1. Introduction. Properties and characterizations for prime and semi-prime rings have been provided by A. W. Goldie (2, 3). In a previous paper (1), the authors used the results of (2) to characterize prime and uniform prime modules. It is the aim of the present paper to generalize Goldie's work on semi-prime rings (3) to modules. In this setting certain new properties will appear.

Notationally, in the work to follow, the symbol $R$ always denotes a ring and all $R$-modules will be right $R$-modules.

In the theory of rings an ideal $C$ is said to be prime if and only if whenever $AB \subseteq C$ for ideals $A$ and $B$, then either $A \subseteq C$ or $B \subseteq C$. A ring is prime if the zero ideal is prime. In order to avoid confusion in definitions for modules it was deemed necessary to introduce the term *closed prime submodule* (see (2.1)) and call a module *annihilator prime* provided the zero submodule is closed prime (see (2.2)).

In §2 the *prime radical* $P(M)$ of an $R$-module $M$ is defined as the intersection of the closed prime submodules. It is shown that the prime radical of $M/P(M)$ is zero. A module is termed *semi-prime* if the prime radical is zero. It is proved in §3 that if $M$ is a faithful semi-prime $R$-module, then $R$ is a semi-prime ring. In addition, (3.3) states that if an $R$-module $M$ is semi-prime, then $M$ is a subdirect sum of modules, each of which is a prime module over the difference ring $R$ modulo the annihilator. This describes semi-prime modules in terms of prime modules as discussed by the authors in (1).

In §4, torsion-free modules over Goldie semi-prime rings are investigated. Theorem (4.8) states that in this case $M$ is a semi-prime module where the zero submodule is the intersection of maximal non-large submodules that are closed prime. Then $M$ is a subdirect sum of uniform annihilator prime modules. If $M$ is finite-dimensional, then a finite number of such submodules will suffice. This analysis enables one to express the zero ideal of a Goldie semi-prime ring as an intersection of an explicit set of prime ideals (see (4.13)).

The last section also includes a characterization for faithful semi-prime modules that generalizes a result obtained in (1) for prime modules.

2. The prime radical. If $N$ is a submodule of an $R$-module $M$, then
\[ \text{cl}(N) = \{ m \in M | m \subseteq N \text{ for some large right ideal } I \text{ of } R \}. \]
If \( \operatorname{cl}(N) = N \), then \( N \) is said to be closed. The singular submodule \( Z(M) \) is defined as \( \operatorname{cl}(0) \). If \( N_1 \) and \( N_2 \) are submodules of \( M \), then \( (N_1 : N_2) \) is the ideal consisting of all \( a \in R \) such that \( N_2 a \subseteq N_1 \).

(2.1) Definition. A submodule \( N \) of an \( R \)-module \( M \) is termed closed-prime provided the following two conditions are satisfied:

(i) if \( N' \) is a submodule such that \( N \subseteq N' \subseteq M \), then \( (N : N') \subseteq (N : M) \),

(ii) \( \operatorname{cl}(N) = N \).

Here the symbol \( \subseteq \) means proper inclusion.

Note that condition (i) is equivalent to the statement that for every non-zero submodule \( N'/N \) of the right \( R/(N : M) \)-module \( M/N \), one has \( (0 : N'/N) = 0 \), where \( 0 \) denotes the zero coset.

The module \( M \) itself is closed-prime since condition (i) is (vacuously) true in the case \( N = M \). It is also clear that maximal submodules that are not large are closed-prime. We shall see later that for torsion-free modules over semi-prime rings with chain conditions, the maximal elements in the set of non-large submodules are closed-prime.

(2.2) Definition. An \( R \)-module \( M \) is termed annihilator-prime provided \( (0) \) is a closed-prime submodule of \( M \).

Thus the module \( M \) is annihilator-prime if \( Z(M) = 0 \) and \( (0 : N') \subseteq (0 : M) \) for all non-zero submodules \( N' \) of \( M \).

From the preceding definitions and remarks we obtain

(2.3) Proposition. A submodule \( N \) of an \( R \)-module \( M \) is closed prime if and only if the \( R \)-module \( M/N \) is annihilator-prime.

If \( M \) is a faithful annihilator-prime module, then \( Z(M) = 0 \) and \( (0 : N') = 0 \) for all non-zero submodules \( N' \) of \( M \). If, in addition, \( R \) is a ring with maximum condition, then \( M \) is a prime \( R \)-module in the sense of (1). For modules over rings without chain conditions we have

(2.4) Definition. A faithful annihilator-prime module is termed a prime module.

(2.5) Theorem. If \( M \) is an annihilator-prime \( R \)-module, then \( M \) is prime as a module over \( R/(0 : M) \).

Proof. Since \( M \) is faithful over \( R/(0 : M) \) we need only show that \( \operatorname{cl}(0) = 0 \).

If \( x \in \operatorname{cl}(0) \), then \( x\tilde{L} = 0 \) where \( \tilde{L} \) is a large right ideal in \( \tilde{R} = R/(0 : M) \). If \( L \) is the inverse image of \( \tilde{L} \) in the natural homomorphism of \( R \) onto \( \tilde{R} \), then \( L \) is a large right ideal and \( xL = 0 \). Since \( M \) is annihilator-prime over \( R \), we obtain \( x = 0 \) and the theorem is proved.

(2.6) Corollary. If \( N \) is a closed-prime submodule of an \( R \)-module \( M \), then \( M/N \) is a prime module over \( R/(N : M) \).

Proof. If \( N \) is closed-prime, then by (2.3) \( M/N \) is an annihilator-prime \( R \)-module. Now employ (2.5).
(2.7) Proposition. If $N$ is a closed-prime submodule of an $R$-module $M$, then $(N : M)$ is a closed prime ideal in $R$.

Remark. An ideal of $R$ is closed if it is closed as a right $R$-module.

Proof. If $N$ is closed-prime, then by (2.6) $M/N$ is a prime module over $R/(N : M)$. We may employ the argument used to establish (1, (1.2)) to prove that $R/(N : M)$ is a prime ring, and hence that $(N : M)$ is a prime ideal of $R$. This implies $Ma \subseteq \text{cl}(N) = N$ or $a \in (N : M)$. Therefore $\text{cl}((N : M)) = (N : M)$.

We now define a radical for modules that is analogous to the prime radical of a ring as given in (7).

(2.8) Definition. The prime radical $P(M)$ of an $R$-module $M$ is the intersection of all the closed-prime submodules of $M$.

It follows from this definition that $Z(M) \subseteq P(M)$ and consequently when $P(M) = 0$, $Z(M) = 0$.

One can obtain a correspondence between closed-prime submodules of an $R$-module $M$ and the closed-prime submodules of a homomorphic image.

(2.9) Proposition. Let $\theta$ be an $R$-homomorphism of an $R$-module $M$ onto an $R$-module $W$ with kernel $K$. If $\{N_i\}$ is the collection of closed-prime submodules of $M$ that contain $K$, then the correspondence $N_i \rightarrow \theta N_i$ is a one-one mapping of $\{N_i\}$ onto the set of all closed-prime submodules of $W$.

Proof. As in (7, (2.45)), the mapping $N \rightarrow \theta N$ is a one-to-one inclusion preserving correspondence between the set of submodules of $M$ containing $K$ onto the totality of submodules of $W$. Thus it suffices to show that under this mapping the closed-prime submodules correspond to one another.

Suppose that $N$ is closed-prime in $M$ with $N \supseteq K$. Let $a \in (\theta N : \theta N')$ where $N'$ is a submodule of $M$ such that $\theta N \subseteq \theta N'$ (and hence $N \subseteq N'$). Then $N'a \subseteq \theta^{-1}(\theta N) = N$, so $a \in (N : N')$. Since $N$ is closed-prime, this implies that $a \in (N : M)$ and we have $(\theta M)a \subseteq \theta N$. Therefore,

$$(\theta N : \theta N') \subseteq (\theta N : W).$$

Now let $\theta m \in \text{cl}(\theta N)$, where $m \in M$. Then $mI \subseteq \theta^{-1}(\theta N) = N$, where $I$ is a large right ideal in $R$. Thus $m \in \text{cl}(N) = N$, or $\theta m \in \theta N$. We have proved that $\text{cl}(\theta N) \subseteq \theta N$ and hence $\theta N$ is closed-prime in $W$.

We complete the proof by showing that if $V$ is a closed-prime submodule of $W$, then $\theta^{-1}V$ is closed-prime in $M$. Let $a \in (\theta^{-1}V : N')$, where $\theta^{-1}V \subseteq N'$. Then $V = \theta(\theta^{-1}V) \subseteq \theta N'$ and since $V$ is closed-prime,

$$a \in (V : W) = (V : \theta M) = (\theta^{-1}V : M),$$

so (2.1), (i) is true for $\theta^{-1}V$.

To show that (2.1), (ii) holds, let $m \in \text{cl}(\theta^{-1}V)$. Then $(\theta m)I \subseteq V$ for a large right ideal $I$ of $R$ and hence $\theta m \in \text{cl}(V) = V$, i.e. $m \in \theta^{-1}V$. 

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(2.10) Proposition. If $N$ is a submodule of an $R$-module $M$ and if $L$ is a closed-prime submodule of $M$, then $N \cap L$ is a closed-prime submodule of $N$.

Proof. We may assume that $N \cap L \neq N$. Let $N'$ be a submodule such that $N \cap L \subseteq N' \subseteq N$. Then $L \subseteq L + N'$ and we have

$$(N \cap L : N') \subseteq (L : L + N') \subseteq (L : M),$$

whence $(N \cap L : N') \subseteq (N \cap L : M)$. It is evident that $\text{cl}(N \cap L) = N \cap L$. Therefore $N \cap L$ is a closed-prime in $N$.

(2.11) Theorem. If $M$ is an $R$-module, then $P(M/P(M)) = 0$.

Proof. Let \( \theta \) be the natural homomorphism of $M$ onto $M/P(M)$. By (2.8), every closed-prime submodule of $M$ contains the kernel of $\theta$. Hence, from (2.9), the correspondence $N_i \to \theta N_i$ is a one-one mapping of the set \( \{N_i\} \) of all closed-prime submodules of $M$ onto the set of all closed-prime submodules of $M/P(M)$. If $\theta m \in P(M/P(M))$ for $m \in M$, then $m \in \theta^{-1}(\theta N_i) = N_i$ for all $i$ and we have $m \in P(M)$. Thus $\theta m = 0$.

3. Semi-prime modules. As in (7, (4.25)), a ring $R$ is semi-prime if and only if the prime radical of $R$ is zero. We shall use this approach to define a semi-prime module.

(3.1) Definition. An $R$-module $M$ is semi-prime if and only if $P(M) = 0$.

Notice that by (2.11), $M/P(M)$ is semi-prime for all $R$-modules $M$.

One can characterize semi-prime modules in terms of subdirect sums.

(3.2) Theorem. An $R$-module $M$ is semi-prime if and only if $M$ is a subdirect sum of annihilator-prime $R$-modules.

Proof. If $M$ is semi-prime, then $0 = \cap N_i$ where \( \{N_i\} \) is the collection of closed-prime submodules of $M$. Then $M$ is isomorphic to a subdirect sum of the modules $M/N_i$, where, by (2.3), each $M/N_i$ is an annihilator-prime $R$-module.

Conversely, if $M$ is isomorphic to a subdirect sum of annihilator-prime modules \( \{H_i\} \), then there exist submodules $N_i$ of $M$ such that $H_i \cong M/N_i$ and $\cap N_i = 0$. Since each $N_i$ is closed-prime by (2.3), we have $P(M) = 0$ and (3.2) is proved.

(3.3) Corollary. If $M$ is a semi-prime $R$-module, then $M$ is a subdirect sum of modules $M_i$, where each $M_i$ is a prime module over $R/(0 : M_i)$.

Proof. Apply (2.5) to the “only if” part of (3.2).

(3.4) Proposition. Let $M$ be a semi-prime $R$-module and let \( \{N_i\} \) be the collection of closed-prime submodules of $M$. Then $M$ is faithful if and only if

$$\cap (N_i : M) = 0.$$
Proof. If \( a \in \bigcap (N_i : M) \), then \( Ma \subseteq N_i \) for all \( i \) and hence \( Ma = 0 \). Consequently, if \( M \) is faithful, then \( \bigcap (N_i : M) = 0 \). Conversely, if \( \bigcap (N_i : M) = 0 \) and \( Ma = 0 \), then \( a \in (N_i : M) \) for all \( i \) and hence \( a = 0 \).

(3.5) Proposition. If \( M \) is a faithful semi-prime \( R \)-module, then \( R \) is a semi-prime ring.

Proof. By (3.4), we have \( \bigcap (N_i : M) = 0 \), where \( \{N_i\} \) is the collection of closed-prime submodules of \( M \). From (2.7), each \( (N_i : M) \) is a prime ideal in \( R \). Hence the intersection of these prime ideals is zero, which implies that \( R \) is a semi-prime ring.

(3.6) Theorem. A complete direct product of semi-prime \( R \)-modules is a semi-prime \( R \)-module.

Proof. It is not difficult to show that given a direct sum \( M = M_1 \oplus M_2 \) of \( R \)-modules, if \( L_1 \) is a closed-prime submodule of \( M_1 \), then \( L_1 \oplus M_2 \) is a closed-prime submodule of \( M \). The methods used in (5, p. 416) may now be used to complete the proof.

4. Modules over semi-prime rings. A ring that satisfies the right quotient conditions of (3) is called a (right) Goldie ring. According to (3, (3.9)), a right ideal of a semi-prime Goldie ring is large if and only if it contains a regular element. It follows directly that a module \( M \) over a Goldie semi-prime ring is torsion-free if and only if \( Z(M) = 0 \). (As in (6), an \( R \)-module \( M \) is torsion-free if, whenever \( mc = 0 \), where \( m \in M \) and \( c \) is regular in \( R \), then \( m = 0 \).) The principal objective of the present section is to investigate the structure of such modules. Accordingly, in order to simplify our statements, we shall, throughout this section, let \( M \) denote a torsion-free module over a Goldie semi-prime ring \( R \) and let \( Q \) denote the (semi-simple) quotient ring of \( R \).

If \( N \) is an \( R \)-submodule of \( M \), then as in (6, (1.5)) \( N \) can be imbedded in the \( Q \)-module \( N \otimes_R Q \cong NQ \). The elements of \( NQ \) may be written in the form \( nc^{-1} \) for \( n \in N \), \( c \) regular in \( R \), and we may assume that \( N \subseteq NQ \) where \( n \in N \) is identified with \( n1 \in NQ \). Moreover, every submodule of \( M \) can be written in this way. Specifically, if \( T \) is a submodule of \( M \), we have
\[
T = (M \cap T)Q.
\]

If \( N_1 \) and \( N_2 \) are \( R \)-submodules of \( M \) we write \( N_1 \sim N_2 \), and say that \( N_1 \) is related to \( N_2 \), if and only if \( N_1 \) and \( N_2 \) meet the same non-zero submodules of \( M \). The following propositions are patterned after those given for rings by Goldie (3).

(4.1) Proposition. If \( N_1 \) and \( N_2 \) are submodules of \( M \), then the following conditions are equivalent:

(i) \( N_1 \sim N_2 \),
(ii) \( N_1 Q = N_2 Q \),
(iii) \( \text{cl}(N_1) = \text{cl}(N_2) \).
Proof. Suppose that \( N_1 \sim N_2 \). For each \( n \in N_1 \) let
\[
(N_2 : n) = \{ a \in R \mid na \in N_2 \}.
\]
As in (4, p. 63) \((N_2 : n)\) is a large right ideal in \( R \). Since \( R \) is semi-prime, 
\((N_2 : n)\) contains a regular element \( c \) and hence \( N_1 \subseteq N_2 Q \). By symmetry, 
\( N_2 \subseteq N_1 Q \), which proves that (i) implies (ii).

Conversely, suppose \( N_1 Q = N_2 Q \) and let \( n \) be a non-zero element of \( N_1 \).
Then \( n = mc^{-1} \) for some \( m \in N_2 \) and \( c \) regular in \( R \). Therefore \( nR \cap N_2 \neq \emptyset \),
which implies that \( N_1 \sim N_2 \).

The equivalence of (i) and (iii) follows from the fact that if \( N_1 \sim N_2 \), then
\( N_1 \subseteq \text{cl}(N_2) \).

(4.2) Corollary. A submodule \( N \) of \( M \) is large if and only if \( NQ = MQ \).

(4.3) Proposition. If \( N_1 \) and \( N_2 \) are submodules of \( M \), then
\[
N_1 Q + N_2 Q = (N_1 + N_2)Q \quad \text{and} \quad N_1 Q \cap N_2 Q = (N_1 \cap N_2)Q.
\]
Proof. Use the method given in (3, p. 214).

(4.4) Proposition. The minimal submodules of \( MQ \) are of the form \( NQ \) where \( N \) is a uniform submodule of \( M \).

Proof. If \( NQ \) is not minimal, there exist non-zero submodules \( N_1, N_2 \) of \( M \) such that \( NQ = N_1 Q \oplus N_2 Q \). Then
\[
N_1 Q \cap N_2 Q = (N_1 \cap N_2)Q = 0
\]
or \( N_1 \cap N_2 = 0 \), so \( N \) is not uniform.

Conversely, if \( NQ \) is minimal, then \( N \) is uniform, for otherwise \( N \) contains non-trivial submodules \( N_1 \) and \( N_2 \) with \( N_1 \cap N_2 = 0 \). Then \( N_1 Q \cap N_2 Q = 0 \),
contrary to the minimality of \( NQ \).

One may easily prove

(4.5) Proposition. If \( N \) is a submodule of \( M \), then
\[
\text{cl}(N) = NQ \cap M \quad \text{and} \quad \text{cl}(NQ \cap M) = NQ \cap M.
\]

In (3, p. 202), Goldie defined the notion of complement submodule of an \( R \)-module \( M \). If \( X \) denotes the set of non-large submodules of \( M \), then a direct argument proves that the maximal complement submodules are precisely the maximal elements of \( X \). We shall term the latter maximal non-large submodules.

(4.6) Theorem. If \( N^* \) is a maximal submodule of \( MQ \), then \( N^* \cap M \) is a maximal non-large submodule of \( M \). If \( N \) is a maximal non-large submodule of \( M \), then \( NQ \) is a maximal submodule of \( MQ \). The correspondence \( N^*_i \to N^*_i \cap M \)
is a one-to-one mapping of the set \( \{ N^*_i \} \) of maximal submodules of \( MQ \) onto the set of maximal non-large submodules of \( M \).
Proof. Suppose \( N^* \) is maximal in \( MQ \) and let \( N \) be a submodule of \( M \) such that \( N \supseteq N^* \cap M \) and \( NQ \neq MQ \). Then \( N^* = (N^* \cap M)Q \subseteq NQ \subset MQ \) and we have \( N^* = NQ \). Hence \( N \subseteq N^* \cap M \), which proves that \( N^* \cap M \) is a maximal non-large submodule of \( M \).

Now suppose that \( N \) is a maximal non-large submodule of \( M \). If \( N^* \) is a \( Q \)-submodule of \( MQ \) such that \( NQ \subset N^* \subset MQ \), then \( NQ \cap M \subset N^* \cap M \) and hence \( N \subset N^* \cap M \). However, \( N^* \cap M \) is non-large since

\[
(N^* \cap M)Q = N^* \neq MQ.
\]

This contradicts the maximality of \( N \), so \( NQ \) is maximal in \( MQ \).

The last statement follows from the fact that if \( N \) is a maximal non-large submodule, then \( N = NQ \cap M \).

(4.7) Theorem. If \( N \) is a maximal non-large submodule of \( M \), then \( N \) is closed-prime and intersection-irreducible. Consequently, \( M/N \) is a uniform annihilator-prime \( R \) module.

Remark. A submodule \( N \) is intersection-irreducible if it cannot be written as an intersection of two submodules that properly contain \( N \).

Proof. From (4.6) we may write \( N = N^* \cap M \) where \( N^* \) is a maximal submodule of \( MQ \). If \( N' \) is a submodule such that \( N \subset N' \subset M \), then by the maximal nature of \( N \) and (4.2), \( N'Q = MQ \). Hence

\[
N'(N : N')Q \subseteq NQ = N^*.
\]

As in (3, (5.2)), \( (N : N')Q \) is an ideal of \( Q \) so that

\[
N'Q(N : N') \subseteq N'(N : N')Q \subseteq N^*.
\]

Therefore \( M(N : N') \subseteq N^* \cap M = N \), or \( (N : N') \subseteq (N : M) \). Thus condition (2.1), (i) is satisfied by \( N \). The fact that \( N = \text{cl}(N) \) is clear since \( N \sim \text{cl}(N) \).

If \( N \) is maximal non-large, it follows readily that \( N \) is intersection-irreducible; see (4, p.60). It is evident that this implies uniformity for \( M/N \).

(4.8) Theorem. The \( R \)-module \( M \) is semi-prime. Specifically, the zero submodule is the intersection of maximal non-large closed-prime submodules, and hence \( M \) is a subdirect sum of uniform annihilator-prime \( R \)-modules.

Proof. Since \( Q \) is semi-simple, \( MQ \) can be expressed as a direct sum of simple submodules \( \{L_i\} \). For each \( i \) let \( N_i^* = \sum_{j \neq i} L_j \). (If there is only one \( L_i \), then \( MQ \) is simple and, by convention, we take \( N_i^* = 0 \).) Then each \( N_i^* \) is a maximal submodule of \( MQ \) and \( \cap N_i^* = 0 \). By (4.6), \( N_i = N_i^* \cap M \) is a maximal non-large submodule and is closed-prime by (4.7). Then

\[
\cap N_i \subseteq \cap N_i^* = 0,
\]

which implies \( P(M) = 0 \). The last statement follows from (2.3) and (4.7).
The next result characterizes faithful semi-prime modules over Goldie rings.

(4.9) **Theorem.** Let $L$ be a faithful module over a Goldie ring $R$. Then $L$ is semi-prime if and only if $L$ is contained in a unitary right $Q$-module where $Q$ is a semi-simple Artinian ring and a right quotient ring for $R$.

**Proof.** If $L$ is semi-prime, then by (3.5) $R$ is a semi-prime ring. Moreover, since $Z(L) \subseteq P(L) = 0$, $L$ is torsion-free over $R$. Then the module $LQ$ fulfills the conditions cited in the statement of the theorem.

Conversely, if an $R$-module $L$ is contained in a unitary right $Q$-module $L'$ and if $Q$ is a right quotient ring for $R$ which is semi-simple Artinian, then by (3), $R$ is a Goldie semi-prime ring. Suppose $xc = 0$ where $x \in L$ and $c$ is regular in $R$. Then $c^{-1}$ exists in $Q$ and, since $L'$ is unitary, $x = x1 = (xc)c^{-1} = 0$. Hence $L$ is torsion-free over $R$ and, by (4.8), $L$ is semi-prime.

Goldie (3) has called a module *finite-dimensional* if every direct sum of non-zero submodules has only a finite number of terms. In our case notice that $M$ is finite-dimensional if and only if $MQ$ is finite-dimensional. This follows from the fact that given a direct sum of submodules $T_1 \oplus \ldots \oplus T_k$ in $MQ$, the sum $(T_1 \cap M) \oplus \ldots \oplus (T_k \cap M)$ is direct in $M$. Likewise a direct sum $L_1 \oplus \ldots \oplus L_k$ in $M$ induces a direct sum $L_1 Q \oplus \ldots \oplus L_k Q$ in $MQ$; cf. the proof of (3, (4.3)).

(4.10) **Proposition.** If $M$ is finitely generated, then $M$ is finite-dimensional.

**Proof.** If $x_1, x_2, \ldots, x_n$ are generators for $M$, then $x_1 1, x_2 1, \ldots, x_n 1$ are generators for $MQ$. Since $Q$ is semi-simple, this implies that $MQ$ is finite-dimensional. Then, from the preceding paragraph, $M$ is finite-dimensional.

(4.11) **Proposition.** Let $M$ be finite-dimensional and let $N$ be a non-large submodule of $M$. Then $N$ is maximal non-large if and only if $N$ is intersection-irreducible.

**Proof.** If $N$ is intersection-irreducible, then by an argument similar to (1, (2.2)), $N$ is closed. From (4, 3.9)), $\text{cl}(N)$ is a finite intersection of maximal non-large submodules and, because of its intersection-irreducibility, $N$ must equal one of these submodules. The converse follows from (4.7).

(4.12) **Theorem.** If $M$ is finite-dimensional, then $M$ is a finite subdirect sum of uniform annihilator-prime submodules.

**First proof.** By our previous remarks, $MQ$ is finite-dimensional. The proof of (4.8) now provides a finite number of maximal non-large closed-prime submodules $\{N_i\}$ with $\bigcap N_i = 0$. Then $M$ is a subdirect sum of the modules $M_i = M/N_i$. Applying (2.3) and (4.7), we have the theorem.

**Second proof.** By (4, (3.7)) we may write $0 = \bigcap N_i$ where the $N_i$ are maximal non-large submodules. The remainder of the proof is the same as that given above.
Let $M$ and $N_i$ be as in the proof of (4.12). Then

$$(0 : M) = (\cap N_i : M) = \cap (N_i : M)$$

where, by (2.7), each $(N_i : M)$ is a closed-prime ideal in $R$. If, in addition, $M$ is faithful, then the zero ideal is a finite intersection of closed-prime ideals of the form $(N_i : M)$. Applying these remarks to a Goldie semi-prime ring $R$, considered as a module over itself, we obtain the following

(4.13) THEOREM. If $R$ is a Goldie semi-prime ring, there exist a finite number of prime ideals $\{P_i\}$ such that $0 = \cap P_i$. Moreover, each $P_i$ is of the form $(J_i : R)$ where $J_i$ is a maximal non-large right ideal which is closed-prime as a right $R$-module.

We shall conclude with a characterization for Noetherian semi-prime modules.

(4.14) THEOREM. Let $R$ be a Goldie semi-prime ring. An $R$-module $L$ is Noetherian semi-prime if and only if $L$ is a finite subdirect sum of uniform Noetherian annihilator-prime $R$-modules.

Proof. If $L$ is Noetherian, it is finite-dimensional. In addition, if $L$ is semi-prime and $R$ is a Goldie semi-prime ring, then $L$ is torsion-free. Using (4.12) we see that $L$ is a finite subdirect sum of uniform annihilator-prime modules of the form $L/N_i$. It is evident that each such module is Noetherian.

Conversely, if $L$ is a subdirect sum of uniform Noetherian annihilator-prime modules $L_1, \ldots, L_k$, then there exist submodules $N_1, \ldots, N_k$ in $M$ such that $L_i \cong L/N_i$ and $\cap N_i = 0$. Since the $N_i$ are closed-prime by (2.3), we have $P(L) = 0$. The fact that $L$ is Noetherian follows as in (1, (3.3)).

References

4. ——— Rings with maximum condition (Yale University, 1961) (multilithed).

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