A BASE NORM SPACE WHOSE CONE IS NOT 1-GENERATING

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Let E be an ordered Banach space with closed positive cone C. A base for C is a convex subset K of C with the property that every non-zero element of C has a unique representation of the form λk with $\lambda > 0$ and $k \in K$. Let S be the absolutely convex hull of K. If the Minkowski functional of S coincides with the given norm on E, then E is called a base norm space. Then K is a closed face of the unit ball of E, and S contains the open unit ball of E. Base norm spaces were first defined by Ellis [**5**, p. 731], although the special case of dual Banach spaces had been studied earlier by Edwards [**4**].

Some authors [**1**, p. 77] also insist that S be radially compact. This is the same as insisting that S coincides with the closed unit ball of E. When this is the case, we will call E a strong base norm space. The purpose of this note is to show that these two definitions are distinct. Although this is essentially a problem concerning real Banach spaces, our solution requires the use of conformal mappings.

Before producing our counterexample, we recall some elementary definitions and results. The positive cone C of an ordered Banach space E is said to be λ-generating if, given $x \in E$, we can find $y, z \in C$ such that $x = y - z$ and $\|y\| + \|z\| \leq \lambda \|x\|$. The norm on E will be called regular (respectively, strongly regular) if, given x in the open (respectively, closed) unit ball of E, we can find y in the closed unit ball with $y \geq x$ and $y \geq -x$. The norm is said to be additive on C if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in C$.

If C is 1-generating, it is easy to show that E is strongly regular. Similarly, if C is λ-generating for all $\lambda > 1$, then E is regular. Additivity of the norm gives the converses to these observations. The proofs are routine.

PROPOSITION 1 [**8**, p. 90]. *If E is an ordered Banach space with closed positive cone C, the following are equivalent:*

(i) *E is a base norm space;*
(ii) *E is regular, and the norm is additive on C;*
(iii) *C is λ-generating for all $\lambda > 1$, and the norm is additive on C.*

PROPOSITION 2. *If E is an ordered Banach space with closed positive cone C, the following are equivalent:*

(i) *E is a strong base norm space;*
(ii) *E is strongly regular, and the norm is additive on C;*
(iii) *C is 1-generating, and the norm is additive on C.*

To show that a base norm space need not be a strong base norm space, it suffices to exhibit a base norm space which is not strongly regular. Simultaneously, this will show that a cone which is λ-generating for all $\lambda > 1$ need not be 1-generating. Another such cone has recently been given by Asimow and Ellis [**3**, Example 2.1.6]. However, their example is not a base norm space, since its norm is not additive on the positive cone.

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EXAMPLE. Let A be the disc algebra, i.e. the sup-normed space of complex-valued functions which are continuous on the closed unit disc, and analytic on the open unit disc [7, Chapter 6]. A result of Hirsberg [6, Theorem 3.1] asserts that $M=\{f\in A: f(1)=0\}$ is an M-ideal in A. The definition of M-ideals need not bother us; what is important is that M-ideals have the 2-ball property [2, Theorem 5.8]. This means that if B_1 and B_2 are closed balls in A, whose intersection has an interior point, and each ball meets M, then $M\cap B_1\cap B_2$ is not empty.

Let $E=\{f\in A: f(1)\in\mathbb{R}\}$. Then E is a real Banach space, which we can partially order by taking as the positive cone

$$C=\{f: f(1)=\|f\|\}=\{f: f(1)\geq\|f\|\}.$$

It is clear that the norm is additive on C, and that M has the 2-ball property in E.

To show that E is a base norm space, we need only establish regularity. So choose $f\in E$ with $\|f\|<1$. Let e denote the constant function $e(z)=1$, and let $B_\pm=B(e\pm f, 1\pm f(1))$. Then $f-f(1)e\in M\cap B_+$ and $f(1)e-f\in M\cap B_-$. Since $e-f(1)f$ is an interior point of $B_+\cap B_-$, the 2-ball property gives us a function $h\in M\cap B_+\cap B_-$. Let $g=e-h$. Then $g(1)=1$ and $\|g\pm f\|\leq 1\pm f(1)$. Hence $g\pm f\geq 0$, and $\|g\|\leq \frac{1}{2}\|g+f\|+\frac{1}{2}\|g-f\|\leq 1$.

To show E is not strongly regular, define $f\in M$ by $f(z)=w(w(z)^{1/2})$, where $w(z)=(i-z)/(1-iz)$. We show that $\|g\|>1$ whenever $g\geq\pm f$. First note that f maps the disc $\{z\in\mathbb{C}: |z|\leq 1\}$ onto $\{z: |z|\leq 1 \text{ and } \operatorname{re} z\leq 0\}$, and that the arc $\{z: |z|=1 \text{ and } \operatorname{re} z\leq 0\}$ is mapped onto itself. It follows [7, pp. 138–139] that f is an extreme point of the unit ball of A. Now suppose that $g\in E$ and $g\pm f\geq 0$. Clearly f cannot be a scalar multiple of $f+g$ or $f-g$, since $f\notin -C\cup C$. However $f=\frac{1}{2}(f+g)+\frac{1}{2}(f-g)$, and f is an extreme point of the unit ball of E. Thus $\|\frac{1}{2}(f+g)\|+\|\frac{1}{2}(f-g)\|>1$, and so $\|g\|=\|\frac{1}{2}(g+f)\|+\|\frac{1}{2}(g-f)\|>1$.

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