular, there exist $u, v \in E$ such that $hv \neq h, v$ and $fu \neq u, f$. Now, $auvb = auv$, uvb or ab. If $auv = auvb$, then $fuv = fuvb$ and hence $fuvh = fuv$. Since $fuvh = fh$ and $fuv = fv$, we have $fh = fv$. Then, $h = hfh = hv$. This contradicts $hv \neq h$. Similarly, if $auvb = uvb$ then we have a contradiction. Hence $auvb = ab$. Since $au = fu$ and $vb = vh$,

$$fh = fuvh = auvb = ab$$

holds. In the case $fh = h$, we can also prove the relation $ab = fh$ by the same method.

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Thus, the problem of determining the structure of exclusive (R)-semigroups is reduced to that of exclusive (R)-semigroups whose idempotents form a one-sided zero semigroup. First, let us study unipotent exclusive semigroups.

Let M be a set, and X a subset of M (X might be empty). Let A be a subset of $M \times M$ such that

(1.2)
(1) $A \not\ni (a, a)$ for all $a \in M$,
(2) if $A \ni (a_i, a_j)$ for all i, j such that $1 \leqq i < j \leqq n$ (n arbitrary) and if $A \ni (a_k, v)$ and $A \ni (v, a_{k+1})$ for an integer k such that $1 \leqq k \leqq n-1$, then $A \ni (a_t, v)$, (v, a_s) for all t, s such that $t \leqq k$ and $k+1 \leqq s$.

Then we have

THEOREM 1.5. *$S = A \cup X \cup \{0\}$ is a unipotent exclusive semigroup under the multiplication $\circ$ defined as follows:*

(1) *For all* $\alpha \in S$, $0 \circ \alpha = \alpha \circ 0 = 0$,

(2) *for* $a, b \in X$, $a \circ b = \begin{cases} (a,b), & \text{if } (a,b) \in A, \\ 0, & \text{otherwise}, \end{cases}$

(3) *for* $a \in X$ *and* $(b, c) \in A$,

$$a \circ (b,c) = \begin{cases} (a,c), & \text{if } (a,b),\ (a,c) \in A, \\ 0, & \text{otherwise}, \end{cases}$$

(4) *for* $a \in X$ *and* $(b, c) \in A$,

$$(b,c) \circ a = \begin{cases} (b,a), & \text{if } (b,a),\ (c,a) \in A, \\ 0, & \text{otherwise}, \end{cases}$$

(5) *for* (a, b), $(c, d) \in A$,

$$(a,b) \circ (c,d) = \begin{cases} (a,d), & \text{if } (a,c),\ (a,d),\ (b,c),\ (b,d) \in A, \\ 0, & \text{otherwise}. \end{cases}$$

PROOF. To prove that $S(\circ)$ is a semigroup, we have to show that $S(\circ)$ satisfies the associative law $a \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$. We shall check this equation only in the most complicated case where all α, β, γ are elements of A since in the other cases we can check the equation by similar methods.

Let

$$\alpha = (a,b) \in A, \quad \beta = (c,d) \in A \quad \text{and} \quad \gamma = (e,f) \in A.$$

Suppose that $(a,b) \circ ((c,d) \circ (e,f)) = 0$. Then, at least one of

$$(c,e),\ (c,f),\ (d,e),\ (d,f),\ (a,c),\ (a,f),\ (b,c),\ (b,f)$$

is not contained in A.

(*i*) *Case* $(c,e) \notin A$: If $((a,b) \circ (c,d)) \circ (e,f) \neq 0$, then

$$(a,c),\ (a,d),\ (b,c),\ (b,d)\ (a,e)\ (a,f),\ (d,e),\ (d,f) \in A.$$

Since (a,d), (a,e), (a,f), (d,e), (d,f), $(e,f) \in A$ and (a,c), $(c,d) \in A$, by the definition of A we have $(c,e) \in A$. Therefore,

$$((a,b) \circ (c,d)) \circ (e,f) = 0.$$

(ii) *Case* $(c,f) \notin A$. In this case, we can prove $((a,b) \circ (c,d)) \circ (e,f) = 0$ by the same method as in (i).

(iii) *Case* $(d,e) \notin A$ or $(d,f) \notin A$: If $((a,b) \circ (c,d)) \circ (e,f) \neq 0$, then

$$((a,b) \circ (c,d)) \circ (e,f) = (a,d) \circ (e,f) \neq 0$$

and hence $(d,e), (d,f) \in A$. Therefore,

$$((a,b) \circ (c,d)) \circ (e,f) = 0.$$

(iv) *Case* $(a,f) \notin A$: In this case, we can prove

$$((a,b) \circ (c,d)) \circ (e,f) = 0$$

by the same method as in (iii).

(v) *Case* $(a,c) \notin A$ or $(b,c) \notin A$: In this case, $(a,b) \circ (c,d) = 0$. Hence,

$$((a,b) \circ (c,d)) \circ (e,f) = 0.$$

(vi) *Case* $(b,f) \notin A$: If $((a,b) \circ (c,d)) \circ (e,f) \neq 0$, then $(a,d) \circ (e,f) \neq 0$. Hence

$$(a,d),\ (a,e),\ (a,f),\ (d,e),\ (d,f),\ (e,f) \in A$$

and (a,b), $(b,d) \in A$. Therefore, by the definition of A, $(b,f) \in A$. Thus,

$$((a,b) \circ (c,d)) \circ (e,f) = 0.$$

By (i)–(vi),

$$(a,b) \circ ((c,d) \circ (e,f)) = 0 \text{ implies } ((a,b) \circ (c,d)) \circ (e,f) = 0.$$

Conversely, by the same method we can prove that

$$((a,b) \circ (c,d)) \circ (e,f) = 0 \text{ implies } (a,b) \circ ((c,d) \circ (e,f)) = 0.$$

Next, let us consider the case where

$$(a,b) \circ ((c,d) \circ (e,f)) \neq 0 \text{ and } ((a,b) \circ (c,d)) \circ (e,f) \neq 0.$$

In this case,

$$(a,b) \circ ((c,d) \circ (e,f)) = (a,f) = ((a,b) \circ (c,d)) \circ (e,f).$$

Hence, in all cases the associativity equation

$$(a,b) \circ ((c,d) \circ (e,f)) = ((a,b) \circ (c,d)) \circ (e,f)$$

is satisfied.

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Next we must also check the exclusiveness of S. We check this also only in the most complicated cases where the three elements are contained in A, because in the other cases we can easily check it by simpler methods.

At first, let us consider the case $(a,b)\circ(c,d)\circ(e,f)=0$. Then, at least one of

$$(c,e),\ (c,f),\ (d,e),\ (d,f),\ (a,c),\ (a,f),\ (b,c),\ (b,f)$$

is not contained in A. For example, let $(c,e)\notin A$. Then $(c,d)\circ(e,f)=0$. Hence,

$$(a,b)\circ(c,d)\circ(e,f)=(c,d)\circ(e,f).$$

In the other cases, we can also prove by the same method that $(a,b)\circ(c,d)\circ(e,f)$ coincides with one of

$$(a,b)\circ(c,d),\ (a,b)\circ(e,f) \text{ and } (c,d)\circ(e,f).$$

Next, consider the case $(a,b)\circ(c,d)\circ(e,f)\neq 0$. In this case,

$$(a,b)\circ(c,d)\circ(e,f)=(a,f).$$

Now, $(a,b)\circ(c,d)\circ(e,f)=(a,b)\circ(c,f)$ since $(c,d)\circ(e,f)=(c,f)$. Since

$$(a,b),(a,c),(a,f),(b,c),(b,f),(c,f)\in A$$

and $(c,e),(e,f)\in A$, it follows that $(a,e),(b,e)\in A$. Therefore, $(a,b)\circ(e,f) = (a,f)$. Thus

$$(a,b)\circ(c,d)\circ(e,f)=(a,b)\circ(e,f).$$

We shall denote the unipotent exclusive semigroup $S=A\cup X\cup\{0\}$ of Theorem 1.5 by $F(X,A;M)$.

LEMMA 1.1. *In a unipotent exclusive semigroup,* $a_1a_2\cdots a_n\neq 0$ *implies* $a_ia_j\neq 0$ *for all* $i<j$.

PROOF. If $n=2$ or 3, then the assertion is clearly satisfied. Suppose that $n>3$, and then use mathematical induction with respect to n. Since

$$a_1(a_2\cdots a_{n-1})a_n\neq 0,$$

it follows that

$$a_1(a_2\cdots a_{n-1})\neq 0,\ a_1a_n\neq 0 \text{ and } (a_2\cdots a_{n-1})a_n\neq 0.$$

Hence, by the induction hypothesis, $a_ia_j\neq 0$ for all $i<j$.

THEOREM 1.6. *Any homomorphic image of* $F(X,A;M)$ *is a unipotent exclusive semigroup. Conversely, any unipotent exclusive semigroup can be obtained as a homomorphic image of some* $F(X,A;M)$.

PROOF. It is obvious that any homomorphic image of an exclusive semigroup is exclusive. Hence to prove the first part of the theorem it is sufficient to show that any homomorphic image of a unipotent exclusive semigroup is unipotent. Let S be a unipotent exclusive semigroup, and $\phi: S \to T$ a homomorphism of S onto an exclusive semigroup T. For the zero element 0 of S, 0ϕ is of course an idempotent of T. Suppose that

$$(x\phi)^2 = x\phi, \ x \in S.$$

Then $0\phi = x^2\phi = (x\phi)^2 = x\phi$. Hence, T has no idempotent except 0ϕ. Next, the second part can be proved as follows. Let S be a unipotent exclusive semigroup. Let $M = S$, and put

$$S \backslash S^2 = X \text{ and } \{(a,b): ab \neq 0 \text{ in } S, a, b \in M\} = A.$$

Then, A clearly satisfies (1) of the condition (1.2). Further, A satisfies the second condition (2). For, suppose that

$$A \ni (a_i, a_j) \ \ (i<j, \ i = 1,2,\cdots,n-1, \ j = 2,3,\cdots,n)$$

and (a_i, v), $(v, a_{i+1}) \in A$. Since $a_i v \neq 0$, $va_{i+1} \neq 0$, and $a_i a_{i+1} \neq 0$ it follows that $a_i v a_{i+1} \neq 0$ and $a_i a_{i+1} = a_i v a_{i+1}$. Let k, t be integers such that $1 \leqq t \leqq i$ and $i+1 \leqq k \leqq n$. Then

$$a_1 a_2 \cdots a_t \cdots a_i a_{i+1} \cdots a_k \cdots a_n \neq 0$$

since $a_s a_r \neq 0$ for all $s<r$. Therefore,

$$a_1 a_2 \cdots a_t \cdots a_i v a_{i+1} \cdots a_k \cdots a_n \neq 0$$

since $a_i a_{i+1} = a_i v a_{i+1}$. By Lemma 1.1, this implies that $a_t v \neq 0$ and $va_k \neq 0$, that is, $(a_t, v) \in A$ and $(v, a_k) \in A$. Now, we can consider $F(X, A; M)$. Define $\phi: F(X,A;M) \to S$ by $0\phi = 0$, $a\phi = a$ for $a \in X$, $(b,c)\phi = bc$ for $(b,c) \in A$. Then, it is easy to see that ϕ is a homomorphism of $F(X,A;M)$ onto S. For example, we can check $(\alpha\beta)\phi = (\alpha\phi)(\beta\phi)$ in the case α, $\beta \in A$ as follows: Let

(a,b), $(c,d) \in A$.

$$((a,b) \circ (c,d))\phi = \begin{cases} (a,d)\phi, & \text{if } (a,c), (a,d), (b,c), (b,d) \in A \\ 0\phi, & \text{otherwise.} \end{cases}$$

Consider the first case. In this case, $abcd \neq 0$ in S since ab, ac, ad, bc, bd, cd are all not 0. Hence $a(bc)d = ad$. Therefore

$$((a,b) \circ (c,d))\phi = (a,d)\phi = ad = abcd = ((a,b)\phi)((c,d)\phi).$$

Consider next the second case. If $abcd \neq 0$, then by Lemma 1.1 the elements ab, ac, ad, bc, bd, cd are all not 0. Hence

$$(a,c), \ (a,d), \ (b,c), \ (b,d) \in A.$$

This is a contradiction. Therefore $abcd = 0$ in S, and hence

$$((a,b)\circ(c,d))\phi = 0\phi = abcd = ((a,b)\phi))((c,d)\phi).$$

For the other cases, we can also prove $(\alpha\beta)\phi = (\alpha\phi)(\beta\phi)$ by simpler methods. Further, it is obvious that ϕ is onto.

Next, we investigate exclusive semigroups whose idempotents form a one-sided zero semigroup. Let S be an exclusive semigroup in which the set E of idempotents is a left zero subsemigroup. Then S is an (R)-semigroup, so E is an ideal of S, and we can consider the Rees factor semigroup S/E of S mod E. The semigroup $T = S/E$ is of course a unipotent exclusive semigroup, and accordingly S is an ideal extension of a left zero semigroup by a unipotent exclusive semigroup. From this point of view, we obtain the following theorem by slightly modifying Theorem 4.21 of [2]:

THEOREM 1.7. *Let E be a left zero semigroup, and T a unipotent exclusive semigroup. Let Λ be the left translation semigroup of E, and put $T\backslash 0 = T^*$ (where 0 is the zero element of T). Let $\{T_e\colon e \in E\}$ be a family of subsemigroups T_e of T (T_e might consist of only the single element 0) such that $T = \cup\{T_e\colon e \in E\}$, $T_e \cap T_f = 0$ and $T_eT_f = 0$ for $e \neq f$. Let θ be a partial anti-homomorphism of T^* into Λ such that (1) $\lambda_A\lambda_B = \lambda_t$, where $t = e\lambda_B$, (where λ_t is the inner left translation induced by t) if $BA = 0$ in T and if $A \in T_e$, (2) $f\lambda_A = f$ if $A \in T_f$, and (3) $f\lambda_B\lambda_A = f\lambda_B$ or $f\lambda_A$ if $AB \neq 0$, where $\lambda_A = A\theta$.*(3) *Then $S = T^* \dotplus E$ (disjoint sum) becomes an exclusive semigroup whose idempotents form a left zero semigroup, under the multiplication $\circ$ defined as follows:*

$$A \circ B = \begin{cases} AB \text{ if } AB \neq 0 \text{ in } T, \\ e\lambda_A \text{ if } AB = 0,\ B \in T_e, \end{cases} \tag{1}$$

$$A \circ f = f\lambda_A,\ (3)\ f \circ A = f,\ (4)\ e \circ f = ef. \tag{2}$$

Further, every exclusive semigroup whose idempotents form a left zero semigroup is constructed in this fashion.

PROOF. First, we shall show that $S(\circ)$ is an ideal extension of E by T. By assumption, $\theta\colon T^* \to \Lambda$ is a partial anti-homomorphism. The right translation semigroup P of E consists of only the identity mapping 1 of E. Define $\nu\colon T^* \to P$ by $A\nu = 1$ for all A. Define

$$\phi\colon \{(A,B)\colon A, B \in T,\ AB = 0\} \to E$$

by $(A,B)\phi = e\lambda_A$ if $B \in T_e$. Then, these θ, ν, ϕ satisfy the conditions (C1)–(C3) given in Theorem 4.21 of [2]. Hence, $S(\circ)$ becomes an ideal extension of E by T. Next, we shall prove the exclusiveness of S. We divide the proof up into several cases:

(3) Capital letters A, B, etc. will denote elements of T, while small letters e, f, etc. will denote elements of E. For $A, B \in T^*$, $(AB)\theta = (B\theta)(A\theta)$ if AB is defined in T^*. For given T and E, there exist such a family $\{T_e\colon e \in E\}$ and such a partial anti-homomorphism θ.

(i) $A \circ e \circ B = A \circ e$;
(ii) $e \circ A \circ B = e \circ A$;
(iii) Case $A \circ B \circ e$. If $AB \neq 0$ in T, then

$$A \circ B \circ e = (AB) \circ e = e\lambda_{AB} = e\lambda_B \lambda_A = \{e\lambda_B \text{ or } e\lambda_A\} = B \circ e \text{ or } A \circ e.$$

If $AB = 0$ in T, then

$$A \circ B \circ e = (A,B)\phi \circ e = (A,B)\phi = A \circ B;$$

(iv) $e \circ f \circ A = e \circ f$;
(v) $e \circ A \circ f = e \circ f$;
(vi) $A \circ e \circ f = A \circ ef = A \circ e$;
(vii) $e \circ f \circ h = efh = e = ef = e \circ f$;
(viii) Case $A \circ B \circ C$. If $ABC \neq 0$ in T, then

$$A \circ B \circ C = ABC = AC = A \circ C.$$

If $ABC = 0$ in T, then AB, BC or $AC = 0$. If $AB = 0$, then

$$(A \circ B) \circ C = A \circ B.$$

Hence, we can assume that $AB \neq 0$. If $AC \neq 0$, then $BC = 0$ and hence

$$A \circ (B \circ C) = A \circ e\lambda_B \text{ (where } C \in T_e) = e\lambda_B \lambda_A = e\lambda_B \text{ or } e\lambda_A.$$

Since $AB \neq 0$, the elements A, B, AB are contained in the same T_f. If $e \neq f$, then $AC = 0$ (since $T_f T_e = 0$). This is a contradiction, and $e = f$ must be satisfied. Therefore, A, B, $C \in T_e$ and hence

$$e\lambda_A = e\lambda_B = e = B \circ C.$$

Thus, $A \circ B \circ C = B \circ C$. Next, let $AC = 0$. If $BC \neq 0$, then

$$A \circ B \circ C = A \circ (BC) = f\lambda_A \text{ (where } B, C \in T_f) = A \circ C$$

since $AC = 0$. Finally, consider the case where $AB \neq 0$, $AC = 0$ and $BC = 0$. In this case,

$$(A \circ B) \circ C = (AB) \circ C = f\lambda_{AB} \text{ (where } C \in T_f) = f\lambda_B \lambda_A = \{f\lambda_B \text{ or } f\lambda_A\} = B \circ C \text{ or } A \circ C.$$

Therefore, in any case we have $A \circ B \circ C = A \circ B$, $A \circ C$ or $B \circ C$.

Next, we shall show that every exclusive semigroup whose idempotents form a left zero semigroup can be obtained in this way. Suppose that S is an exclusive ideal extension of a left zero semigroup E by a unipotent exclusive semigroup T. Let 0 be the zero element of T. We can assume that $T = S/E$. Let

$$S_e = \{x \in S : x^2 = e\},$$

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and put $(S_e \cup E)/E = T_e$. Then, (1) $T = \cup \{T_e : e \in E\}$, (2) $T_e \cap T_f = 0$ for $e \neq f$, and (3) $T_e T_f = 0$ for $e \neq f$. (Let a, b be elements of S_e, S_f respectively. $af = ab^2 = ab$ or b^2. If $af = b^2$, then $af = f$ and hence $ef = a^2 f = af = f$. Hence, $e = ef = f$, which contradicts our assumption $e \neq f$. Therefore, $af = ab \in E$. Thus, $a \in T_e$ and $b \in T_f$ imply $ab = 0$ in T.) Put $T \setminus 0 = T^*$, and define a partial anti-homomorphism $\theta: T^* \to \Lambda$, where Λ is the left translation semigroup of E, as follows: $A\theta = \lambda_A$, where λ_A is the left translation defined by $e\lambda_A = Ae$, $e \in E$ (capital letters A, B, C, etc. denote elements of T^* and small letters e, f, etc. denote elements of E). It is easy to see that the family $\{\lambda_A : A \in T^*\}$ satisfies the conditions (1)–(3) of the theorem. Hence, we can consider the exclusive ideal extension $S(\circ)$ of E by T in which the multiplication is given by (1)–(4) of the theorem. Now, it is easily proved that this $S(\circ)$ coincides with S.

2. Primitive exclusive semigroups with zero

In section 1, the structure of primitive exclusive semigroups without zero has been clarified. In this section, we shall deal with primitive exclusive semigroups with zero.

Let S be a primitive exclusive semigroup with zero 0. Let

$$S_0 = \{x \in S : x^2 = 0\}.$$

Then it is obvious from (1) of Theorem 1.2 that S_0 is a unipotent exclusive subsemigroup of S. Further, we have

THEOREM 2.1. (1) *S_0 is an ideal of S;*

(2) *$T = S/S_0$ (the Rees factor semigroup of S mod S_0) is a primitive exclusive semigroup with zero $\underset{\sim}{0}$ such that*

(i) *the set B of idempotents of T is a subsemigroup of T, and*

(ii) $T_0 = \{\alpha \in T : \alpha^2 = \underset{\sim}{0}\} = \{\underset{\sim}{0}\}$;

(3) *the set $B \setminus \underset{\sim}{0}$ = the set $E \setminus 0$, where E is the set of idempotents of S.*

PROOF. First, we shall show that S_0 is an ideal of S. Take elements x, y from S_0 and S respectively. By using the exclusiveness of S, it can be shown that $xyx = 0$. Hence $(xy)^2 = 0$, and hence $xy \in S_0$. Similarly, $yx \in S_0$. Thus, S_0 is an ideal of S. It is obvious that $T = S/S_0$ is a primitive exclusive semigroup with zero $\underset{\sim}{0}$ and satisfies (3) and (ii) of (2). Hence, we need to prove only part (i) of (2). Let α, β be non-zero idempotents of T. Then there exist $e, f \in E$ such that $e \neq 0, f \neq 0$, $\alpha = \bar{e}$ and $\beta = \bar{f}$, where $\bar{a}$ means the congruence class containing $a \bmod S_0$. It is easy to see that efe is an idempotent of S. Hence $efe = 0$ or e. If $efe = e$, then $efef = ef$ and accordingly ef is an idempotent. Therefore, in this case $\alpha\beta$ is an idempotent. If $efe = 0$, then $(ef)^2 = 0$ and hence $ef \in S_0$. Therefore, $\alpha\beta = \underset{\sim}{0}$ and accordingly $\alpha\beta$ is an idempotent. Thus, in any case $\alpha\beta$ must be an idempotent. Therefore, B is a subsemigroup of T.

A primitive exclusive semigroup T with zero is called a *basic primitive exclusive semigroup* if T satisfies (i), (ii) of Theorem 2.1. Under this definition, we can say that a primitive exclusive semigroup S with zero is an exclusive ideal extension of a unipotent exclusive semigroup by a basic primitive exclusive semigroup. Further, it is easily seen that the converse of this result is also satisfied. That is, we have

THEOREM 2.2. *A semigroup S with zero is primitive exclusive if and only if S is an exclusive ideal extension of a unipotent exclusive semigroup by a basic primitive exclusive semigroup.*

The structure of unipotent exclusive semigroups has been clarified in section 1. We shall next investigate the structure of basic primitive exclusive semigroups T. Since the set B of idempotents of T is a primitive regular subsemigroup of T, it follows from Preston [4] that there exists a family $\{B_\gamma : \gamma \in \Gamma\}$ of rectangular subsemigroups B_γ of B such that (i)

$$B = \sum \{B_\gamma : \gamma \in \Gamma\} \dotplus \{0\}$$

(where Σ, $\dotplus$ denote the disjoint sum and 0 denotes the zero element of T) and (ii)

$$B_\alpha B_\beta = 0 \text{ for all } \alpha, \beta \in \Gamma,\ \alpha \neq \beta .$$

For each $\gamma \in \Gamma$, put $\{a \in T : a^2 \in B_\gamma\} = T_\gamma$. First, we shall show that each T_γ is a subsemigroup of T. Now,

$$T_\gamma = \sum \{T_e : e \in B_\gamma\},$$

where $T_e = \{a \in T : a^2 = e\}$. Let $x, y \in T_\gamma$. There exist e, $f \in B_\gamma$ such that $T_e \ni x$ and $T_f \ni y$. If $e = f$, then $xy \in T_e = T_f \subset T_\gamma$ follows from (1) of Theorem 1.2. Assume that $e \neq f$. It is easily seen by simple calculation that xf, fx, ey and ye are idempotents. Since $ef \neq f$ or $\neq e$ suppose, without loss of generality, that $ef \neq f$. If $xf = f$, then $x^2 f = xf$ and hence $ef = f$. This contradicts our assumption. Therefore, we have $xf \neq f$. Hence $xf = xyy = xy$. If $xf \in B_\beta$, $\beta \neq \gamma$, then $xff \in B_\beta B_\gamma = 0$, i.e. $xf = 0$. Hence, $ef = x^2 f = 0$. This contradicts the fact that B_γ is a rectangular band $\not\ni 0$ and $e, f \in B_\gamma$. Therefore $xf \in B_\gamma$, and accordingly $xy \in B_\gamma$. Thus, T_γ is a subsemigroup of T. Next, let $x \in T_\alpha$ and $y \in T_\beta$, $\alpha \neq \beta$. Since $x^2 = e \in B_\alpha$ and $y^2 = f \in B_\beta$, we have $ef = 0$. Since $ef = x^2 y^2 = xy$, e or f and since $ef = 0$, we have $xy = 0$. Thus, $T_\alpha T_\beta = 0$. It is obvious that T_α is of course an exclusive (R)-subsemigroup of T. Therefore, we have

THEOREM 2.3. *Let T be a basic primitive exclusive semigroup, and 0 the zero element of T. Then there exists a family $\{T_\gamma : \gamma \in \Gamma\}$ of exclusive (R)-subsemigroups $T_\gamma \not\ni 0$ such that $T = \sum \{T_\gamma : \gamma \in \Gamma\} \dotplus \{0\}$ and $T_\alpha T_\beta = 0$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$.*

The structure of exclusive (R)-semigroups has been completely determined in section 1. Conversely, let $\{T_\gamma: \gamma \in \Gamma\}$ be a family of exclusive (R)-semigroups T_γ. Then, it is easily checked that $T = \sum \{T_\gamma: \gamma \in \Gamma\} \dotplus \{0\}$ becomes a basic primitive exclusive semigroup under the multiplication $\circ$ defined by

(1) $0 \circ a = a \circ 0 = 0$ for all $a \in T$,

(2) $a \circ b = 0$ for $a \in T_\alpha$, $b \in T_\beta$, $\alpha \neq \beta$,

(3) $a \circ b = ab$ (the product of a, b in T_α) for $a, b \in T_\alpha$, $\alpha \in \Gamma$. Accordingly, the structure of basic primitive exclusive semigroups has now been also clarified. Thus, by Theorem 2.2, the problem of constructing all primitive exclusive semigroups with zero is reduced to the problem of determining all possible exclusive ideal extensions of S by T for a given unipotent exclusive semigroup S and a given basic primitive exclusive semigroup T. This problem can be solved by slightly modifying Theorem 1.1 of Yoshida [10], which shows how to construct all the ideal extensions of S by T for a given semigroup S and a given semigroup T with zero, but it is too complicated and is somewhat tedious to give all the details of this approach. Therefore, we shall give here only the result, without a proof. Let S and T be a unipotent exclusive semigroup and a basic primitive exclusive semigroup respectively. Let 0, $\underset{\sim}{0}$ be the zero elements of S, T. Put $T \backslash \underset{\sim}{0} = T^*$. Hereafter, for every notation and symbol the reader is referred to [10].

If a mapping ζ of a semigroup M into M satisfies $(st)\zeta = s\zeta$, $t\zeta$ or st for all $s, t \in M$, then ζ is said to be *semi-identity*. Let

$$[\{\lambda(A): A \in T^*\}, \{\rho(A): A \in T^*\}, \phi]$$

be a system of mappings $\lambda(A)$, $\rho(A)(A \in T^*)$, ϕ satisfying (C1)–(C5) of Theorem 1.1 of [10] and the following conditions I–IV:

I. Each $\lambda(A)$ and each $\rho(A)$ are semi-identity mappings.

II. For any $s \in S$,
 (i) $s\rho(A)\rho(B) = (A, B)\phi$, $s\rho(A)$ or $s\rho(B)$,
 (ii) $s\lambda(A)\lambda(B) = (B, A)\phi$, $s\lambda(A)$ or $s\lambda(B)$,
 (iii) $s\rho(B)\lambda(A)$ $(= s\lambda(A)\rho(B)) = (A, B)\phi$, $s\lambda(A)$ or $s\rho(B)$.

III. For any s, $t \in S$,

$$s\rho(A) \neq 0,\ t\lambda(A) \neq 0 \text{ implies } s\rho(A)t\ (= s(t\lambda(A))) = st.$$

IV. (i) $(A, B)\phi\rho(C) = (A, B)\phi, (B, C)\phi$, or $(A, C)\phi$, if $AB = \underset{\sim}{0}$,
 (ii) $(B, C)\phi\lambda(A) = (A, B)\phi$, $(B, C)\phi$, or $(A, C)\phi$, if $BC = \underset{\sim}{0}$,
 (iii) $(AB, C)\phi$ $(= (A, BC)\phi) = (A, C)\phi$, if $AB \neq \underset{\sim}{0}$ and $BC \neq \underset{\sim}{0}$.

Then, $\Sigma = S \dotplus T^*$ becomes an exclusive ideal extension of S by T under the multiplication $\circ$ defined by (N1)–(N4) of Theorem 1.1 of [10]. Further, every exclusive ideal extension of S by T can be constructed in this fashion. It is also noted that for given S, T there exists at least one such system

$$[\{\lambda(A): A \in T^*\},\ \{\rho(A): A \in T^*\}, \phi]$$

satisfying I–IV above and (C1)–(C5) of Theorem 1.1 of [10]. For example, for every $A \in T^*$ define $\lambda(A)$, $\rho(A)$ as follows: $s\lambda(A) = 0$ and $s\rho(A) = 0$ for all $s \in S$. Define ϕ by $(A,B)\phi = 0$ for A, $B \in T^*$, $AB = \underset{\sim}{0}$. Then, the system

$$[\{\lambda(A): A \in T^*\},\ \{\rho(A): A \in T^*\}, \phi]$$

satisfies the conditions I–IV and the conditions (C1)–(C5) of Theorem 1.1 of [10].

3. Exclusive homobands

We have already seen that if S is an exclusive (R)-semigroup whose idempotents form a purely rectangular band, then the set E of idempotents of S is an ideal of S and S satisfies the following condition:

(3.1) $$(xy)^2 = x^2y^2 \text{ for all } x, y.$$

It is easy to see that, for exclusive semigroups, the condition (3.1) is equivalent to the following:

(3.1)* $$(xy)^n = x^n y^n \text{ for all } n \geqq 1 \text{ and for all } x, y.$$

Further, we have

THEOREM 3.1. *For an exclusive semigroup S, the condit on* (3.1) (*hence* (3.1)*) *is equivalent to the following*:

(3.2) *The set E of idempotents of S is a band, and there exists a homomorphism $\xi: S \to E$ such that $e\xi = e$ for all $e \in E$.*

PROOF. (3.1) $\Rightarrow$ (3.2): Take two idempotents e, f. Then, $(ef)^2 = e^2f^2 = ef$. Hence ef is an idempotent. Let $\xi: S \to E$ define by $x\xi = x^2$, $x \in S$. Then,

$$(xy)\xi = (xy)^2 = x^2y^2 = (x\xi)(y\xi).$$

Hence ξ is a homomorphism. It is obvious that $e\xi = e$ for all idempotents ef.

(3.2) $\Rightarrow$ (3.1): For any $x \in S$, $x^2 = x^2\xi = (x\xi)^2 = x\xi$. Hence, $(xy)^2 = (xy)\xi = (x\xi)(y\xi) = x^2y^2$.

An exclusive semigroup S is called an *exclusive homoband* if S satisfies condition (3.1) and the following condition:

(3.3) The set E of idempotents of S is an ideal of S.

REMARK. In an exclusive semigroup, the set of idempotents is not necessarily a subsemigroup. This can be seen from the example given by O'Carroll and Schein [5]. Also, the set of idempotents is not necessarily an ideal even when it is a subsemigroup.

This can also be seen from Tamura's paper [6]. Therefore, each of our conditions (3.1) and (3.3) seems to be strong. However, in general, for any two idempotents e, f of an exclusive semigroup at least one of ef and fe is an idempotent. Further, both a commutative semigroup and a completely non-commutative semigroup[4] satisfy the condition (3.1). Moreover, it is easily seen that in an exclusive semigroup S whose idempotents form a band, for any $x, y \in S$ the relation $(xy)^2 = x^2y^2$ is satisfied except in the case $x^2 = e$, $y^2 = f$, $ef = e$ or f, $e \neq f$. It is also easily proved that for any idempotent f and any element x of an exclusive semigroup S, at least one of xf and fx is contained in the set E of idempotents of S except in the case $x^2 = e$, $ef = fe = e$, $e \neq f$. Therefore, the conditions (3.1) and (3.3) are not such strong conditions for the class of exclusive semigroups as at first appears.

Now, we have

THEOREM 3.2. *Let S be an exclusive homoband, and E the band of idempotents of S. Let $S_e = \{x \in S : x^2 = e\}$, $e \in E$. Then,*

(1) $S = \Sigma\{S_e : e \in E\}$,

(2) *$S_eS_f = ef$ for all $e, f \in E$, $e \neq f$,*

(3) *S_e is a unipotent exclusive semigroup, and in particular S_e is a null semigroup if there exists $f \in E$ such that $efe = e$, $ef \neq e$ and $fe \neq e$,*

(4) *E is an exclusive band.*

PROOF. Parts (1), (4) and the first half of (3) are obvious. Therefore, next we prove the latter half of (3). Let e, f be idempotents such that $efe = e$, $ef \neq e$ and $fe \neq e$. Take any elements x, y of S_e. Then,

$$xfy = xf,\ fy \text{ or } xy.$$

Since $xfy = efe$, $xf = ef$ and $fy = fe$, we have $xfy \neq xf$ and $xfy \neq fy$. Hence

$$e = efe = xfy = xy.$$

This means that S_e is a null semigroup. Finally, we prove part (2). Since $ef \neq e$ or $ef \neq f$, we can assume that $ef \neq e$ without loss of generality. Take any elements x, y from S_e, S_f, respectively. Then,

$$(ey)^2 = e^2y^2 = ef$$

by condition (3.1). Since $ey \in E$, $(ey)^2 = ey$. Hence $ey = ef$. Therefore,

$$ef = ey = xxy = x^2 \text{ or } xy.$$

Since $ef \neq e$ we have $xy = ef$. Thus, $S_eS_f = ef$.

By using the theorem above, we have

(4) A semigroup S is said to be completely non-commutative if it satisfies the following condition: For any $x, y \in S$, $xy = yx$ implies $x = y$.

THEOREM 3.3. *Let E be an exclusive band. For every $e \in E$ let S_e be a unipotent exclusive semigroup having e as its zero element, and moreover let Se be a null semigroup having e as its zero element if there exists $f \in E$ such that $efe = e$, $ef \neq e$ and $fe \neq e$. Then $S = \Sigma\{S_e: e \in E\}$ becomes an exclusive homoband having E as the band of its idempotents under the multiplication $\circ$ defined by*

$$(3.4) \qquad x \circ y = \begin{cases} xy \text{ if } x,\ y \in S_e \text{ for some } e \in E, \\ ef \text{ if } x \in S_e,\ y \in S_f \text{ and } e \neq f. \end{cases}$$

Further, every exclusive homoband is constructed in this fashion.

PROOF. First, we prove that $S(\circ)$ is a semigroup. Take any x, y, z from S_e, S_f, S_h respectively. If $e = f = h$, then

$$(x \circ y) \circ z = (xy)z = x(yz) = x \circ (y \circ z).$$

Hence in this case, the associative law

$$(x \circ y) \circ z = x \circ (y \circ z)$$

is satisfied. Next, consider the case where one of e, f, h is different from each of the others. In this case,

$$x \circ (y \circ z) = efh = (x \circ y) \circ z.$$

Therefore, $S(\circ)$ is a semigroup. Next, we prove the exclusiveness of $S(\circ)$. Take x, y, z from S_e, S_f, S_h, respectively.

(i) The case $e = f = h$: In this case

$$x \circ y \circ z = x \circ y,\ x \circ z \text{ or } y \circ z,$$

as is obvious.

(ii) The case where two of e, f, h are same and one of e, f, h is different from the other two elements: In this case,

$$x \circ y \circ z = efh.$$

If $e = f$ and $f \neq h$, then $efh = eh$. Hence

$$x \circ y \circ z = y \circ z.$$

If $f = h$ and $f \neq e$, then $efh = ef$. Hence

$$x \circ y \circ z = x \circ y.$$

If $e = h$ and $e \neq f$, then $efh = ef, fh$ or e. In the case $efh = ef$, we have

$$x \circ y \circ z = x \circ y.$$

If $efh = fh$; then we have

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$$x \circ y \circ z = y \circ z.$$

Finally in the case efe $(= efh) \neq ef$, $efe \neq fe$ and $efe = e$, by the hypothesis S_e is a null subsemigroup. Therefore, $x \circ y \circ z = e = xz = x \circ z$.

(iii) The case where e, f, h are distinct: In this case,

$$x \circ y \circ z = x \circ y, \; y \circ z \text{ or } x \circ z$$

is easily verified since $efh = ef, fh$ or eh and since $x \circ y = ef$, $y \circ z = fh$ and $x \circ z = eh$.

Thus, in any case

$$x \circ y \circ z = x \circ y, \; y \circ z \text{ or } x \circ z$$

is satisfied. The latter half of the theorem is obvious from Theorem 3.2.

COROLLARY. *If S is an exclusive homoband and if the idempotents E of S form a semilattice of purely rectangular bands, then S is an inflation of E. Conversely, if a semigroup S is an inflation of an exclusive band E then S is an exclusive homoband having E as the band of its idempotents.*

PROOF. Obvious from Theorems 1.4 and 3.3.

From the theorem above, the problem of describing the structure of exclusive homobands is reduced to that of exclusive bands. Recently, the author heard that O'Carroll and Schein [5] have completely described the structure of exclusive bands. They have also independently obtained Theorem 1.2 and the parts (2)–(4) of Theorem 1.3 and Theorems 4.1 and 4.2 below.

4. Medial exclusive semigroups

If a [left, right] normal band S (see [7], [8]) is exclusive, then S is called a *[left, right] normal exclusive band.*

For left [right] normal bands we have

THEOREM 4.1. *Let S be a left [right] normal band, and $S \sim \Sigma\{S_\gamma : \gamma \in \Gamma\}$ the structure decomposition of S (see [7], [8]). Let $\Omega = \{\psi^\alpha_\beta : \alpha \leqq \beta, \alpha, \beta \in \Gamma\}$ be the characteristic family of S (where $\alpha \leqq \beta$ if and only if $\alpha\beta = \beta\alpha = \beta$) (see [7], [8]). Then S is exclusive if and only if*

(1) *Γ is exclusive, and*

(2) *for α, β, $\gamma \in \Gamma$ such that $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ are mutually distinct $S_\alpha\psi^\alpha_{\alpha\beta\gamma} = S_\beta\psi^\beta_{\alpha\beta\gamma} = S_\gamma\psi^\gamma_{\alpha\beta\gamma} =$ a single element.*

PROOF. First we prove the "only if" part. Since Γ is a homomorphic image of S, it is also exclusive. To prove part (2), let α, β, γ be elements of Γ such that $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ are all mutually distinct. Since Γ is exclusive, $\alpha\beta\gamma = \alpha\beta$, $\beta\gamma$ or $\alpha\gamma$. We can assume without loss of generality that $\alpha\beta\gamma = \beta\gamma$, because, for example, if

$\alpha\beta\gamma = \alpha\gamma$ then this implies $\beta\alpha\gamma = \alpha\gamma$. For any $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$ and $c_\gamma \in S_\gamma$, we have $a_\alpha b_\beta c_\gamma = b_\beta c_\gamma$. Hence

$$a_\alpha \psi^\alpha_{\alpha\beta\gamma} = b_\beta \psi^\beta_{\alpha\beta\gamma} = b_\beta \psi^\beta_{\alpha\beta\gamma}.$$

Similarly, $a_\alpha c_\gamma b_\beta = c_\gamma b_\beta$ implies $a_\alpha \psi^\alpha_{\alpha\beta\gamma} = c_\gamma \psi^\gamma_{\alpha\beta\gamma}$. Therefore,

$$a_\alpha \psi^\alpha_{\alpha\beta\gamma} = b_\beta \psi^\beta_{\alpha\beta\gamma} = c_\gamma \psi^\gamma_{\alpha\beta\gamma}.$$

This implies

$$S_\alpha \psi^\alpha_{\alpha\beta\gamma} = S_\beta \psi^\beta_{\alpha\beta\gamma} = S_\gamma \psi^\gamma_{\alpha\beta\gamma} = \text{a single element.}$$

Next, we prove the "if" part. Since Γ is exclusive, $\alpha\beta\gamma = \alpha\beta$, $\beta\gamma$ or $\alpha\gamma$. Let a_α, b_β, c_γ be elements of S_α, S_β, S_γ respectively, and consider $a_\alpha b_\beta c_\gamma$.

(i) Case $\alpha\beta\gamma = \alpha\beta$: $a_\alpha b_\beta c_\gamma = a_\alpha b_\beta(a_\alpha b_\beta c_\gamma) = a_\alpha b_\beta$ (by the left normality of S).

(ii) Case $\alpha\beta\gamma = \alpha\gamma$: $a_\alpha b_\beta c_\gamma = a_\alpha c_\gamma b_\beta$ (by the left normality of S) $= a_\alpha c_\gamma$ (by (i)).

(iii) Case $\alpha\beta\gamma \neq \alpha\beta, \alpha\gamma$: In this case, it is easy to see that $\alpha\beta\gamma = \beta\gamma$ and $\alpha\beta$, $\beta\gamma$, $\alpha\gamma$ are mutually distinct. Now,

$$a_\alpha b_\beta c_\gamma = a_\alpha \psi^\alpha_{\alpha\beta\gamma} = b_\beta \psi^\beta_{\alpha\beta\gamma} \quad \text{(by condition (2))} = b_\beta \psi^\beta_{\beta\gamma} = b_\beta c_\gamma.$$

Thus S is exclusive.

Next, let N be a normal band. Then N is isomorphic to a spined product of a left normal band L and a right normal band R with respect to a semilattice Γ; that is, $N \cong L \bowtie R(\Gamma)$ (see [7], [8]). Let

$$L \sim \Sigma\{L_\gamma : \gamma \in \Gamma\}, \; R \sim \Sigma\{R_\gamma : \gamma \in \Gamma\}$$

be the structure decompositions of L, R respectively, and let

$$\Omega = \{\psi^\alpha_\beta : \alpha \leqq \beta, \alpha, \beta \in \Gamma\}, \quad \Delta = \{\phi^\alpha_\beta : \alpha \leqq \beta, \alpha, \beta \in \Gamma\}$$

be the characteristic families of L, R.

Then, we have

THEOREM 4.2. *N is exclusive if and only if*

(1) *Γ is exclusive,*

(2) *if $\beta \in \Gamma$ is not a minimal element, then L_β = a single element or R_β = a single element, and*

(3) *for α, β, $\gamma \in \Gamma$ such that $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ are mutually distinct,*

$$L_\alpha \psi^\alpha_{\alpha\beta\gamma} = L_\beta \psi^\beta_{\alpha\beta\gamma} = L_\gamma \psi^\gamma_{\alpha\beta\gamma} = \textit{a single element}$$

and $R_\alpha \phi^\alpha_{\alpha\beta\gamma} = R_\beta \phi^\beta_{\alpha\beta\gamma} = R_\gamma \phi^\gamma_{\alpha\beta\gamma}$ = a single element.

PROOF. First we prove the "only if" part. Conditions (1), (3) follow from Theorem 4.1 and the fact that Γ, L, R are homomorphic images of the exclusive band N. Hence, we need only prove (2). Let $\beta \in \Gamma$ be not a minimal element.

Then, there exists $\alpha \in \Gamma$ such that $\alpha < \beta$. Take any

$$a_\alpha,\ c_\alpha \in L_\alpha,\ b_\beta \in L_\beta,\ a'_\alpha,\ c'_\alpha \in R_\alpha,\ b'_\beta \in R_\beta.$$

Consider the three elements (a_α, a'_α), (b_β, b'_β), $(c_\alpha, c'_\alpha) \in L \bowtie R(\Gamma)$. Since

$$(a_\alpha, a'_\alpha)(b_\beta, b'_\beta)(c_\alpha, c'_\alpha) = (a_\alpha, a'_\alpha)(b_\beta, b'_\beta) \text{ or } (b_\beta, b'_\beta)(c_\alpha, c'_\alpha),$$

we have $(a_\alpha b_\beta c_\alpha, a'_\alpha b'_\beta c'_\alpha) = (a_\alpha b_\beta, a'_\alpha b'_\beta)$ or $(b_\beta c_\alpha, b'_\beta c'_\alpha)$. On the other hand,

$$(a_\alpha b_\beta c_\alpha, a'_\alpha b'_\beta c'_\alpha) = (a_\alpha b_\beta, b'_\beta c'_\alpha)$$

since $L_{\alpha\beta}$, $R_{\alpha\beta}$ are a left zero semigroup and a right zero semigroup, respectively.

If $(a_\alpha b_\beta, b'_\beta c'_\alpha) = (a_\alpha b_\beta, a'_\alpha b'_\beta)$, then $c'_\alpha \phi^\alpha_\beta = b'_\beta$.

If $(a_\alpha b_\beta, b'_\beta c'_\alpha) = (b_\beta c_\alpha, b'_\beta c'_\alpha)$, then $a_\alpha \psi^\alpha_\beta = b_\beta$.

Hence, we have $a_\alpha \psi^\alpha_\beta = b_\beta$ or $c'_\alpha \phi^\alpha_\beta = b'_\beta$. Suppose that each of L_β and R_β contains at least two elements. Then, there exist $d_\beta \in L_\beta$, $d'_\beta \in R_\beta$ such that $d_\beta \neq b_\beta$ and $d'_\beta \neq b'_\beta$. By the same method, we have

$$a_\alpha \psi^\alpha_\beta = d_\beta \text{ or } c'_\alpha \phi^\alpha_\beta = d'_\beta.$$

Therefore, either

$$\text{“}a_\alpha \psi^\alpha_\beta = b_\beta \text{ and } c'_\alpha \phi^\alpha_\beta = d'_\beta\text{” or “}a_\alpha \psi^\alpha_\beta = d_\beta \text{ and } c'_\alpha \phi^\alpha_\beta = b'_\beta\text{”}.$$

Suppose that $a_\alpha \psi^\alpha_\beta = b_\beta$ and $c'_\alpha \phi^\alpha_\beta = d'_\beta$. Then,

$$(a_\alpha, a'_\alpha)(d_\beta, b'_\beta)(c_\alpha, c'_\alpha) = (a_\alpha d_\beta, a'_\alpha b'_\beta) \text{ or } (d_\beta c_\alpha, b'_\beta c'_\alpha).$$

On the other hand, $(a_\alpha d_\beta c_\alpha, a'_\alpha b'_\beta c'_\alpha) = (a_\alpha d_\beta, b'_\beta c'_\alpha)$. Hence,

$$a_\alpha \psi^\alpha_\beta = d_\beta \text{ or } c'_\alpha \phi^\alpha_\beta = b'_\beta.$$

This is a contradiction. Similarly, if we assume “$a_\alpha \psi^\alpha_\beta = d_\beta$ and $c'_\alpha \phi^\alpha_\beta = b'_\beta$” then we also get a contradiction. Therefore, $L_\beta =$ a single element or $R_\beta =$ a single element. Since the “if” part can be proved by a method almost identical to that of the proof of the “if” part of Theorem 4.1, we omit its proof.

COROLLARY. *If a normal exclusive band S is a semilattice of purely rectangular bands, then S is necessarily a purely rectangular band.*

PROOF. Obvious from the theorem above.

Since the structure of exclusive semilattices has been completely described by [6], the structure of normal exclusive bands is now also clarified by the theorem above. A semigroup is said to be *medial* if it satisfies the identity $xyzw = xzyw$[5]. It is easy to see that a medial exclusive archimedean semigroup (see [1]) is a medial exclusive homoband whose idempotents form a rectangular band. Next, we shall study the structure of medial exclusive homobands.

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(5) Hence, a medial band is just the same as a normal band.

THEOREM 4.3. *Let E be a normal exclusive band, and $E \sim \Sigma\{E_\gamma : \gamma \in \Gamma\}$ the structure decomposition of E (see [7], [8]). For every $e \in E$, let S_e be a unipotent medial exclusive semigroup having e as its zero element. Moreover, let S_e be a null semigroup having e as its zero element if there exists $f \in E$ such that $efe = e$, $ef \neq e$ and $fe \neq e$. Then, $S = \Sigma\{S_e : e \in E\}$ becomes a medial exclusive homoband under the multiplication $\circ$ defined by*

$$(4.1) \qquad x \circ y = \begin{cases} xy \text{ if } x, y \in S_e \text{ for some } e \in E, \\ ef \text{ if } x \in S_e,\ y \in S_f \text{ and } e \neq f. \end{cases}$$

Further, every medial exclusive homoband can be constructed in this fashion.

PROOF. Obvious from Theorem 3.3 and the fact that every subsemigroup of a medial semigroup is medial.

Thus the problem of describing the structure of medial exclusive homobands is reduced to that of describing the structure of unipotent medial exclusive semigroups. Next, we consider this problem.

Let X be a set, and A a subset of $X \times X = \{(x, y) : x, y \in X\}$ such that

$$(4.2) \qquad \begin{cases} (1)\ A \not\ni (a, a) \text{ for all } a \in X, \\ (2)\ A \ni (x, y),\ (y, z),\ (x, z) \text{ implies that} \\ \quad \text{(i) } (x, v) \notin A \text{ or } (v, y) \notin A \text{ for all } v \in X, \text{ and} \\ \quad \text{(ii) } (y, w) \notin A \text{ or } (w, z) \notin A \text{ for all } w \in X. \end{cases}$$

Then, of course A satisfies condition (1.2). Hence, $S = A \cup X \cup \{0\}$ becomes a unipotent exclusive semigroup under the multiplication $\circ$ defined by

$$(4.3) \qquad \begin{cases} (1)\ \text{for } a, b \in X,\ a \circ b = (a, b) \text{ if } (a, b) \in A; \\ (2)\ \text{for } a \in X \text{ and } (b, c) \in A,\ a \circ (b, c) = (a, c) \\ \quad \text{if } (a, b),\ (a, c) \in A; \\ (3)\ \text{for } c \in X \text{ and } (a, b) \in A,\ (a, b) \circ c = (a, c) \\ \quad \text{if } (a, c),\ (b, c) \in A; \\ (4)\ \alpha \circ \beta = 0 \text{ otherwise (where } \alpha, \beta \in S). \end{cases}$$

Further, $S(\circ)$ is medial For, let α, β, γ, δ be any elements of $S(\circ)$. Then,

$$\alpha \circ \beta \circ \gamma \circ \delta = (\alpha \circ \beta) \circ (\gamma \circ \delta)$$

and each of $\alpha \circ \beta$ and $\gamma \circ \delta$ is 0 or an element of A. Hence, by the definition of the multiplication in S, we have

$$(\alpha \circ \beta) \circ (\gamma \circ \delta) = 0.$$

Similarly, $\alpha \circ \gamma \circ \beta \circ \delta = 0$. Therefore, $S(\circ)$ is medial. We shall denote this $S(\circ)$ by $\mathrm{FM}(X, A)$.

Then, we have

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THEOREM 4.4. *Any homomorphic image of $FM(X, A)$ is a unipotent medial exclusive semigroup. Conversely, every unipotent medial exclusive semigroup can be obtained as a homomorphic image of some $FM(X, A)$.*

PROOF. The first half of the theorem follows from Theorem 1.6 and the fact that any homomorphic image of a medial semigroup is medial. To prove the latter half, let S be a unipotent medial exclusive semigroup, and put

$$S \backslash S^2 = X \text{ and } \{(x, y): xy \neq 0 \text{ in } S, x, y \in X\} = A.$$

Then, it is easy to see that A satisfies condition (4.2). Therefore, we can consider $FM(X, A)$. Now, define $\phi: FM(X, A) \to S$ by $0\phi = 0$, $x\phi = x$ for $x \in X$ and $(x, y)\phi = xy$ for $(x, y) \in A$. Then, this ϕ is a homomorphism of $FM(X, A)$ onto S.

REMARK. Let S be a unipotent medial exclusive semigroup. Then by the theorem above, S is a homomorphic image of some $FM(X, A)$. Since for any $\alpha, \beta, \gamma, \delta \in FM(X, A)$ $\alpha\beta\gamma\delta = 0$ is satisfied as was seen above, the relation $abcd = 0$ is also satisfied for any a, b, c, d of S. Hence, we have: If S is a unipotent medial exclusive semigroup, then $abcd = 0$, for all a, b, c, $d \in S$.

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