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EXCLUSION REGIONS FOR EIGENVALUES OF LINEAR OPERATORS

STEPHEN C. HENNAGIN and PETER LINZ

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Abstract

The question of the location of the eigenvalues of a linear operator is considered. In particular, a numerical technique is developed which can be used to demonstrate the absence of eigenvalues in certain segments of the real line.

1. Introduction

The eigenvalue problem

$$Lu = \lambda u \tag{1}$$

arises in many applications and is therefore of considerable interest in numerical analysis. For approximate computation the problem is generally replaced by its discretized version

$$L_n \mathbf{u}_n = \Lambda \mathbf{u}_n, \tag{2}$$

where L_n is an $n \times n$ matrix. The basic problem is to determine the relation between the eigenvalues of L and those of L_n. Much of the work that has been done in this connection is concerned with *convergence analysis*; that is, if $\lambda \in \sigma(L)$, one shows that for sufficiently large n there is a $\Lambda \in \sigma(L_n)$ arbitrarily close to λ. A variety of results for integral operators can be found in Anselone [1], Atkinson [2] and Spence [12], while Chatelin [4] and Grigorieff [5] present a more general theory.

Convergence analysis is the first question to consider since it gives us some assurance that the method works. The associated order of convergence gives some indication of effectiveness of the method, but it rarely provides a way for computing useful error estimates. Thus the need arises for easily *computable* and *realistic error bounds.* For integral equations Wielandt [15] and Brakhage [3] provide some answers, while Hubbard [7], Wendroff [14], Weinberger [13] and Kuttler [9]

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consider certain differential operators. A general approach is described in [10], where it is shown how to use the computed eigenvector $\mathbf{u}_n$ to find an ε such that $[\Lambda-\varepsilon, \Lambda+\varepsilon]$ contains at least one eigenvalue of L. Some examples indicate that the ε so computed is quite realistic, that is, it is of the same order of magnitude as the actual error.

These results, while useful, are incomplete since the question of the exact number of eigenvalues in an interval is left open. Of particular concern is whether the absence of an eigenvalue of L_n in some interval can be used to prove that L does not have any eigenvalues in a corresponding interval. This is the problem we consider here by examining *exclusion regions*, defined to be intervals of the real line known not to contain eigenvalues of L. The more difficult question of the exact multiplicity of a known eigenvalue is beyond the scope of this paper, but some results may be found in Hennagin [6].

We will make a few simplifying assumptions, which, in the light of the proposed applications, are quite reasonable. It will be assumed that L is a linear operator with domain and range in some (generally infinite-dimensional) linear space X. For the purpose of discussion we take X to be $C[0, 1]$, but obviously other choices are possible. We also assume that, with the possible exception of $\lambda=0$, the spectrum $\sigma(L)$ contains only eigenvalues, that is, it is a pure point spectrum $\sigma_p(L)$. Furthermore, we consider only the case of real eigenvalues. The matrices L_n will be assumed to be symmetric or symmetrizable, which is a natural requirement in light of the assumptions on L.

To connect the space X with R^n we use linear operators $r_n\colon X \to R^n$, called *restrictions*. A general discussion and motivation for these operators can be found in Linz [11]; here we simply define r_n by

$$r_n x = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_n) \end{bmatrix}, \tag{3}$$

where $t_1, t_2, \ldots, t_n$ are distinct points in $[0, 1]$.

2. Computation of exclusion regions

The following theorem is used to develop results on exclusion regions. The theorem first appeared in Keller [8], where it was used to investigate the spectrum of differential and integral operators. For our discussion the norm on R^n will be defined as

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{n} (x_i)^2.$$

THEOREM 1. *Let L_n be a matrix which can be symmetrized by the transformation $D_n L_n D_n^{-1}$. Let $\lambda \in \sigma_p(L)$ and let u be an eigenfunction corresponding to λ. Then there exists at least one $\Lambda \in \sigma(L_n)$ such that*

$$|\Lambda - \lambda| \leqslant \beta_n \frac{\| \delta_n(u) \|}{\| r_n(u) \|}, \tag{4}$$

where

$$\delta_n(u) = r_n Lu - L_n r_n u, \tag{5}$$

and

$$\beta_n = \| D_n^{-1} \| \, \| D_n \|.$$

PROOF. Substituting (1) into (5) we have

$$\delta_n(u) = \lambda r_n u - L_n r_n u,$$

$$D_n \delta_n(u) = \lambda D_n r_n u - S_n D_n r_n u,$$

where $S_n = D_n L_n D_n^{-1}$. If $\lambda \in \sigma(L_n)$ there is nothing to prove. Otherwise, since $\sigma(L_n) = \sigma(S_n)$, $(\lambda I - S_n)$ is invertible and

$$D_n r_n u = (\lambda I - S_n)^{-1} D_n \delta_n(u).$$

Since S_n is symmetric it follows that

$$\| (\lambda I - S_n)^{-1} \| = \Big(\min_{\Lambda \in \sigma(S_n)} |\lambda - \Lambda| \Big)^{-1}.$$

Thus

$$\| r_n u \| \leqslant \| D_n^{-1} \| \, \| D_n \| \, \| \delta_n(u) \| \Big(\min_{\Lambda \in \sigma(L_n)} |\lambda - \Lambda| \Big)^{-1},$$

and (4) follows.

This result is similar to one stated in [10]. However, in [10] the factor β_n was erroneously omitted so that the results there hold only if L_n is symmetric.

The difficulty in applying Theorem 1 lies in the quantity $\alpha = \| \delta_n(u) \| / \| r_n(u) \|$, since it involves the unknown eigenfunction u. For many problems of interest α can be bounded above by a function which depends on the eigenvalue λ but not on the eigenfunction. Therefore, we assume the existence of a function $F(\lambda)$ such that

$$\beta_n \frac{\| \delta_n(u) \|}{\| r_n(u) \|} \leqslant F(\lambda), \tag{6}$$

for each $\lambda \in \sigma_p(L)$, where $Lu = \lambda u$. Note that the linearity of δ_n and r_n makes $F(\lambda)$ independent of the normalization of u.

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THEOREM 2. *Let $a \leqslant b$ be any numbers and define*

$$\eta = \sup\{F(\mu)\colon a \leqslant \mu \leqslant b\}. \tag{7}$$

If L_n has no eigenvalues in $[a-\eta, b+\eta]$ then L has no eigenvalues in $[a,b]$.

PROOF. Assume that there exists a $\lambda \in \sigma_p(L)$ such that $a \leqslant \lambda \leqslant b$. By Theorem 1 this implies that there exists a $\Lambda \in \sigma(L_n)$ such that

$$\begin{aligned} |\Lambda - \lambda| &\leqslant \beta_n \| \delta_n(u) \| / \| r_n(u) \| \\ &\leqslant F(\lambda) \\ &\leqslant \eta. \end{aligned}$$

Thus there must exist a $\Lambda \in \sigma(L_n)$ such that

$$\lambda - \eta \leqslant \Lambda \leqslant \lambda + \eta,$$

and, since $a \leqslant \lambda \leqslant b$,

$$a - \eta \leqslant \Lambda \leqslant b + \eta.$$

This contradiction proves the theorem.

In order to apply this theorem, we must be able to establish a relation between λ and $\| \delta_n(u) \| / \| r_n(u) \|$ through (6). We now present some examples to show how this can be done.

3. Some numerical examples

EXAMPLE 1. The integral operator eigenvalue problem

$$Lx(s) = \int_0^1 e^{st} x(t)\, dt = \lambda x(s) \tag{8}$$

has been used as an example by several authors (for instance, Brakhage [3] and Linz [10]). To approximate the eigenvalues of L we use equidistant points $t_i = (i-1)h$, $h = 1/(n-1)$, replace the integral by Simpson's rule on these points, and satisfy the equation at t_i, $i = 1, 2, \ldots, n$. The resulting matrix is easily seen to be symmetrizable with

$$D_n = \begin{bmatrix} 1 & & & & & & \\ & 2 & & & & & \\ & & \sqrt{2} & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot\ 2 & \\ & & & & & & 1 \end{bmatrix}.$$

Thus $\beta_n = 2$. With $n = 21$ the two largest eigenvalues of L_n are 1.35303 and 0.10598.

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To determine exclusion regions we obtain an upper bound on $\| \delta_n(u) \|$ and a lower bound on $\| r_n(u) \|$. First, from the definition of δ_n we have

$$(\delta_n u)_i = \int_0^1 \exp(t_i t)\, u(t)\, dt - h \sum_{j=1}^{n} w_j \exp(t_i t_j)\, u(t_j), \tag{9}$$

where $(\delta_n u)_i$ denotes the ith component of $\delta_n(u)$ and $\{w_i\} = \frac{1}{6}\{1, 4, 2, \ldots, 4, 1\}$ is the set of Simpson's rule weights. From elementary results on Simpson's rule we have

$$\begin{aligned} |(\delta_n u)_i| &\leqslant \frac{h^4}{180} \left| \frac{d}{dt^4} \{\exp(t_i t)\, u(t)\} \right| \\ &\leqslant \frac{eh^4}{180} \{|u^{(\mathrm{iv})}(t)| + |4u'''(t)| + |6u''(t)| + |4u'(t)| + |u(t)|\}. \end{aligned} \tag{10}$$

To bound the various derivatives of u we simply differentiate (8) repeatedly to obtain

$$|u^{(i)}(t)| \leqslant \frac{1}{\lambda} e \| u \|_\infty, \quad i = 1, 2, \ldots .$$

With u normalized by $\| u \|_\infty = 1$, the inequality (10) becomes

$$|(\delta_n u)_i| \leqslant \frac{16e^2 h^4}{180\lambda},$$

so that

$$\| \delta_n(u) \| \leqslant 0.66 h^4 \sqrt{(n)}/\lambda. \tag{11}$$

To bound $\| r_n(u) \|$, we use the assumed normalization $\| u \|_\infty = 1$ to conclude that

$$\begin{aligned} |u(t_i)| &\geqslant 1 - \tfrac{1}{2} h \| u' \|_\infty \\ &\geqslant 1 - \frac{eh}{2\lambda}, \quad \text{for some } i. \end{aligned}$$

Thus

$$\begin{aligned} \| r_n(u) \| &= \left(\sum_{i=1}^{n} u(t_i)^2 \right)^{\frac{1}{2}} \\ &\geqslant 1 - \frac{eh}{2\lambda}. \end{aligned}$$

This completes the steps necessary to compute $F(\lambda)$ and we get

$$F(\lambda) \leqslant \frac{1.32 h^4 \sqrt{n}}{\lambda(1 - eh/2\lambda)}.$$

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With $n=21$, $h=0.05$ we now apply Theorem 2, choosing $a=1.35307$ and $b=+\infty$. This yields

$$\eta<3.0\times 10^{-5}.$$

Since the computations showed no eigenvalues of L_n in $(1.353031, +\infty)$ we can conclude that L has no eigenvalues in $(1.35307, +\infty)$.

Next, if we take $a=0.107$, $b=1.352$, we find that

$$\eta<1.01\times 10^{-3},$$

indicating that there are no eigenvalues of L in $(0.107, 1.352)$.

The details of computation carried out above were deliberately kept simple in order not to obscure the underlying idea. Somewhat lengthier computations yielding more precise answers can be found in [6].

EXAMPLE 2. The technique is also applicable to differential equations. For example, to approximate the eigenvalues of

$$\left.\begin{aligned} u'' &= \lambda u, \\ u(0) &= u(1) = 0, \end{aligned}\right\} \tag{12}$$

we can use the standard technique of replacing the second derivative by a three-point centred difference. The error in such an approximation is bounded by $h^2 \| u^{(\text{iv})} \|_\infty / 12$, so that we can find $F(\lambda)$ by expressing $u^{(\text{iv})}$ in terms of u and λ. This can be done simply by differentiating (12); the rest of the computations are then analogous to those shown in Example 1. This example is of course too simple to be of practical interest. However, the technique can be applied to more general cases, such as the Sturm–Liouville equation. The computation of bounds on the higher derivatives requires a few manipulations, but is manageable. A detailed discussion may be found in [6].

4. Conclusions

We have shown how exclusion regions can be found for eigenvalues of linear operators. This, together with the technique outlined in [10] for finding errors in computed eigenvalues, allows a complete description of the spectrum of certain linear operators. The method presented here is useful whenever the function $F(\lambda)$ defined in (6) can be found. This is generally the case whenever higher derivatives of the eigenfunctions u can be expressed in terms of λ and u. For integral equations and ordinary differential and integro-differential equations this requirement is usually met by elementary techniques, as demonstrated by the examples. For partial differential operators we know of no comparable method. Thus applicability is restricted to special cases where bounds for the higher derivatives of the eigenfunctions are known.

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Department of Mathematics
St. Mary's College
Moraga
California
U.S.A.
and
Department of Mathematics
University of California
Davis
California
U.S.A.