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GROUP C^*-ALGEBRAS AND THE SPECTRUM OF A PERIODIC SCHRÖDINGER OPERATOR ON A MANIFOLD

TOSHIKAZU SUNADA

0. **Introduction.** The spectrum of the Laplacian or more generally of a Schrödinger operator on an *open* manifold may have possibly a complicated aspect. For example, a Cantor set in the real axis may appear as the spectrum even for an innocent looking potential on a standard Riemannian manifold (see J. Moser [10]). The fundamental result of the spectral theory of *periodic* Schrödinger operators, however, says that the picture of the spectrum of a Schrödinger operator on $\mathbb{R}^n$ with a periodic potential is simple; indeed the spectrum consists of a series of closed intervals of the real axis without accumulation, separated in general by gaps outside the spectrum (see M. Reed and B. Simon [13] or M. M. Skriganov [15] for instance). For brevity, such a spectrum will be said to have the *band structure.* It turns out that the action of a lattice (i.e., a finitely generated subgroup of rank n) on $\mathbb{R}^n$ by translation imposes such a restriction on the structure of the spectrum. It seems likely that the same is true for a general periodic Schrödinger operator defined on a manifold with a *co-compact action* of a discontinuous group. For instance, if the group acting on a manifold is *abelian,* the situation is much the same as in the classical case (except for eigenvalues possibly existing, [8]). In this paper, we shall observe that the spectrum of a period Schrödinger operator on a manifold has band structure in the presence of a certain property of orthogonal projections in (matrix algebras over) the reduced C^*-algebra of the transformation group.

Let Γ be a finitely generated discrete group. We denote by $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ the tensor product of the *reduced group C^*-algebra* of Γ with the algebra of compact operators on a separable Hilbert space, and by tr_Γ the canonical trace on $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ (see § 1 for the definitions). Let X be a Riemannian manifold with an isometric, properly discontinuous Γ-action with compact quotient $\Gamma\backslash X$. Let q be a potential function on X which is smooth and *periodic* under the Γ-action.

THEOREM 1. *Suppose that there exists a positive constant C such that* $\mathrm{tr}_\Gamma P \geq C$ *for every non-trivial orthogonal projection P in $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$. Then the spectrum of the Schrödinger operator $-\Delta_X + q$ acting in $L^2(X)$ has the band structure (possibly with degenerate intervals, corresponding to isolated eigenvalues). Furthermore if the* Γ*-action*

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on X is free, then $N(\lambda)$, the number of components of the spectrum of $-\Delta_X + q$ which intersects the interval $(-\infty, \lambda]$, has the following asymptotic estimate:

$$\lim_{\lambda\to\infty} \sup \frac{CN(\lambda)}{(2\pi)^{-n}\omega_n \operatorname{vol}(\Gamma\backslash X)\lambda^{n/2}} \leqq 1,$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

A large number of discrete groups seem to satisfy the condition in Theorem 1 (cf. M. V. Pimsner [12]). An example of such a discrete group is the *free product* of a finite number of *finite* groups and *infinite cyclic* groups (in this case, a proof using the K-theory was given by C. Lance in [9]). It should be noted that, in the case of free groups, this is just a consequence of the K-theoretic formulation of the Kadison conjecture established by M. Pimsner and D. Voiculescu in [11]. For convenience of the reader (particularly, of geometers), we shall give an elementary proof of this fact in the appendix, by modifying slightly the elegant proof of the Kadison conjecture due to A. Connes and J. Cuntz, outlined skillfully by E. Effros in [7].

In the proof of Theorem 1, we shall observe that, for a general discrete group Γ, the operator semigroup $\exp\big(-t(-\Delta_X + q)\big)$, as well as the spectral projection of $-\Delta_X + q$ corresponding to a closed interval with end points in the resolvent set, lies in $C^*_{\text{red}}(\Gamma, \mathcal{K})$. It is interesting to note that this is derived from the finite propagation property for the wave equation (see § 2).

Throughout we shall follow the convention that the sign of the Laplacian, Δ_X, is such that the spectrum of $-\Delta_X$ is in the interval $[0, \infty)$. The spectrum of an operator T will be denoted by $\sigma(T)$. We shall write H_X for the self-adjoint extension of the Schrödinger operator $-\Delta_X + q$ on $L^2(X)$.

It should be noted that the theory of C^*-algebras has also been employed in the study of almost periodic Schrödinger operators in a somewhat different way (cf. J. Bellissard, P. Lima and D. Testard [3]).

A discrete (graph-theoretical) analogue of periodic Schrödinger operators can be treated in much the same way. Actually, the proof of an analogue of Theorem 1 is almost self-evident since the discrete Schrödinger operator itself lies in $C^*_{\text{red}}(\Gamma, \mathcal{K})$.

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1. **Reduced group C^*-algebras.** In this section we present the rudiments of the theory of group C^*-algebras, which are needed in the next section (see J. Diximier [3], M. Atiyah [2] and W. Arveson [1]).

Let Γ be a discrete group and let $C^*_{\text{red}}(\Gamma)$ be the reduced group C^*-algebra of Γ. We set $C^*_{\text{red}}(\Gamma, \mathcal{K}) = C^*_{\text{red}}(\Gamma) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators of a separable

Hilbert space, say V. To give a concrete description of $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$, we put

$$\begin{aligned}\mathcal{L} &= \text{ the algebra of all bounded linear operators on } V,\\ \mathcal{K} &= \text{ the set of } \textit{compact} \text{ operators in } \mathcal{L}\\ \mathcal{F} &= \text{ the set of operators } \textit{with finite rank,}\\ L^2(\Gamma, V) &= L^2(\Gamma) \otimes V = \{\, V\text{-valued square summable functions}\}.\end{aligned}$$

We regard $L^2(\Gamma, V)$ as a Γ-module by the left regular representation of Γ on $L^2(\Gamma)$ extended by the identity on V. It is known that the set

$$\begin{aligned}W^*(\Gamma, \mathcal{L}) = \{A \colon L^2(\Gamma, V) \longrightarrow L^2(\Gamma, V);\, &A \text{ a bounded linear operator with}\\ &A\sigma = \sigma A \text{ for all } \sigma \in \Gamma\}\end{aligned}$$

forms a *semifinite von Neumann algebra* of type II_∞ (not necessarily a factor). We denote by π the natural action of $W^*(\Gamma, \mathcal{L})$ on $L^2(\Gamma, V)$.

We set

$$\delta^v_\sigma(\mu) = \begin{cases} v & \text{if } \sigma = \mu\\ 0 & \text{otherwise.}\end{cases}$$

If $\{\, v_i\}_{i=1}^\infty$ is a complete orthonormal basis, then $\{\,\delta^{v_i}_\sigma : i \in \mathbb{N}, \sigma \in \Gamma\}$ forms a complete orthonormal basis of $L^2(\Gamma, V)$.

Let $A \in W^*(\Gamma, \mathcal{L})$. We define the *Fourier coefficient* $\hat{A}(\sigma) \in \mathcal{L}$ at σ by

$$\hat{A}(\sigma)v = (A\delta^v_1)(\sigma).$$

If $\hat{A}(\sigma) = 0$ for all $\sigma \in \Gamma$, then $A = 0$.

Let $C^*_0(\Gamma, \mathcal{F})$ be the set of $A \in W^*(\Gamma, \mathcal{L})$ with $\hat{A}(\sigma) \in \mathcal{F}$ and $\hat{A}(\sigma) = 0$ for all but finitely many $\sigma \in \Gamma$. We may identify $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ with the completion of $C^*_0(\Gamma, \mathcal{F})$ with respect to the operator norm. Since $\mathcal{F}$ is dense in $\mathcal{K}$ in the uniform operator topology, it follows that $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ coincides with the completion of the subalgebra

$$\begin{aligned}C^*_0(\Gamma, \mathcal{K}) = \{A \in W^*(\Gamma, \mathcal{L}) : \hat{A}(\sigma) \in \mathcal{K} &\text{ and } \hat{A}(\sigma) = 0\\ &\text{for all but finitely many } \sigma\}.\end{aligned}$$

An operator $A \in W^*(\Gamma, \mathcal{L})$ is said to be of *Γ-Hilbert-Schmidt class,* or ΓHS class, if

$$\sum_{\sigma\in\Gamma} \|\hat{A}(\sigma)\|^2_{\mathrm{HS}} < \infty,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. If there are operators B and C of ΓHS class such that $A = BC$, then A is said to be of *Γ-trace class.* If A is of Γ-trace class, then $\hat{A}(\sigma)$ is of trace class for all σ. We define the Γ-trace of A to be $\mathrm{tr}_\Gamma A = \mathrm{tr}\,\hat{A}(1)$. If $A \le B$ and B is of Γ-trace class, then so is A. If $P \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ is an orthogonal projection, then P is of Γ-trace class.

Let X be a Riemannian manifold with an isometric discontinuous Γ-action with compact quotient. To define a representation of $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ on $L^2(X)$, fix a *compact* fundamental domain $\mathcal{D}$ in X for the Γ-action, and identify $L^2(\mathcal{D})$ with V. Then the correspondence

$$L^2(X) \longrightarrow L^2(\Gamma, V)$$

given by $f \mapsto \varphi$ with $\varphi(\sigma)(x) = f(\sigma x)$, $x \in \mathcal{D}$, is a Γ-isomorphism. From now on, identifying $L^2(X)$ with $L^2(\Gamma, V)$, we regard $L^2(X)$ as a Hilbert space on which the algebra $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ acts.

Consider a Γ-equivariant operator A acting in $L^2(X)$ with a smooth kernel $k(x,y)$ such that $k(x,y) = 0$ for $x, y \in X$ with $d(x,y) \geq C > 0$ for some constant C. Then $A \in C^*_0(\Gamma, \mathcal{K})$. In fact, for $x \in X$,

$$\begin{aligned}(Af)(\sigma x) &= \int_X k(\sigma x, y) f(y)\, dy \\ &= \sum_\mu \int_{\mathcal{D}} k(\sigma x, \mu y) f(\mu y)\, dy \\ &= \sum_\mu \int_{\mathcal{D}} k(\sigma x, \mu y)\varphi(\mu)(y)\, dy \\ &= \sum_\mu \int_{\mathcal{D}} k(\mu^{-1}\sigma x, y)\varphi(\mu)(y)\, dy.\end{aligned}$$

Thus, the operator $\hat{A}(\sigma)$ acting on $L^2(\mathcal{D})$ has the kernel function $k(\sigma x, y)$. Since the set $\{\sigma \in \Gamma : d(\sigma\mathcal{D}, \mathcal{D}) < C\}$ is finite, we find that A is in $C^*_0(\Gamma, \mathcal{K})$. Moreover, A is of Γ-trace class, and

$$\operatorname{tr}_\Gamma A = \int_{\mathcal{D}} k(x,x)\, dx.$$

REMARK. It is a classical fact that, if Γ is abelian, then the correspondence

$$A \in C^*_0(\Gamma, \mathcal{K}) \longrightarrow F \in C^0(\hat{\Gamma}, \mathcal{K})$$

given by $F(\chi) = \sum \chi(\sigma)\hat{A}(\sigma)$ is extended to an isomorphism of C^*-algebras onto the space of continuous functions on the character group $\hat{\Gamma}$ of Γ with values in $\mathcal{K}$. Furthermore, we have

$$\operatorname{tr}_\Gamma A = \int_{\hat{\Gamma}} \operatorname{tr} F(\chi)\, d\chi$$

where $d\chi$ is the normalized Haar density on $\hat{\Gamma}$. Therefore, if P is an orthogonal projection in $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$, then $\operatorname{tr}_\Gamma P$ is an integral multiple of $(\#\Gamma^{\mathrm{tor}})^{-1}$ where Γ^{tor} is the torsion subgroup of Γ.

2. **Heat kernels.** We keep the situation of § 1. Let $q \in C^\infty(X)$ with $q(\sigma x) = q(x)$ for all $x \in X$ and $\sigma \in \Gamma$. Without loss of generality, we may assume that H_X is positive. Consider the spectral resolution

$$H_X = \int \lambda\, dE(\lambda).$$

Then

$$\exp(-\sqrt{-1}t\sqrt{H_X}) = \int \exp(-\sqrt{-1}\sqrt{\lambda}\, t)\, dE(\lambda)$$

is a unitary transformation of $L^2(X)$. For an *even* $f \in C_0^\infty(\mathbb{R})$, we find

$$\begin{aligned}\hat{f}(\sqrt{H_X}) &= \int \hat{f}(\sqrt{\lambda})\, dE(\lambda) \\ &= \iint \exp(-\sqrt{-1}\sqrt{\lambda}\, t) f(t)\, dt\, dE(\lambda) \\ &= 2\int_0^\infty f(t)\cos(t\sqrt{H_X})\, dt.\end{aligned}$$

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It is a standard fact that $\hat{f}(\sqrt{H_X})$ has a smooth kernel, which we denote by $k_f(x,y)$. We let $U(t,x,y)$ be the fundamental solution of the wave equation:

$$\Big(\frac{\partial^2}{\partial t^2} + H_{X,x}\Big)U(t,x,y) = 0,$$
$$U(0,x,y) = \delta_y(x),$$
$$U_t(0,x,y) = 0.$$

Then the support of $U(t,\cdot,\cdot)$ is contained in the set $\{(x,y) : d(x,y) \le t\}$ (a consequence of finite propagation speed for hyperbolic partial differential equations; see, for instance, M. E. Taylor [18]). Since $k_f(x,y) = \int f(t)U(t,x,y)\,dt$, we find that $k_f(x,y) = 0$ if $d(x,y)$ is large enough. Therefore, $\hat{f}(\sqrt{H_X}) \in C_0^*(\Gamma, \mathcal{K})$.

Note that $\hat{f}(\sqrt{H_X})$ makes sense for a rapidly decreasing function $f \in \mathcal{S}(\mathbb{R})$. For instance, if

$$f(t) = \frac{1}{(4\pi s)^{1/2}} \exp(-t^2/4s),$$

then $\hat{f}(\sqrt{H_X}) = \exp(-sH_X)$, so that $k_f(x,y)$ equals the heat kernel function $k_X(s,x,y)$. Furthermore, $\exp(-sH_X)$ is of Γ-trace class and

$$\mathrm{tr}_\Gamma \exp(-sH_X) = \int_{\mathcal{D}} k_X(s,x,x)\,dx.$$

LEMMA 1. *For an even $f \in \mathcal{S}(\mathbb{R})$, $\hat{f}(\sqrt{H_X}) \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$. In particular,* $\exp(-tH_X) \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$.

PROOF. Let $\{f_n\} \in C_0^\infty(\mathbb{R})$ a sequence of even functions with $\|f_n - f\|_1 \longrightarrow 0$. Then

$$\sup|\hat{f}_n - \hat{f}| \le \|f_n - f\|_1 \longrightarrow 0.$$

Note that if $\|\hat{g}\|_\infty < \varepsilon$, then $\|\hat{g}(\sqrt{H_X})\| \le \varepsilon$. Indeed,

$$\begin{aligned}\Big\|\int \hat{g}(\sqrt{\lambda})\,dE(\lambda)x\Big\|^2 &= \int |\hat{g}(\sqrt{\lambda})|^2\,d\|E(\lambda)x\|^2\\ &\le \varepsilon^2 \int d\|E(\lambda)x\|^2\\ &= \varepsilon^2\|x\|^2.\end{aligned}$$

This implies that

$$\hat{f}_n(\sqrt{H_X}) \longrightarrow \hat{f}(\sqrt{H_X})$$

in the uniform operator topology. Since $\hat{f}_n(\sqrt{H_X}) \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$, the assertion follows.

LEMMA 2. *Let $H_X = \int \lambda\, dE(\lambda)$ be the spectral resolution. Then each $E(\lambda)$ is of Γ-trace class.*

PROOF. We have

$$\begin{aligned}E(\mu) = \int_0^\mu dE(\lambda) &\le e^{\mu t}\int_0^\mu e^{-\lambda t}\,dE(\lambda)\\ &\le e^{\mu t}\int_0^\infty e^{-\lambda t}\,dE(\lambda) = e^{\mu t}\exp(-tH_X).\end{aligned}$$

Since $\exp(-tH_X)$ is of Γ-trace class, so is $E(\mu)$.

LEMMA 3. *Let $a < b$, and $a, b \notin \sigma(H_X)$. Then $E(b) - E(a) \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$.*

PROOF. Choose $f \in C_0(\mathbb{R})$ with

$$f = \begin{cases} 1 & \text{on } I = [e^{-bt}, e^{-at}], \\ 0 & \text{outside a sufficiently small neighborhood of } I. \end{cases}$$

Then

$$E(b) - E(a) = \int_a^b dE(\lambda) = \int f(e^{-t\lambda})\, dE(\lambda) = f\big(\exp(-tH_X)\big).$$

Given a positive ε, we may choose a polynomial p on $\mathbb{R}$ with

$$\sup_{[0,1]} |p - f| < \varepsilon.$$

Then

$$\|p\big(\exp(-tH_X)\big) - f\big(\exp(-tH_X)\big)\| < \varepsilon.$$

Since $p\big(\exp(-tH_X)\big) \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$, this means that $E(b) - E(a) \in C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$.

PROOF OF THEOREM 1. Let $\lambda > 0$. Let $a_1 < a_2 < \cdots < a_n$ be a sequence in the resolvent set such that $a_i < \lambda$ and $E(a_{i+1}) - E(a_i)$ is a nontrivial projection for all i. Since $\sum\big(E(a_{i+1}) - E(a_i)\big) \leq E(\lambda)$, one has

$$(n-1)C \leq \mathrm{tr}_\Gamma\, E(\lambda).$$

This means that $(-\infty, \lambda]$ intersects only finitely many components of the resolvent set. The rest of the statements in Theorem 1 are consequences of the following proposition.

PROPOSITION 1. *If Γ acts freely on X, then*

$$\mathrm{tr}_\Gamma\, E(\lambda) \sim \frac{\omega_n \operatorname{vol}(\Gamma \backslash X)}{(2\pi)^n} \lambda^{n/2} \text{ as } \lambda \uparrow \infty.$$

PROOF. We set $M = \Gamma \backslash X$ and $\varphi(\lambda) = \mathrm{tr}_\Gamma\, E(\lambda)$. The manifold M has a metric induced from the metric on X. Since

$$\exp(-tH_X) = \int e^{-t\lambda}\, dE(\lambda),$$

by taking the Γ-trace of both sides, we obtain,

$$\mathrm{tr}_\Gamma \exp(-tH_X) = \int e^{-t\lambda}\, d\varphi(\lambda).$$

Note the following relation between the kernel function $k_M(t, x, y)$ of $\exp(-tH_M)$ and $k_X(t, x, y)$ of $\exp(-tH_X)$:

$$k_M(t, x, y) = \sum_{\sigma \in \Gamma} k_X(t, x, \sigma y).$$

It follows that

$$\operatorname{tr}\big(\exp(-tH_M)\big) = \int_M k_M(t,x,x)\,dx = \sum_{\sigma\in\Gamma}\int_{\mathcal{D}} k_X(t,x,\sigma x)\,dx$$
$$= \operatorname{tr}_\Gamma\big(\exp(-tH_X)\big) + \sum_{\sigma\neq 1}\int_{\mathcal{D}} k_X(t,x,\sigma x)\,dx.$$

Since in the asymptotic expansion as $t\downarrow 0$, all terms in the righthand side except for $\sigma = 1$ are exponentially small, $\operatorname{tr}\big(\exp(-tH_M)\big)$ and $\operatorname{tr}_\Gamma \exp(-tH_X)$ $\big(= \int_{\mathcal{D}} k_X(t,x,x)\,dx\big)$ have the same asymptotic expansions. In particular, one has

$$\int e^{-\lambda t}\,d\varphi(\lambda) \sim (4\pi t)^{-n/2}\operatorname{vol}(M) \text{ as } t\downarrow 0.$$

The Tauberian theorem (W. Feller [21, p. 446]) leads us to the assertion.

REMARK. Let P be the projection onto an eigenspace of H_X. From Lemma 3, it follows that, if the eigenvalue is isolated, then $P\in C^*_{\mathrm{red}}(\Gamma,\mathcal{K})$. One may ask if the same is true for embedded eigenvalues. Probably this is false in general, but it is still likely that $\operatorname{tr}_\Gamma P \geq C$ for some positive constant C not depending on the choice of eigenvalues (this is actually true for abelian covering spaces; see H. Donnelly [4]).

APPENDIX. We shall prove, in an elementary way, that, if Γ is a free product of finite groups and infinite cyclic groups, then $\operatorname{tr}_\Gamma P$ is an integral multiple of a rational number for all projections P in $C^*_{\mathrm{red}}(\Gamma,\mathcal{K})$.

Let $\Gamma = \mathbb{Z}*\cdots*\mathbb{Z}*G_1*\cdots*G_n = G_0*G_1*\cdots*G_n$, where $G_0 = F_k = \langle\alpha_1,\ldots,\alpha_k\rangle$ is a free group on k-generators and G_i is a finite group for each $i = 1,\ldots,n$. We shall identify each G_i with a subgroup of Γ. By adding generators of each G_i to $\{\alpha_1,\ldots,\alpha_k\}$, we construct the *Cayley graph* associated to Γ. Given $\sigma,\mu\in\Gamma$, we denote by $d(\sigma,\mu)$ the distance between σ and μ in the graph; that is, $d(\sigma,\mu)$ is the minimum of the number of symbols in words expressing $\sigma^{-1}\mu$ in the generators and their inverses.

A special feature of the free product is that each element in Γ is expressed in a unique way as a *reduced* word, i.e., as

$$\sigma_1\sigma_2\cdots\sigma_N$$

with $\sigma_i\in G_{j(i)}\setminus\{1\}$ and $j(i)\neq j(i+1)$ (see, for instance, J. P. Serre [12]). From this fact, it follows that

$$d(\sigma_1\cdots\sigma_N,1) = \sum_{i=1}^N d(\sigma_i,1),$$

provided that $\sigma_1\cdots\sigma_N$ is a reduced word.

We let Γ_i^+ (resp. Γ_i^-) $(i\geq 1)$ be the set of elements σ represented by reduced words of the form $\sigma_1\sigma_2\cdots\sigma_N$ with $\sigma_1\in F_k$ and σ_1 being expressed in F_k by a reduced word $\alpha_i^{+n}\alpha_{j_1}^{\pm1}\cdots\alpha_{j_l}^{\pm1}$ (resp. $\sigma_1 = \alpha_i^{-n}\alpha_{j_1}^{\pm1}\cdots\alpha_{j_l}^{\pm1}$) $(j_1\neq i, n>0)$. We put $\Gamma_i = \Gamma_i^+\cup\Gamma_i^-$ and $\Gamma_0 = \{1\}$.

Given $\sigma\in G_i$ $(\neq 1)$, we define Γ_σ^+ as the set of elements represented by reduced words of the form

$$\sigma_1\cdots\sigma_N,\quad \sigma_1 = \sigma.$$

We put $\Gamma_{G_i} = \bigcup_{\sigma\in G_i\setminus\{1\}} \Gamma_\sigma^+$, and $\Gamma_\sigma^- = \Gamma_{G_i}\setminus\Gamma_\sigma^+$. We then obtain

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k \cup \Gamma_{G_1} \cup \cdots \cup \Gamma_{G_n} \quad \text{(disjoint).}$$

The following lemma is easy to check.

LEMMA 4. *$\Gamma = \alpha_i\Gamma_i^- \cup \Gamma_i^+$ (disjoint), and $\Gamma = \sigma\Gamma_{G_i} \cup \Gamma_\sigma^+$ (disjoint) for $\sigma \in G_i$.*

A key point in our proof is the following lemma.

LEMMA 5. *(1) If $\sigma \in \Gamma_i^+$ and $\mu \in \Gamma_i^-$, then*

$$d(\sigma,\mu) = d(\mu,1) + d(\sigma,1).$$

If $\sigma \in \Gamma_i^+$ and $\mu \in \alpha_i\Gamma_i^-$, then

$$d(\sigma,\mu) = d(\mu,1) + d(\sigma,1).$$

(2) Let $\sigma \in G_i$. If $\theta \in \Gamma_\sigma^+$ and $\mu \in \Gamma_\sigma^-$, then

$$d(\theta,\mu) \geq d(\theta,1) + d(1,\mu) - 2d_i,$$

where d_i is the diameter of G_i.

If $\theta \in \Gamma_{G_i}$ and $\mu \in \Gamma_0 \cup \cdots \cup \Gamma_k \cup \left(\bigcup_{j\neq i}\Gamma_{G_j}\right)$, then

$$d(\theta,\mu) = d(\theta,1) + d(\mu,1).$$

PROOF. (1) Let $\sigma = \alpha_i^x\sigma_1\cdots\sigma_s \in \Gamma_i^+$ and $\mu = \alpha_i^{-y}\mu_1\cdots\mu_t \in \Gamma_i^-$ be the shortest expressions by words in the generators and their inverses, so that $d(\sigma,1) = x+s$ and $d(\mu,1) = y+t$. Then

$$\sigma_s^{-1}\cdots\sigma_1^{-1}\alpha_i^{-x-y}\mu_1\cdots\mu_t$$

is the shortest expression for $\sigma^{-1}\mu$. Therefore, we have

$$d(\sigma,\mu) = d(\sigma^{-1}\mu,1) = x+y+s+t = d(\sigma,1) + d(\mu,1).$$

If $\mu \in \alpha_i\Gamma_i^-$ is not the identity element, then μ has the shortest expression by a word of the form

$$\mu = \mu_1\cdots\mu_t$$

where $\mu_1 \in \bigcup_{i=1}^n G_i$ or $\mu_1 =$ a power of $\alpha_j, j \neq i$, or $\mu_1 = \alpha_i^{-1}$. Hence if $\alpha_i^x\sigma_1\cdots\sigma_s$ is the shortest expression of σ, then

$$\sigma_s^{-1}\cdots\sigma_1^{-1}\alpha_i^{-x}\mu_1\cdots\mu_t$$

is the shortest expression for $\sigma^{-1}\mu$. We thus have

$$d(\sigma,\mu) = d(\sigma,1) + d(\mu,1).$$

(2) Let $\theta = \sigma\theta_1 \cdots \theta_s \in \Gamma_\sigma^+$ and $\mu = \sigma'\mu_1 \cdots \mu_t \in \Gamma_\sigma^-$ be the reduced expressions where $\sigma, \sigma' \in G_i$ and $\sigma' \neq \sigma$. If we put $\sigma'' = \sigma^{-1}\sigma'$, then

$$\theta_s^{-1} \cdots \theta_1^{-1}\sigma''\mu_1 \cdots \mu_t$$

is the reduced expression of $\theta^{-1}\mu$. Therefore we have

$$d(\theta, \mu) = d(\theta^{-1}\mu, 1) = d(\theta, 1) + d(\mu, 1) + d(\sigma, \sigma') - d(\sigma, 1) - d(\sigma', 1).$$

Since $d(\sigma, \sigma') - d(\sigma, 1) - d(\sigma', 1) \geq -2d_i$, we obtain the desired inequality.

If $\theta = \sigma\theta_1 \cdots \theta_s \in \Gamma_{G_i}$ and $\mu = \sigma'\mu_1 \cdots \mu_t \in \Gamma_1 \cup \cdots \cup \Gamma_k \cup \left(\bigcup_{j\neq i} \Gamma_{G_j}\right)$ are the reduced expressions, then

$$\theta_s^{-1} \cdots \theta_1^{-1}\sigma^{-1}\sigma'\mu_1 \cdots \mu_t$$

is the reduced expression of $\theta^{-1}\mu$. Thus we have the equality

$$d(\theta, \mu) = d(\theta, 1) + d(\mu, 1).$$

We denote by g_i the order of G_i, and by h the least common multiple of $\{g_1, \ldots, g_n\}$. We now take h copies of Γ, which we denote by

$$\Gamma(1), \Gamma(2), \ldots, \Gamma(h).$$

We also take a copy $\Gamma(\sigma, m)$ of Γ for each $\sigma \in G_i \setminus \{1\}$ and $m \in \{0, 1, \ldots, (h/g_i) - 1\}$. To each $\Gamma(\ell)$, we denote by $\Gamma_i(\ell)$, $\Gamma_i^\pm(\ell)$, $\Gamma_{G_i}(\ell)$, $\Gamma_\sigma^\pm(\ell)$ the subsets in $\Gamma(\ell)$ corresponding to Γ_i, $\Gamma_i^\pm$, Γ_{G_i}, $\Gamma_\sigma^\pm$, respectively.

Consider the disjoint union

$$\begin{aligned} \Gamma(1) \cup \cdots \cup \Gamma(h) = {} & \Gamma_0(1) \cup \cdots \cup \Gamma_0(h) \\ & \cup \Gamma_1(1) \cup \cdots \cup \Gamma_1(h) \cup \cdots \cup \Gamma_k(1) \cup \cdots \cup \Gamma_k(h) \\ & \cup \Gamma_{G_1}(1) \cup \cdots \cup \Gamma_{G_1}(h) \cup \cdots \cup \Gamma_{G_n}(1) \cup \cdots \cup \Gamma_{G_n}(h). \end{aligned}$$

We put

$$\begin{aligned} H_i &= L^2\big(\Gamma_i(1)\big) \oplus \cdots \oplus L^2\big(\Gamma_i(h)\big), \\ H_{G_i} &= L^2\big(\Gamma_{G_i}(1)\big) \oplus \cdots \oplus L^2\big(\Gamma_{G_i}(h)\big), \end{aligned}$$

so that

$$L^2\big(\Gamma(1)\big) \oplus \cdots \oplus L^2\big(\Gamma(h)\big) = H_0 \oplus H_1 \oplus \cdots \oplus H_h \oplus H_{G_1} \oplus \cdots \oplus H_{G_n}.$$

For $i \geq 1$, we define the map $\Gamma_i(\ell) \longrightarrow \Gamma(\ell) = \Gamma$ which agrees with the inclusion $\Gamma_i^+(\ell) \subset \Gamma(\ell)$ on $\Gamma_i^+(\ell)$, and agrees with the map $\sigma \longmapsto \alpha_i\sigma$ on $\Gamma_i^-(\ell)$. It is clear (see Lemma 4) that this map is a bijection. Using this map, we have an identification $H_i = \oplus h \cdot L^2(\Gamma)$.

Let $m \in \{0, 1, \ldots, (h/g_i) - 1\}$. We define a bijection

$$\Phi\colon \Gamma_{G_i}(mg_i + 1) \cup \Gamma_{G_i}(mg_i + 2) \cdots \cup \Gamma_{G_i}(mg_i + g_i) \simeq \bigcup_{\sigma \in G_i \setminus \{1\}} \Gamma(\sigma, m)$$

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in the following way. On $\Gamma_\sigma^+(mg_i+1)$, Φ is defined to be the inclusion:

$$\Gamma_\sigma^+(mg_i+1) \simeq \Gamma_\sigma^+ \subset \Gamma \simeq \Gamma(\sigma, m).$$

On $\Gamma_{G_i}(mg_i+j)$, $2 \leq j \leq g_i$, we define Φ by setting

$$\Phi(\mu) = \sigma_j\mu \in \sigma_j\Gamma_{G_i}(mg_i+j) \simeq \sigma_j\Gamma_{G_i} \subset \Gamma \simeq \Gamma(\sigma_j, m),$$

where $G_i \backslash \{1\} = \{\sigma_2, \ldots, \sigma_{g_i}\}$. In view of Lemma 4, we find that Φ is bijection. Thus we have a bijection

$$\bigcup_{\ell=1}^{h} \Gamma_{G_i}(\ell) \simeq \bigcup_{m=0}^{(h/g_i)-1} \bigcup_{\sigma \in G_i \backslash \{1\}} \Gamma^\sigma,$$

which gives rise to an identification

$$H_{G_i} = \oplus\{(h/g_i) - 1\}(g_i - 1) \cdot L^2(\Gamma).$$

We shall define two representations ρ_0 and ρ_1 of $W^*(\Gamma, \mathcal{L})$ on the Hilbert space

$$\Big(L^2\big(\Gamma(1)\big) \oplus \cdots \oplus L^2\big(\Gamma(h)\big)\Big) \otimes V$$

in the following way. The first one, ρ_0, is defined as the direct sum of the natural representation π of $W^*(\Gamma, \mathcal{L})$ on $L^2(\Gamma, V)$. The second one, ρ_1, is defined by

$$\rho_1 \equiv \begin{cases} 0 & \text{on } H_0 \otimes V \simeq \mathbb{C}^h \otimes V, \\ \text{the direct sum of copies of } \pi & \text{on } H_i \otimes V = \oplus h \cdot L^2(\Gamma, V), \\ \text{the direct sum of copies of } \pi & \text{on } H_{G_i} \otimes V = \oplus\{(h/g_i) - 1\}(g_i - 1) \\ & \qquad \cdot L^2(\Gamma, V). \end{cases}$$

From now on, we set $a(\sigma) = \hat{A}(\sigma)$ for brevity.

LEMMA 6. *Let $\sigma \in \Gamma_i^+(\ell)$, so that $\delta_\sigma^v \in H_i$. Then*

$$(1) \qquad \rho_1(A)\delta_\sigma^v = \sum_{\mu \in \Gamma_i^+(\ell)} \delta_\mu^{a(\sigma^{-1}\mu)v} + \sum_{\mu \in \Gamma_i^-(\ell)} \delta_\mu^{a(\sigma^{-1}\alpha_i\mu)v}.$$

If $\sigma \in \Gamma_i^-$, then

$$(2) \qquad \rho_1(A)\delta_\sigma^v = \sum_{\mu \in \Gamma_i^+(\ell)} \delta_\nu^{a(\sigma^{-1}\alpha_i^{-1}\mu)v} + \sum_{\mu \in \Gamma_i^-(\ell)} \delta_\mu^{a(\sigma^{-1}\mu)v}.$$

PROOF. Define the unitary operator $U: L^2(\Gamma_i, V) \longrightarrow L^2(\Gamma, V)$ by

$$U(\delta_\sigma^v) = \begin{cases} \delta_\sigma^v & \text{if } \sigma \in \Gamma_i^+(\ell), \\ \delta_{\alpha_i\sigma}^v & \text{if } \sigma \in \Gamma_i^-(\ell). \end{cases}$$

Then, on $L^2\big(\Gamma_i(\ell), V\big)$,

$$\rho_1(A) = U^{-1}\pi(A)U.$$

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Therefore one has, if $\sigma \in \Gamma_i^+(\ell)$,

$$\begin{aligned}\rho_1(A)\delta_\sigma^v &= U^{-1}\rho(A)U\delta_\sigma^v = U^{-1}\rho(A)\delta_\sigma^v\\ &= U^{-1}\Big(\sum_{\mu\in\Gamma(\ell)}\delta_\mu^{a(\sigma^{-1}\mu)v}\Big)\\ &= U^{-1}\Big(\sum_{\mu\in\Gamma_i^+(\ell)}\delta_\mu^{a(\sigma^{-1}\mu)v}\Big)+U^{-1}\Big(\sum_{\mu\in\Gamma_i^-(\ell)}\delta_{\alpha_i\mu}^{a(\sigma^{-1}\alpha_i\mu)v}\Big)\\ &= \sum_{\mu\in\Gamma_i^+(\ell)}\delta_\mu^{a(\sigma^{-1}\mu)v}+\sum_{\mu\in\Gamma_i^-(\ell)}\delta_\mu^{a(\sigma^{-1}\alpha_i\mu)v}.\end{aligned}$$

Similarly one gets the equality (2).

In particular, if $\sigma \in \Gamma_i^+$ and $d(\sigma,1)$ is large enough, the, in view of Lemma 5 (1), we have

$$a(\sigma^{-1}\mu)=0, \text{ and } a(\sigma^{-1}\alpha_i\mu)=0$$

for every $\mu \in \Gamma_i^-$ and

$$(a)\qquad \begin{aligned}\rho_1(A)\delta_\sigma^v &= \sum_{\mu\in\Gamma_i^+(\ell)}\delta_\mu^{a(\sigma^{-1}\mu)v}\\ &= \sum_{\mu\in\Gamma_i^+(\ell)}\delta_\mu^{a(\sigma^{-1}\mu)v}+\sum_{\mu\in\Gamma_0(\ell)\cup\Gamma_i^-(\ell)}\delta_\mu^{a(\sigma^{-1}\mu)v}\\ &= \rho_0(A)\delta_\sigma^v.\end{aligned}$$

Similarly, for $\sigma \in \Gamma_i^-$ with sufficiently large $d(\sigma,1)$,

$$(b)\qquad \rho(A)\delta_\sigma^v = \rho_0(A)\delta_\sigma^v.$$

In a similar way as in the proof of the above lemma, we may prove

LEMMA 7. *Let $\sigma=\sigma_j \in G_i$. We put $\ell = mg_i+1$ and $\ell' = mg_i+j$.*
(1) If $\theta \in \Gamma_\sigma^+(\ell)$ (so that $\delta_\theta^v \in H_{G_i}$), then

$$\rho_1(A)\delta_\theta^v = \sum_{\mu\in\Gamma_\sigma^+(\ell)}\delta_\mu^{a(\theta^{-1}\mu)v}+\sum_{\mu\in\Gamma_{G_i}(\ell')}\delta_\mu^{a(\theta^{-1}\sigma\mu)v}.$$

(2) If $\theta \in \Gamma_{G_i}(\ell')$, then

$$\rho_1(A)\delta_\theta^v = \sum_{\mu\in\Gamma_\sigma^+(\ell)}\delta_\mu^{a(\theta^{-1}\sigma^{-1}\mu)v}+\sum_{\mu\in\Gamma_{G_i}(\ell')}\delta_\mu^{a(\theta^{1}\mu)v}.$$

REMARK. In the above, the operations $\theta^{-1}\mu$, $\theta^{-1}\sigma\mu$, $\theta^{-1}\sigma^{-1}\mu$ are executed in $\Gamma(\sigma,m)$.

Now if θ is as above in (1) and if $d(\theta,1)$ is large enough, then

$$a(\theta^{-1}\sigma\mu)=0$$

for every $\mu \in \Gamma_{G_i}(\ell')$, and

$$a(\theta^{-1}\mu)=0$$

for every $\mu \in \sigma\Gamma_{G_i}(\ell) = \Gamma_\sigma^-(\ell) \cup \left(\bigcup_{j=0}^h \Gamma_j(\ell)\right) \cup \left(\bigcup_{j\neq i} \Gamma_{G_j}(\ell)\right)$ (see Lemma 5, (2)). Thus,

$$(c) \qquad \rho_1(A)\delta_\theta^v = \rho_0(A)\delta_\theta^v.$$

Similarly, we find that if $\theta \in \Gamma_{G_i}(\ell')$ and $d(\theta, 1)$ is sufficiently large, then we get the same equality as (c).

Putting (a) (b) (c) together, we are led to the following lemma.

LEMMA 8. *If $A \in C_0^*(\Gamma, \mathcal{F})$, then $\rho_0(A)$ and $\rho_1(A)$ coincides on the complement of a finite dimensional space, so that the operator $\rho_0(A) - \rho_1(A)$ is of trace class.*

It is easy to check that, for $A \in W^*(\Gamma, \mathcal{L})$

$$\langle \rho_0(A)\delta_\sigma^v, \delta_\sigma^v\rangle = \langle a(1)v, v\rangle \quad \sigma \in \Gamma(\ell)$$

and from Lemma 6 and Lemma 7, we deduce that

$$\langle \rho_1(A)\delta_\sigma^v, \delta_\sigma^v\rangle = \begin{cases} \langle a(1)v, v\rangle & \sigma \neq 1 \\ 0 & \sigma = 1. \end{cases}$$

Therefore, if $A \in C_0^*(\Gamma, \mathcal{F})$, then

$$\begin{aligned} \operatorname{tr}\big(\rho_0(A) - \rho_1(A)\big) &= \sum_{\ell=1}^h \sum_{i=1}^\infty \sum_{\sigma\in\Gamma(\ell)} \langle \big(\rho_0(A) - \rho_1(A)\big)\delta_\sigma^{v_i}, \delta_\sigma^{v_i}\rangle \\ &= \sum_{\ell=1}^h \sum_{i=1}^\infty \langle a(1)v_i, v_i\rangle \\ &= h \cdot \operatorname{tr}_\Gamma A. \end{aligned} \tag{3}$$

If we put $\mathcal{A} = \{A \in W^*(\Gamma, \mathcal{L}) : \rho_0(A) - \rho_1(A) \text{ is of trace class}\}$, then (3) is valid for $A \in \mathcal{A}$. What we have proved above is that $C_0^*(\Gamma, \mathcal{F}) \subset \mathcal{A}$.

We now let $P \in C_{\mathrm{red}}^*(\Gamma, \mathcal{K})$ be a projection. Given a positive ε, we may find a self-adjoint $A \in C_0^*(\Gamma, \mathcal{F})$ such that $\|A - P\| < \varepsilon$. Note that the spectrum of A is located near the two points $\{0, 1\}$, providing ε is small enough. Let C be a circle in $\mathbb{C}$ with the centre 1 which does not intersect the spectrum of A. Then

$$E = \frac{1}{2\pi i}\oint_C \frac{1}{z - A}\, dz$$

is a projection, and $\|E - P\| < 1$ for sufficiently small ε. We observe that

$$\rho_0\big((z-A)^{-1}\big) - \rho_1\big((z-A)^{-1}\big) = \big(z - \rho_0(A)\big)^{-1}\big(\rho_0(A) - \rho_1(A)\big)\big(z - \rho_1(A)\big)^{-1},$$

so that $(z - A)^{-1}$ is in $\mathcal{A}$.

The function $z \longmapsto \big(z - \rho_0(A)\big)^{-1}\big(\rho_0(A) - \rho_1(A)\big)\big(z - \rho_1(A)\big)^{-1}$ on the circle C with value in $\mathcal{C}_1$ is continuous. Here $\mathcal{C}_1$ is the Banach space of operators of trace class acting in $L^2(\Gamma, V)$. The norm $\|\cdot\|_1$ is defined by

$$\|T\|_1 = \operatorname{tr}(T^*T)^{1/2}.$$

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In fact, it is enough to observe that, if $T \in \mathcal{C}_1$, then for bounded operators S, S_0, U, U_0 of $L^2(\Gamma, V)$

$$\begin{aligned}\|STU - S_0TU_0\|_1 &= \|(S - S_0)TU + S_0T(U - U_0)\|_1 \\ &\leq \|S - S_0\| \, \|T\|_1 \|U\| + \|S_0\| \, \|T\|_1 \|U - U_0\|.\end{aligned}$$

Thus the integral

$$\frac{1}{2\pi i} \oint_C \big(z - \rho_0(A)\big)^{-1} \big(\rho_0(A) - \rho_1(A)\big) \big(z - \rho_1(A)\big)^{-1} dz$$

exists and in $\mathcal{C}_1$. This means that E is a projection in $\mathcal{A}$. Further $h \cdot \mathrm{tr}_\Gamma(E) = \mathrm{tr}\big(\rho_0(E) - \rho_1(E)\big) \in \mathbb{Z}$ since $\rho_0(E)$ and $\rho_1(E)$ are projections. On the other hand, we have

$$\mathrm{tr}_\Gamma E = \mathrm{tr}_\Gamma P.$$

In fact the estimate $\|E - P\| < 1$ guarantees that the restriction $E|$ Image P: Image $P \longrightarrow$ Image E is injective and has a dense image. The polar decompostion of $E|$ Image P allows us to construct a partial isometry $U \in W^*(\Gamma, \mathcal{L})$ with $E = U^*U$ and $P = UU^*$.

We therefore have proved

PROPOSITION 2. *Let $\Gamma = F_k * G_1 * \cdots * G_n$, where $\#G_i = g_i < \infty$. If we put $h =$* l.c.m.$\{g_1, \ldots, g_n\}$, *then* $\mathrm{tr}_\Gamma P \in h^{-1}\mathbb{Z}$ *for every orthogonal projection in $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$.*

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Department of Mathematics
Nagoya University
Nagoya 464
Japan

Present address:

Department of Mathematics
University of Tokyo
Tokyo 113
Japan