S. Mori
Nagoya Math. J.
Vol. 98 (1985), 43–66

ON 3-DIMENSIONAL TERMINAL SINGULARITIES

SHIGEFUMI MORI

Introduction

Canonical and terminal singularities are introduced by M. Reid [5], [6]. He proved that 3-dimensional terminal singularities are cyclic quotient of smooth points or cDV points [6].

Let (X, p) be a 3-dimensional terminal singularity of index m with the associated Z_m-cover $(\tilde{X}, \tilde{p}) \to (X, p)$. If (X, p) is a cyclic quotient singularity (i.e. if $(\tilde{X}, \tilde{p})$ is smooth), then it is known as Terminal Lemma (Danilov [3], D. Morrison-G. Stevens [4]) that there exist an integer a prime to m and coordinates x, y, z of $(\tilde{X}, \tilde{p})$ which are Z_m-semi-invariants such that $\sigma(x) = \zeta x$, $\sigma(y) = \zeta^{-1} y$, $\sigma(x) = \zeta^a z$ for the standard generator σ of Z_m, where ζ is a primitive m-th root of 1. In this paper, we consider the case where $(\tilde{X}, \tilde{p})$ is a singular point and $m > 1$. The main results are Theorems 12, 23, 25 and Remarks 12.2, 23.1, 25.1. These, together with the Terminal Lemma above, almost classify 3-dimensional terminal singularities.

Since $(\tilde{X}, \tilde{p})$ is an isolated singularity (or smooth) and is a hypersurface defined by a Z_m-semi-invariant power series (say φ), all deformations of (X, p) are induced by deformations of φ as a Z_m-semi-invariant power series [2, §§ 9–10]. By Theorems 12, 23 and 25, one can see that there is a semi-invariant coordinate which has the same character as φ (e.g. z in Theorem 12, (1)), and hence every terminal singularity can be deformed to a cyclic quotient singularity (e.g. by $\varphi + \lambda z$ with parameter λ for the case Theorem 12, (1)). This is not necessarily the case with canonical singularities.

The author expresses his hearty thanks to Professor S. Iitaka who allowed him to use his word processor, and to Professor M. Ishida for his helpful comments and pointing out mistakes in the original version.

As for the notation, we say that a monomial (say u^2) appears in a

Received December 21, 1983.

power series (say φ) or φ contains u^2 if u^2 appears with a non-zero coefficient in the power series expansion of φ.

After having written up the paper, we learnt that Pinkham had proved similar (but slightly weaker) results (unpublished). We are grateful to Professor Kawamata, Pinkham, and Tsunoda for this information.

§1. Criteria for terminal and canonical singularities

Let $\boldsymbol{C}$ be the field of complex numbers, and $\boldsymbol{C}\{x\}$ denotes the ring of convergent power series in variables x.

LEMMA 1. *Let (X, p) be the germ of an n-dimensional terminal (resp. canonical) singularity of index m. Let (X', p') be the germ of an n-dimensional reduced Gorenstein variety and f: $(X', p') \to (X, p)$ a morphism such that f factors as*

$$X' \xrightarrow{g} Y \xrightarrow{h} X\,,$$

where h is a blow-up of X and g is quasi-finite. Let ω be a generator of $\omega_X^{(m)}$ at p. Then $f^*\omega$, as a meromorphic section of $\omega_{X'}^{\otimes m}$, vanishes (resp. is regular) along an arbitrary irreducible divisor $D \ni p$ such that $\dim f(D) < n-1$.

Proof. Let π: $\tilde{X}' \to X'$ be the normalization, and $X'' \subset \tilde{X}'$ the complement of the singular locus of $\tilde{X}'$. Since $\operatorname{codim}_{\tilde{X}'}(\tilde{X}' - X'') \geqq 2$ and since $(\pi^*\omega_{X'})|_{X''} \supset \Omega^n_{X''}$, we may replace (X', p') by (X'', p'') for some smooth p'' such that $\pi(p'') \in D$. In other words, we may assume that X' is smooth. Hence in the factorization $X' \to Y \to X$, we may assume that Y is normal and $q = g(p)$ is a smooth point of Y (by moving p in D if necessary). Then $h^*\omega$ vanishes (resp. is regular) along $g(D)$. Since g: $(X', p') \to (Y, q)$ is a morphism of manifolds, $f^*\omega = g^*h^*\omega$ vanishes (resp. is regular) along D. q.e.d.

Let (X, p) be a 3-dimensional canonical singularity of index m such that, for the associated $\boldsymbol{Z}_m$-cover π: $(X', p') \to (X, p)$ [5, 6], (X', p') is a hypersurface singularity. Then there exist $\boldsymbol{Z}_m$-semi-invariants $x_1, \cdots, x_4$ in the analytic local ring $\mathcal{O}^h_{X',p'}$ such that

$$\rho(x_i) = \zeta^{c_i} x_i \ (i = 1, \cdots, 4) \tag{1.1}$$

$$\mathcal{O}^h_{X',p'} \cong \boldsymbol{C}\{x_1, \cdots, x_4\}/(\varphi)\,, \tag{1.2}$$

where ρ (resp. ζ) is a generator of Z_m (resp. μ_m), $c = (c_1, \cdots, c_4) \in Z^4$ be such that $gcd(c, m) = g.c.d.\{c_1, \cdots, c_4, m\} = 1$, and φ is a semi-invariant. Since π is unramified outside p', one has

$$\{q \in X' \mid x_i(q) = 0 \quad \text{if } c_i \not\equiv 0 \,(\mathrm{mod}\, d)\} = \{p'\}$$

for any divisor $d > 1$ of m. Hence, for any divisor $d > 1$ of m, one has

$$\text{(1.3)} \qquad gcd(c, m) = 1, \qquad \sharp\{i \in [1, 4] \mid c_i \equiv 0 \,(\mathrm{mod}\, d)\} \leqq 1.$$

Now we reverse the process:

NOTATION (2.0). Let Z_m act on $C\{x_1, \cdots, x_4\}$ by (1.1) with ρ (resp. ζ) a generator of Z_m (resp. μ_m), where $c \in Z^4$ satisfies (1.3) for an arbitrary divisor $d > 1$ of m. Let φ be a semi-invariant of $C\{x_1, \cdots, x_4\}$ such that $C\{x_1, \cdots, x_4\}/(\varphi)$ is normal, and let (X', p') be the germ of a hypersurface at 0 defined by (1.2), and $(X, p) = (X', p')/Z_m$.

Then we have

THEOREM 2. *Under Notation* (2.0), *let σ be an arbitrary element of Z_m, and $a = (a_1, \cdots, a_4)$ a 4-ple of arbitrary integers $\geqq 0$ such that $\sigma(x_i) = \zeta^{a_i} x_i$ $(i = 1, \cdots, 4)$ and that at least three of $a_1, \cdots, a_4$ are positive. Let $e(a) = \max\{j \mid \varphi(x_1 t^{a_1}, \cdots, x_4 t^{a_4}) \equiv 0 \pmod{t^j}\}$ and $|a| = a_1 + \cdots + a_4$. Then if (X, p) is terminal* (*resp. canonical*), *then $|a| - m - e(a) > 0$* (*resp.* $\geqq 0$).

Proof. (2.1) Let X'_0 be C^4 with global coordinates $x_1, \cdots, x_4$, and let Z_m act on X'_0 by (1.1). Then $T' = (C^*)^4 \cap C^4 = X'_0$ is an affine torus embedding, and to the affine torus embedding $T' \subset X'_0$ correspond the group $\Gamma(T') \cong Z^4$ of 1 parameter subgroups of T' and a cone $C(X'_0)$ of $\Gamma(T') \otimes_Z Q$. Let $\Gamma(T') = Z^4$ and $C(X'_0) = Q^4_+$ in the standard way, where $Q_+ = \{q \in Q \mid q \geqq 0\}$. Then, to $T = T'/Z_m \to X_0 = X'_0/Z_m$, correspond $\Gamma(T) = Z^4 + Z c/m$ and $C(X_0) = Q^4_+$. By the definition of a, there exist integers β, γ such that $\beta c_i \equiv \gamma a_i \bmod m$, where γ is prime to m. Hence $a/m \in \Gamma(T)$.

(2.2) Let φ' be a convergent power series defined by

$$\varphi'(w, y_1, \cdots, y_4) = w^{-e(a)} \cdot \varphi(y_1 w^{a_1}, \cdots, y_4 w^{a_4}) .$$

Then $\varphi'(0, y)$ is a non-zero weighted homogeneous polynomial of weight $e(a)$ in $y_1, \cdots, y_4$ for weight $y_i = a_i$ (cf. $(*)$ below). Since $y_1 y_2$ and $y_3 y_4$ are coprime, $\varphi'(0, y)$ has a prime factor which is prime to $y_1 y_2$ or $y_3 y_4$. By symmetry, we may assume that $\varphi'(0, y)$ has a prime factor prime to $y_1 y_2$ and $a_1 > 0$. Since

https://doi.org/10.1017/S0027763000021358 Published online by Cambridge University Press

$$(*) \qquad \varphi'(0, y_1 t^{a_1}, \cdots, y_4 t^{a_4}) = t^{e(a)} \varphi'(0, y_1, \cdots, y_4) ,$$

one can find $r_2, r_3, r_4 \in \boldsymbol{C}$ such that

$$\varphi'(0, 1, r_2, r_3, r_4) = 0 .$$

(2.3) Let $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1) \in \boldsymbol{Z}^4$, and let C be the cone spanned by a, e_2, e_3, e_4 in $\boldsymbol{Q}^4$ and $M = \boldsymbol{Z}e_2 \oplus \boldsymbol{Z}e_3 \oplus \boldsymbol{Z}e_4 \subset \boldsymbol{Q}^4$. Then the commutative diagram

$$\begin{array}{ccc} \boldsymbol{Z}a \oplus M & \longrightarrow & \boldsymbol{Z}\dfrac{a}{m} \oplus M \\ \downarrow & & \downarrow \\ \boldsymbol{Z}^4 & \longrightarrow & \boldsymbol{Z}^4 + \boldsymbol{Z}\dfrac{c}{m} \end{array}$$

gives a commutative diagram of tori:

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

and $C \subset \Gamma(R') \oplus \boldsymbol{Q}$ gives a torus embedding

$$R' \subset Z'_0 = \operatorname{Spec} \boldsymbol{C}[w, \bar{x}_2, \bar{x}_3, \bar{x}_4] ,$$

where a, e_2, e_3, e_4 of $\Gamma(R')$ correspond to $w, \bar{x}_2, \bar{x}_3, \bar{x}_4$. Then

$$R \subset Z_0 = \operatorname{Spec} \boldsymbol{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4] ,$$

with $\bar{x}_1 = w^m$, is the torus embedding corresponding to $C \subset \Gamma(R) \otimes \boldsymbol{Q}$, and commutative diagrams

$$\begin{array}{ccccc} \Gamma(R') \otimes \boldsymbol{Q} & \longrightarrow & \Gamma(R) \otimes \boldsymbol{Q} & \longrightarrow & \Gamma(T) \otimes \boldsymbol{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & C & \longrightarrow & \boldsymbol{Q}^4_+ \end{array}$$

and

$$\begin{array}{ccccc} \Gamma(R') \otimes \boldsymbol{Q} & \longrightarrow & \Gamma(T') \otimes \boldsymbol{Q} & \longrightarrow & \Gamma(T) \otimes \boldsymbol{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & \boldsymbol{Q}^4_+ & \longrightarrow & \boldsymbol{Q}^4_+ \end{array}$$

give a commutative diagram

$$\begin{array}{ccc} Z_0' & \longrightarrow & Z_0 \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ X_0' & \longrightarrow & X_0 \end{array},$$

where f' is given by

$$x_1 = w^{a_1}, \quad x_2 = \bar{x}_2 w^{a_2}, \quad x_3 = \bar{x}_3 w^{a_3}, \quad x_4 = \bar{x}_4 w^{a_4}.$$

If $T \subset Y_0$ is the torus embedding corresponding to $C \subset \Gamma(T) \otimes \boldsymbol{Q}$, then $Z_0 \to Y_0$ is finite and $Y_0 \to X_0$ is a blow-up.

(2.4) Let $s' = V(w, \bar{x}_2 - r_2, \bar{x}_3 - r_3, \bar{x}_4 - r_4) \in Z_0'$ (resp. $s =$ the image of s' in Z_0), and let

$$\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = w^{-e(a)} \cdot \varphi(w^{a_1}, \bar{x}_2 w^{a_2}, \bar{x}_3 w^{a_3}, \bar{x}_4{}^{a_4}),$$

(note that the right hand side is a holomorphic function in $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$ defined near s.) Let $(Z, s) \subset (Z_0, s)$ (resp. $(Z', s') \subset (Z_0', s')$) be defined by $\psi = 0$. Then f (resp. f') induces $f\colon (Z, s) \to (X, p)$ (resp. $f'\colon (Z', s') \to (X', p')$), which satisfies the conditions of Lemma 1 by (2.3). We have the following commutative diagram of natural morphisms:

$$\begin{array}{ccc} Z' & \xrightarrow{\tau} & Z \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{\pi} & X \end{array}.$$

By Poincaré residue formula,

$$\omega_{X'} = \operatorname{Res} \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\varphi} = -\frac{dx_1 \wedge dx_2 \wedge dx_3}{\varphi_4},$$

$$f'^*\omega_{X'} = -\frac{d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3})}{f'^*\varphi_4},$$

where $\varphi_4 = \partial\varphi/\partial x_4$. By calculation, one sees

$$d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3}) = a_1 \cdot w^{a_1 + a_2 + a_3 - 1} \cdot dw \wedge d\bar{x}_2 \wedge d\bar{x}_3.$$

Since $f'^*\varphi = w^{e(a)}\psi$, it follows from the chain rule that

$$(f'^*\varphi_4) \cdot w^{a_4} = w^{e(a)}\psi_4,$$

where $\psi_4 = \partial\psi/\partial\bar{x}_4$. Thus

$$f'^*w_{X'} = -a_1 \cdot w^{|a|-e(a)-1} \cdot \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$

One also has

$$\omega_{Z'} = \operatorname{Res} \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3 \wedge d\bar{x}_4}{\psi} = -\frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$

One has

$$f'^*\omega_{X'} = a_1 \cdot w^{|a|-e(a)-1}\omega_{Z'}.$$

By construction, we have

$$\tau^*\omega_Z = m \cdot w^{m-1}\omega_{Z'}.$$

Hence we have

$$f'^*\omega_{X'} = (a_1/m) \cdot w^{|a|-e(a)-m}\tau^*\omega_Z.$$

Since $\pi\colon X' \to X$ is unramified in codimension 1 by (1.3), one has $\pi^*\omega_X{}^{(m)} = (\text{unit}) \cdot \omega_{X'}{}^{\otimes m}$ near p' (by abuse of language, $\omega_X^{(m)}$ denotes one of its generators at p), and

$$f^*\omega_X^{(m)} = (\text{unit}) \cdot \bar{x}_1{}^{|a|-e(a)-m}\omega_Z^{\otimes m} \qquad \text{near } s.$$

Since $\{\bar{x}_1 = 0\}$ is a divisor through s collapsed by f, one has $|a| - e - m > 0$ (resp. $\geqq 0$). q.e.d.

Under Notation (2.0), let $\chi_1, \cdots, \chi_r \in C\{x_1, \cdots, x_4\}$ ($r \geqq 2$) be Z_m-semi-invariants with the same character such that $\chi_i C\{x_1, \cdots, x_4\} = u_i C\{x_1, \cdots, x_4\}$ for some monomial u_i in $x_1, \cdots, x_4$ ($i = 1, \cdots, r$) and that $u_1, \cdots, u_r$ are linearly independent over C and the locus defined by $\chi_1 = \cdots = \chi_r = 0$ is of dimension $\leqq 1$. Let Φ be the linear system generated by $\chi_1, \cdots, \chi_r$, and assume that our φ is written as

$$\varphi = \sum_{i=1}^{r} \lambda_i \chi_i$$

for some $\lambda = (\lambda_1, \cdots, \lambda_r) \in (C^*)^r$. By Bertini's theorem, $\varphi = 0$ defines a normal variety for general λ since the base locus of Φ is of dimension $\leqq 1$. Let σ, a, ζ be as in Theorem 2, then the value of $e(a)$ given in Theorem 2 does not depend on the choice of $\lambda \in (C^*)^r$. Then under the notation of Theorem 2, the following is the corollary to the proof of Theorem 2.

https://doi.org/10.1017/S0027763000021358 Published online by Cambridge University Press

COROLLARY 2.1. *If* $|a| - m - e(a) > 0$ *(resp.* $\geqq 0$*) for arbitrary* σ, a, ζ *as in Theorem* 2, *then* (X, p) *is terminal (resp. canonical) for general* λ. *If* (X, p) *has an isolated singular point at* p *(resp. canonical singularity outside* p*) for general* λ, *then one can add the extra conditions* $a_1, \cdots, a_4 > 0$ *on* a *in the statement above.*

Proof. Let $X_0 = C^4/Z_m$ with respect to the action given above. Then Φ induces a linear system Φ_0 (of Weil divisors) in a neighborhood of 0 in X_0 which is free from fixed components. By the conditions on χ_i's, there exists a toric resolution $h: U_0 \to X_0$ such that the proper transform Ψ_0 of Φ_0 to U_0 is free from base points (principalizer of the coherent sheaf associated to Φ_0). Then a general member X of Φ_0 is normal at 0 and its proper transform $U = h^{-1}[X]$ is smooth in a neighborhood of $h^{-1}(0)$. Thus it is enough to show that $h^*\omega_X^{(m)}$, as a meromorphic section of $\omega_U^{\otimes m}$, vanishes (resp. is regular) along $D_0 \cap U$, where $D_0 \subset U_0$ is an arbitrary exceptional divisor of h. We now use the notation of the proof of Theorem 2. Let L_+ be the 1-simplex $\subset Q_+^4 \subset \Gamma(T) \otimes Q$ corresponding to D_0. Let $a = (a_1, \cdots, a_4)$ be a 4-ple of integers $\geqq 0$ such that $Za/m = QL_+ \cap \Gamma(T)$. By (1.3), the singular locus of X_0 is of dimension $\leqq 1$, and the base locus of Φ_0 is of dimension $\leqq 1$. Thus one may assume that the image of D_0 to X_0 is of dimension $\leqq 1$, whence at least 3 of $a_1, \cdots, a_4$ are positive. Then the notation of (2.3) can be used, and the torus embedding $T \subset V_0 = T \cup D_0$ corresponds to $L_+ \subset \Gamma(T) \otimes Q$. $L_+ \subset \Gamma(R) \otimes Q$ corresponds to the open subset W_0 of Z_0 defined by $\bar{x}_3\bar{x}_4 \neq 0$. Since $Za/m = Qa/m \cap \Gamma(T)$, $\{\bar{x}_1 = 0\}$ is not in the branch locus of $W_0 \to V_0$. Let V, W be the proper transforms of X to V_0 and W_0, respectively. Then, since Ψ_0 is free from base points, $g: W \to V$ is unramified over general points of arbitrary irreducible components of $V \cap D_0$. Thus by

$$g^*h^*\omega_X^{(m)} = (\text{unit}) \cdot \bar{x}_1^{|a|-e(a)-m} \omega_W^{\otimes m} \quad \text{along} \quad W \cap D_0 ,$$

(2.4), we have Corollary 2.1. q.e.d.

COROLLARY 2.2. *Under the assumptions of Theorem* 2, *assume that* m *is odd, and*

$$\varphi = x_1^2 + f(x_2, x_3, x_4) \qquad (f \in C\{x_2, x_3, x_4\}) .$$

Let $n = \max\{j \,|\, \varphi(x_2 t^{a_2}, x_3 t^{a_3}, x_4 t^{a_4}) \equiv 0\,(t^j)\}$. *Then*

$$a_2 + a_3 + a_4 > (\text{resp. } \geqq) \begin{cases} m + n/2 & \text{if } n \text{ is even} \\ m/2 + n/2 & \text{if } n \text{ is odd} . \end{cases}$$

https://doi.org/10.1017/S0027763000021358 Published online by Cambridge University Press

Proof. One has $2\cdot wt\, x_1 \equiv n \pmod m$. If n is even, then we choose $a_1 = n/2$, keeping a_2, a_3, a_4 the same. Then $n/2 + a_2 + a_3 + a_4 >$ (resp. $\geqq$) $m + n$. If n is odd, then we choose $a_1 = (m+n)/2$, keeping a_2, a_3, a_4 the same. Then $(n+m)/2 + a_2 + a_3 + a_4 >$ (resp. $\geqq$) $m + n$. q.e.d.

For the approximation of φ, we need the standard:

THEOREM 3. *Let $\varphi \in \boldsymbol{C}\{x\}$, where $x = (x_1, x_2, x_3, x_4)$. Assume that φ has an isolated singular point at* (0), *and that $\boldsymbol{Z}_m$ acts on $\boldsymbol{C}\{x\}$ in such a way that x_1, x_2, x_3, x_4, and φ are semi-invariants. Then for an arbitrary integer $b > 0$, there exists an integer $n > 0$ such that, for an arbitrary semi-invariant $\psi \in \boldsymbol{C}\{x\}$ with the property $\psi \equiv \varphi\ (x)^n$, there exists an analytic $\boldsymbol{C}$-automorphism σ of $\boldsymbol{C}\{x\}$ commuting with $\boldsymbol{Z}_m$-action (will be called a $\boldsymbol{Z}_m$-automorphism, for short) such that $\sigma(\varphi) = \psi\boldsymbol{C}$, $\sigma \equiv$* id modulo $(x)^b$.

This can be proved by applying the argument of [1, Lemma (5.11)] to the equation $\varphi(y) - \psi(x) = 0$ in unknown variables $y = (y_1, y_2, y_3, y_4)$ with approximate solution $y^0 = x$, where $m = 1$ and $N = 4$.

COROLLARY 4. *Under the notation and the assumptions of Theorem* 3, *if $\varphi \in (x)^2$ and if x_1x_2 appears in φ and*

$$(\partial^2\varphi/\partial x_1^2(0))(\partial^2\varphi/\partial x_2^2(0)) - (\partial^2\varphi/\partial x_1\partial x_2(0))^2 \neq 0\,,$$

then there exist a $\boldsymbol{Z}_m$-automorphism σ of $\boldsymbol{C}\{x\}$ such that

$$\sigma(x_3) - x_3\,,\quad \sigma(x_4) - x_4 \in (x)^2\,,$$
$$\sigma(\varphi) = x_1x_2 + f(x_3, x_4) \qquad \text{for some } f \in \boldsymbol{C}\{x_3, x_4\}\,.$$

§ 2. Notation and terminal singularities of type cA

ASSUMPTION 5. Let φ be an element of $(x, y, z, u)^2\boldsymbol{C}\{x, y, z, u\}$ which has a $\boldsymbol{Z}_m$-action $(m > 1)$ such that x, y, z, u, φ are semi-invariants. Assume that φ has an isolated cDV singularity at the origin (0), that the quotient of $\{\varphi = 0\}$ by $\boldsymbol{Z}_m$ has a terminal singularity at (0), and that the action of $\boldsymbol{Z}_m$ is free on $U - (0)$, where $U \ni (0)$ is an open set of $\{\varphi = 0\}$. By a $\boldsymbol{Z}_m$-automorphism, we mean an analytic $\boldsymbol{C}$-automorphism of $\boldsymbol{C}\{x, y, z, u\}$ commuting with $\boldsymbol{Z}_m$-action unless otherwise mentioned. We will keep these assumptions and notation, unless otherwise mentioned.

NOTATION 6. Fixing a primitive m-th root ζ of 1, and given the $\boldsymbol{Z}_m$-action above, we associate to each $\sigma \in \boldsymbol{Z}_m$ a weight modulo m (denoted by σ-wt mod m); σ-$wt(x) \equiv$ a$(\sigma), \cdots, \sigma$-$wt(u) \equiv d(\sigma)$ (mod m) are determined by $\sigma(x) = \zeta^{a(\sigma)} \cdot x, \cdots, \sigma(u) = \zeta^{d(\sigma)} \cdot u$. If σ is a generator of Z_m, we may simply call σ-wt a wt, if there is no danger of confusion. *Order* v associates numbers $a, b, c, d > 0$ to x, y, z, u such that $a \equiv \sigma$-$wt(x), \cdots, d \equiv \sigma$-$wt(u)$ mod m for some $\sigma \in Z_m$. Then the order of f, or $v(f)$ is, by definition, $\max\{n \mid f(xt^a, yt^b, zt^c, ut^d) \equiv 0\,(t^n)\}$. We write $v \equiv \sigma$-wt (mod m) if $v(x) \equiv wt\, x, \cdots, v(u) \equiv wt\, u$ (mod m). For two orders v and v', we write $v \equiv v'$ (mod m) if $v(x) \equiv v'(x), \cdots, v(u) \equiv v'(u)$ (mod m). For a positive integer a and a number b (or, $b \in \boldsymbol{R}/a\boldsymbol{Z}$), $(b)_a$ denotes the number c such that $c - b \in a\boldsymbol{Z}$ and $0 < c \leqq a$. We define order $v = (\sigma\text{-}wt)_m$ by $v(x) = (\sigma\text{-}wt\, x)_m, \cdots, v(u) = (\sigma\text{-}wt\, u)_m$.

Remark 7. Assumption 5 implies:

(1) if $e = (m, wt\, x) > 1$, then some power of x appears in φ and $wt\, \varphi \equiv 0$ (mod e). Similar assertion holds also for y, z, u. (Since the action on x-axis is not free, $\{\varphi = 0\}$ does not contain x-axis.)

(2) $(m, wt\, x, wt\, y) = 1$. Similar assertion holds also for any other two distinct coordinate functions. (Otherwise the action is not free on $\{\varphi = 0\} \cap xy$-plane which has dimension > 0 at (0)).

THEOREM 8. *If xy appears in φ and*

$$(\partial^2\varphi/\partial x^2(0))(\partial^2\varphi/\partial y^2(0)) - (\partial^2\varphi/\partial x\partial y(0))^2 \neq 0\,,$$

then one of the following holds (after exchanging z, u if necessary):

(1) *$wt\, x + wt\, y \equiv wt\, z \equiv 0$ mod m, and $wt\, u$, $wt\, x$, $wt\, y$ are prime to m.*

(2) *$m = 4$ and there exists a generator σ of Z_4 such that σ-wts of x, y, z, u are $1, 1, 2, 3$ mod 4 (see Supplement 8.1).*

Proof. By Corollary 4, we may assume $\varphi = xy + f(z, u)$, where $f \in \boldsymbol{C}\{z, u\}$. Let $e = (wt\, x + wt\, y)_m$, $p = (e, m)$.

(8.1) *Claim*: One of $wt\, z, wt\, u$ is a multiple of p, and the other is prime to m. $wt\, x, wt\, y$ are prime to m.

Let wt be σ-wt. Assume that σ-$wt\, z, \sigma$-$wt\, u \not\equiv 0\,(p)$ (and hence $p > 1$). If necessary, we will replace σ by $-\sigma$ to get

$$(\sigma\text{-}wt\, z)_p + (\sigma\text{-}wt\, u)_p \leqq p\,.$$

By Remark 7, (2), and

$$\sigma\text{-}wt\, x + \sigma\text{-}wt\, y \equiv 0\ \ (p)\,,$$

one sees that $\sigma\text{-}wt\,x$ and $\sigma\text{-}wt\,y$ are prime to p. Hence

$$(\sigma\text{-}wt\,x)_p + (\sigma\text{-}wt\,y)_p = p\ .$$

Let v be the order $(m/p)\cdot(\sigma\text{-}wt)_p = ((m/p)\cdot\sigma\text{-}wt)_m$. Then $v(\varphi) = m$ and $v(x) + v(y) = m$, $v(z) + v(u) \leqq m$. Thus $v(x) + v(y) + v(z) + v(u) \leqq m + v(\varphi)$, which is a contradiction to Theorem 2, and hence $\sigma\text{-}wt\ z \equiv 0$ or $\sigma\text{-}wt\,u \equiv 0\,(p)$. By symmetry of z, u, we may assume that $\sigma\text{-}wt\,z \equiv 0\,(p)$. Let $n = (\sigma\text{-}wt\,u, m)$. We will show that $n = 1$. If $n > 1$ then by Remark 7, (1), $wt\ \varphi \equiv 0\,(n)$. Thus $n|p$ and $wt\,z \equiv wt\,u \equiv 0\,(n)$. This contradicts Remark 7, (2). Thus $n = 1$. By Remark 7, (1), (2), if $n = (m, \sigma\text{-}wt\,x) > 1$, then $wt\,x + wt\,y \equiv wt\,\varphi \equiv 0\,(n)$ and $n = (m, \sigma\text{-}wt\,x, \sigma\text{-}wt\,y)$. Thus $n = 1$ by Remark 7, (2). Similar argument shows $(m, \sigma\text{-}wt\,y) = 1$. This proves (8.1).

(8.2) *Claim*: By symmetry of z, u, we may assume that $wt\ z \equiv 0\,(p)$ and $wt\,u$ is prime to m. Then $p = (wt\,z, m)$, and $f(z, u) = g(z, u^p)$ for some convergent power series g.

Let $n = (wt\,z, m)$. Then $p\,|\,n$. By Remark 7, (1), one has $wt\,x + wt\,y = 0\,(n)$, whence $n\,|\,p$ and $n = p$.

(8.3) *Claim*: If $p < m$ and $\rho \in Z_m$ satisfies

$$\rho\text{-}wt\,x + \rho\text{-}wt\,y \equiv \pm\,p\,(m)\ ,$$

then $\rho\text{-}wt\,x$, $\rho\text{-}wt\,y$, $\rho\text{-}wt\,z$, $\rho\text{-}wt\,u \not\equiv 0\,(m)$.

If $\rho\text{-}wt \equiv i\cdot wt\,(m)$, then $\mathrm{i}\cdot\mathrm{e} \equiv \pm\,p\,(m)$. Thus $i \not\equiv 0\,(m)$, whence $i\cdot wt\,x$, $i\cdot wt\,y$, $i\cdot wt\,u \not\equiv 0\,(m)$ by (8.1), (8.2). One also sees that i is prime to $m/p > 1$. Thus by (8.2), $\rho\text{-}wt\,z \equiv i\cdot(wt\,z) \not\equiv 0\,(m)$.

(8.4) *Claim*: Assume $p < m$. The number of elements $\rho \in Z_m$ such that

$$(*) \qquad \rho\text{-}wt\,x + \rho\text{-}wt\,y \equiv \pm\,p\ (m)\ , \quad \text{and}$$

$$(**) \qquad (\rho\text{-}wt\ z)_m + (\rho\text{-}wt\ u)_m \leqq m$$

is $\geqq p$ if $m > 2p$, and $\geqq p/2$ if $m = 2p$.

As in (8.3), the number of ρ's with $(*)$ is the number of solutions i $(0 \leqq i < m)$ for $i\cdot e \equiv \pm\,p\ (m)$, which is $2p$ if $m > 2p, p$ if $m = 2p$. Now involution ι: $\rho \to -\,\rho$ acts on $\sum = \{\rho \text{ with } (*)\}$ and ρ or $-\,\rho$ satisfies $(**)$ by (8.3). Thus (8.4) is settled.

https://doi.org/10.1017/S0027763000021358 Published online by Cambridge University Press

(8.5) If $p = m$, then (1) holds. We may now assume $p < m$, and let ρ be an arbitrary element with (*), (**). Let $v = (\rho\text{-}wt)_m$, then by (8.3),

$$v(x) + v(y) = p, m \pm p\,, \quad \text{or} \quad 2m - p\,,$$
$$v(z) + v(u) \leqq m\,.$$

By Theorem 2,

$$m + v(x) + v(y) \geqq v(x) + v(y) + v(z) + v(u) > m + v(\varphi)\,,$$

i.e. $v(\varphi) < v(x) + v(y)$. Thus there are two cases: (8.5.1) $v(\varphi) = p$, $v(x) + v(y) = m + p$, and (8.5.2) $v(\varphi) = m - p$, $v(x) + v(y) = 2m - p$.

(8.5.1) Assume that $v(\varphi) = p$. Then $v(g(z, u^p)) = p$ by (8.2). Since $g \in (z, u)^2$ and $v(z) \geqq p$ (8.2), one has $p = v(g(z, u^p)) = p \cdot v(u)$ and $p \geqq 2$, hence $v(u) = 1$ and $\rho\text{-}wt\, u \equiv 1\ (m)$. Thus there is at most one such ρ.

(8.5.2) Assume that $m > 2p$ and $v(\varphi) = m - p$. Then $v(x) = m - j$ and $v(y) = m - p + j$ $(j = 1, \cdots, p - 1)$ by (8.3). Since $(\rho\text{-}wt\, x, \rho\text{-}wt\, y, m) = 1$, there are at most $p - 1$ such p's.

(8.6) If $m = 2p$, then by (8.4) and (8.5.1), one sees that $m = 2p = 4$. Then by choosing wt, one has wts of x, y, z, u equal to 1, 1, 2, α ($\alpha = 1, 3$). One sees $\alpha = 3$ by applying Theorem 2. Whence one gets (2).

We now assume that $m > 2p$ and will derive a contradiction to finish the proof. Then by (8.4), (8.5.1) and (8.5.2), one sees that $p \geqq 2$ and there are exactly one ρ in case (8.5.1) and exactly $p - 1$ ρ's in case (8.5.2). We claim that $p = 2$. If $p > 2$, then let ρ_i $(i = 1, 2)$ in case (8.5.2) be such that $\rho_i\text{-}wt\, x \equiv m - i$ $(i = 1, 2)$. Then $\rho_2 = 2\rho_1$ and whence $-p \equiv \rho_2\text{-}wt\, \varphi \equiv 2(\rho_1\text{-}wt\, \varphi) \equiv -2p\ (m)$. This means that $p \equiv 0\ (m)$ and $m = p$, which contradicts $m > 2p$. Thus our claim that $p = 2$ is proved. Let $\rho_1 \in \boldsymbol{Z}_m$ be in case (8.5.1) and $\rho_2 \in \boldsymbol{Z}_m$ in case (8.5.2). One has $\rho_2\text{-}wt\, x \equiv \rho_2\text{-}wt\, y \equiv m - 1\ (m)$ and ρ_2 is a generator of Z_m. Thus by $(\rho_1\text{-}wt\, x)_m + (\rho_1\text{-}wt\, y)_m = m + 2$, one has $\rho_1\text{-}wt\, x \equiv \rho_1\text{-}wt\, y \equiv m/2 + 1 \pmod m$. Hence $\rho_1 = (m/2 - 1)\rho_2$. Let $\rho_3 = 2\rho_1$. Then $\rho_3 = -2\rho_2$, and ρ_3-wts of x, y, u are $2, 2, 2 \bmod m$. Let $w = (\rho_3\text{-}wt)_m$. Then by $m \equiv 0\ (2)$ and $m > 2p = 4$, one has $m \geqq 6$. Thus $w(\varphi) = w(x) + w(y) = 4$. Applying Theorem 2 to w, one has $w(z) > m - 2$. Hence $w(z) = m$ because m is even and $\rho_3 \equiv 0\ (2)$. Since $\rho_3 = -2\rho_2$, one has $\rho_2\text{-}wt\, z \equiv 0$ or $m/2$. Since ρ_2 is a generator of Z_m, one has $p = (\rho_2\text{-}wt\, z, m) = m$ or $m/2$ by (8.2). Thus $m = p$, or $2p$ ocntradicting our assumption $m > 2p$. q.e.d.

SUPPLEMENT 8.1. *In case* (2), *one sees that* $g(z, u^2)$ *contains* u^2 *in* (8.5). *Thus one sees*: *in case* (2) *of Theorem* 8, *modulo* Z_4-*automorphism, one has*

$$\varphi = xy + z^n + u^2 \qquad (n: odd \geqq 3)\ ,$$

where wts of x, y, z, u *are* 1, 1, 2, 3 (mod 4).

Proof. By Z_4-automorphism: $z \to (\text{const})\cdot z$, $u \to (\text{const})\cdot u$, keeping x and y, one has $g = u^2(1 + \alpha(z, u^2)) + z^n(1 + \beta(z))$, where $\alpha \in (z, u)$, $\beta \in (z)$, $n > 1$. Since α, β are Z_4-invariant, one can change $u\cdot\{1 + \alpha(z, u^2)\}^{(1/2)} \to u$, $z\cdot\{1 + \beta(z)\}^{(1/n)} \to z$, to get $g = u^2 + z^n$. q.e.d.

LEMMA 9. *If* x^2 *and* y^2 *appear in* φ *and* $\varphi_2 \in kx^2 + ky^2 + kz^2 + ku^2$, *then* $m = 2, 4$.

Proof. (9.1) *Claim*: m is a power of 2.

If m is not a power of 2, we may assume that m is odd and > 1 by replacing m by its odd factor. Then $wt\,x \equiv wt\,y$, and hence by Z_m-automorphism: $x \to x + \alpha\cdot y$, keeping y, z, u, we may assume that φ contains xy, x^2, y^2. Thus by Theorem 8, $wt\,x + wt\,y \equiv 0\ (m)$, $wt\,x$ and $wt\,y$ are prime to m, which contradicts $wt\,x \equiv wt\,y$. Thus $m = 1$, and hence m is proved to be a power of 2.

(9.2) *Claim*: If $m = 8$, then degree 2 part φ_2 of φ is a quadratic form of rank 3.

By $2\cdot wt\,x \equiv 2\cdot wt\,y\ (8)$, one has $wt\,x \equiv wt\,y\ (4)$. Thus by Z_4-automorphism: $x \to x + \alpha y$ keeping y, z, u, one may assume x^2, y^2, xy appear in φ and $m = 4$. Then by Theorem 8 and Supplement 8.1, φ_2 has rank 3.

(9.3) *Claim*: $m|4$.

If $8|m$ can occur, the case $m = 8$ occurs. if $m = 8$, then φ_2 contains x^2, y^2, z^2 (by exchanging z, u if necessary.) By $2wt\,x \equiv 2wt\,y \equiv 2wt\,z\ (8)$, one sees that two of $wt\,x$, $wt\,y$, $wt\,z$ are congruent mod 8. We may assume $wt\,x \equiv wt\,y\ (8)$, without loss of generality. By Z_8-automorphism: $x \to x + \alpha y$, keeping y, z, u, one may assume that φ contains x^2, y^2, xy for $m = 8$. This contradicts Theorem 8. Thus Lemma 9 is proved.

LEMMA 10. *If* x^2 *and* y^2 *appear in* φ *and* $\varphi \in kx^2 + ky^2 + kz^2 + ku^2$ *and* $m = 4$, *then*

(1) *if* $wt\,x \equiv wt\,y$ mod 4, *then after exchanging* z, u (*if necessary*) *and choosing a generator* σ *of* Z_4, *one has* σ-*wts of* x, y, z, u *equal to* 1, 1, 2, 3 mod 4, *and modulo* Z_4-*automorphism,*

$$\varphi = x^2 + y^2 + z^n + u^2 \, (n\colon odd \geqq 3)\,, \quad and$$

(2) *if* $wt\,x \not\equiv wt\,y$ (4), *then after exchanging* z, u *and exchanging* x, y (*if necessary*) *and choosing a generator* σ *of* Z_4, *one has* σ-*wts of* x, y, z, u *equal to* 1, 3, 2, 1 mod 4.

Proof. *Case* (1) is due to Supplement 8.1 (cf. argument of Lemma 9).

Case (2): By $2wt\,x \equiv 2wt\,y$ (4), one sees that $wt\,x, wt\,y$ are odd by Remark 7, (2). Hence $wt\,\varphi \equiv 2$ (4). Applying Theorem 8 to $m = 2$, $wt\,z$ or $wt\,u$ is even (cf. argument of Lemma 9). Let us assume that $wt\,z$ is even. By Remark 7, (1), φ contains some power of z, whence $wt\,z \equiv 2$ (4). Changing σ to $-\sigma$ if necessary, one gets (2). q.e.d.

LEMMA 11. *If* x^2 *and* y^2 *appear in* φ *and* $\varphi \in kx^2 + ky^2 + kz^2 + ku^2$ *and* $m = 2$, *then*

(1) *if* $wt\,x \equiv wt\,y$ mod 2, *then after exchanging* z, u (*if necessary*), *one has wts of* x, y, z, u *equal to* 1, 1, 0, 1 mod 2, *and modulo* Z_2-*automorphism*, $\varphi = x^2 + y^2 + f(z, u^2)$ *for some* $f \in C\{z, u\}$,

(2) *if* $wt\,x \not\equiv wt\,y$ mod 2, *then after exchanging* x, y (*if necessary*), *one has wts of* x, y, z, u *equal to* 1, 0, 1, 1 mod 2, *and modulo* Z_2-*automorphism*, $\varphi = x^2 + y^2 + f(z, u)$ *for some* $f \in C\{z, u\}$.

Proof. By Remark 7, (2), there is at most one even number among *wts* of x, y, z, u. On the other hand, by Theorem 2 applied to $v = (wt)_2$, one has $v(x) + \cdots + v(u) > 2 + v(\varphi) \geqq 4$. Hence exactly one of *wts* of x, y, z, u is congruent to 0 mod (2). Then Lemma 11 is clear.

THEOREM 12. *If* φ_2 *has rank* $\geqq 2$, *then after permutation of* x, y, z, u (*if necessary*), *one of the following holds.*

(1) $wt\,x + wt\,y \equiv 0$; $wt\,z \equiv wt\,\varphi \equiv 0$; $wt\,x, wt\,y, wt\,u$ *are prime to* m, *and modulo* Z_m-*automorphism*, $\varphi = xy + f(z, u^m)$ *for some* $f \in C\{z, u^m\}$.

(2) $m = 4$; *wts of* x, y, z, u, φ *are* 1, 3, 1, 2 mod 4 (*after choosing a generator of* Z_4), *and modulo* Z_4-*automorphism*, $\varphi = x^2 + y^2 + f(z, u^2)$ *for some* $f \in C\{z, u^2\}$.

(3) $m = 2$; *wts of* x, y, z, u, φ *are* 1, 0, 1, 1, 0 mod 2, *and modulo* Z_2-*automorphism*, $\varphi = x^2 + y^2 + f(z, u)$ *for some* $f \in (z, u)^4 C\{z, u\}$.

By Lemma 12.2, this follows from Theorem 8, Lemmas 9, 10, 11. (For case (3), one may set $f \in (z, u)^4 C\{z, u\}$, because otherwise it is reduced to case (1)).

Remark 12.1. In case (1) (resp. (2), (3)) of Theorem 12, if f is a general linear combination of a finite number of monomials in (z^2, u^m) $\boldsymbol{C}\{z, u^m\}$ (resp. $(z^3, u^2)\boldsymbol{C}\{z^2, zu^2, u^4\}$, $(z^4, z^3u, z^2u^2, zu^2, u^4)\boldsymbol{C}\{z^2, zu, u^2\}$) and if f has an isolated singular point at 0, then (X, p) is a terminal singularity. This follows from Corollary 2.1. For example, in case (1), if v is an order, then $v(x) + v(y) + v(z) + v(u) - m - v(\varphi) = (v(x) + v(y) - v(\varphi)) + (v(z) - m) + v(u) > 0$.

LEMMA 12.2. *If* $\mathrm{rk}\,\varphi_2 \geqq 2$, *then modulo permutation of* x, y, z, u *and* $\boldsymbol{Z}_m$*-automorphism, one has either* (1) $\varphi_2 = xy + g(z, u)$ *for some quadratic form* g *in* z, u *or* (2) $\varphi_2 \in kx^2 + ky^2 + kz^2 + ku^2$.

Proof. If none of x^2, y^2, z^2, u^2 appears in φ_2, then one has case (1) by Theorem 8. If for example x^2 appears in φ_2, then one has $\varphi_2 = x^2 + h(y, z, u)$ modulo $\boldsymbol{Z}_m$-automorphism for some quadratic form h. One can repeat similar argument to $h(y, z, u)$, either to obtain case (1) by applying Theorem 8, or end up with case (2). q.e.d.

COROLLARY 13. *If* $\mathrm{rk}\,\varphi_2 \geqq 3$, *then after permutation of* x, y, z, u *and choice of generator* $\sigma \in \boldsymbol{Z}_m$, *one has*

(1) $\mathrm{rk}\,\varphi_2 = 4$; $m = 2$, *wts of* x, y, z, u, φ *are* $1, 1, 0, 1, 0 \bmod 2$ *and* $\varphi = xy + z^2 + u^2$ *modulo* $\boldsymbol{Z}_2$*-automorphism,*

(2) $\mathrm{rk}\,\varphi_2 = 3$; *either.*

(2.1) $m \geqq 2$, $wt\,x + wt\,y \equiv 0$, $wt\,z \equiv wt\,\varphi \equiv 0\ (m)$, $wt\,x$, $wt\,y$, $wt\,u$ *are prime to* m, *and* $\varphi = xy + z^2 + u^{im}$ $(im \geqq 3)$ *modulo* $\boldsymbol{Z}_m$*-automorphism,*

(2.2) $m = 2$, *wts of* x, y, z, u, φ *are* $1, 1, 0\ 1, 0$, *and* $\varphi = xy + z^i + u^2$ $(i \geqq 3)$ *modulo* $\boldsymbol{Z}_2$*-automorphism, or*

(2.3) $m = 4$, σ*-wts of* x, y, z, u, φ *are* $1, 1, 2, 3, 2$, *and* $\varphi = xy + z^i + u^2$ $(i \geqq 3$: odd) *modulo* $\boldsymbol{Z}_4$*-automorphism.*

This follows from Theorem 12.

§ 3. Terminal singularities of type *cD*

LEMMA 14. *If* $\mathrm{rk}\,\varphi_2 \leqq 1$ *and* u^2 *appears in* φ_2, *then one has*

$$\varphi = u^2 + f(x, y, z) \quad \textit{modulo } \boldsymbol{Z}_m\textit{-automorphism},$$

where $f \in (x, y, z)^3\boldsymbol{C}\{x, y, z\}$ *has non-zero cubic term* f_3.

This follows from the definition of *cDV* points.

LEMMA 15. *Under the assumptions and notation of Lemma* 14, *if* f_3 *contains* xyz *and* m *is a power of* 2, *then* $m = 2$ *and after permutation of* x, y, z (*if necessary*), *one has wts of* x, y, z, u, f *equal to* $1, 1, 0, 1, 0 \bmod 2$.

Proof. Since $wt\, x + wt\, y + wt\, z \equiv 2wt\, u \equiv 0\ (2)$, at least one of $wt\, x$, $wt\, y$, $wt\, z$ is even. Without loss of generality, we may assume $wt\, z$ is even. Then $wt\, x$, $wt\, y$, $wt\, u$ are odd (Remark 7, (2)). If $m = 2$, then Lemma 15 is proved. It remains to disprove the case $m = 4$. If $m = 4$, then $wt\, f \equiv 2\ (4)$ because $wt\, u$ is odd. Since f contains a power of z (Remark 7, (1)), $wt\, z \equiv 2\ (4)$. Choosing $\sigma \in Z_4$, one may assume that σ-*wts* of x, y, z, u are $a, b, 2, 3$ ($a, b = 1, 3$). By $a + b + 2 \equiv 2\ (4)$, one has $a + b \equiv 0\ (4)$, whence $a + b = 4$. Since $wt\, \varphi \equiv 2$, one sees that the order v induced by this weight satisfies $v(f) = 6$. Then $v(x) + v(y) + v(z) + v(u) < 4 + v(\varphi)$ contradicts Theorem 2. q.e.d.

LEMMA 16. *Under the assumptions and notation of Lemma* 14, *if* f_3 *contains* xyz, *then* m *is a power of* 2.

Proof. Assuming that m is odd, we will derive a contradiction.

(16.1) *Claim*: $wt\, x$, $wt\, t$, $wt\, z$ are prime to m.

For example, assume $n = (wt\, x, m) > 1$. Then $wt\, f \equiv 0\ (n)$ by Remark 7, (1). Since n is odd, one has $wt\, u \equiv 0\ (n)$. This means $n = (wt\, x, wt\, u, m) > 1$, which contradicts Remark 7, (2).

(16.2) *Claim*: $wt\, x + wt\, y + wt\, z$ and $wt\, u$ are prime to m.

Since m is odd and $wt\, u^2 \equiv wt\, xyz\ (m)$, it is enough to show $(wt\, xyz, m) = 1$. For this, it is enough to derive a contradiction by assuming that $wt\, xyz \equiv 0\ (m)$, and $m > 1$ is odd. By replacing wt by $-wt$ if necessary, one may assume (by (16.1))

$$(wt\, x)_m + (wt\, y)_m + (wt\, z)_m = m\,.$$

Hence, for the order $v = (wt)_m$, one has $v(\varphi) = m$ and $v(x) + v(y) + v(z) + v(u) \leqq m + v(\varphi)$. This is a contradiction to Theorem 2.

(16.3) By choosing $\rho \in Z_m - \{0\}$, one can assume

$$(\rho\text{-}wt\, x)_m + (\rho\text{-}wt\, y)_m + (\rho\text{-}wt\, z)_m \equiv \pm 1\, (m), \quad \text{and} \quad < 3m/2\,.$$

Then $(\rho\text{-}wt\, x)_m + (\rho\text{-}wt\, y)_m + (\rho\text{-}wt\, z)_m = m \pm 1$, and for $v = (\rho\text{-}wt)_m$, $v(f) = m \pm 1$ is even. By Corollary 2.2, $0 < m \pm 1 - m - (m \pm 1)/2 = (-m \pm 1)/2$. This is a contradiction, and Lemma 16 is proved.

PROPOSITION 17. *Under the assumptions and notation of Lemma* 14, *if* f_3 *contains* xyz, *then* $m = 2$, *and after permutation of* x, y, z, *one has wts of* x, y, z, u, f *equal to* $1, 1, 0, 1, 0 \bmod 2$.

This follows from Lemmas 15, 16.

LEMMA 18. *Under the assumptions and notation of Lemma* 14, *if* f_3 *contains* xy^2, *and if* $m > 1$ *is a power of* 2, *then* $m = 2$ *and wts of* x, y, z, u, f *are* $0, 1, 1, 1, 0 \bmod 2$.

Proof. Assume $m = 4$. By $2wt\, u = wt\, x + 2wt\, y$, $wt\, x$ is even whence $wt\, y$ and $wt\, u$ are odd (Remark 7, (2)). Thus $2wt\, u \equiv 2wt\, y\ (4)$ and $wt\, x \equiv 0\ (4)$. Thus by Remark 7, (1), $wt\, f \equiv 0\ (4)$, which contradicts $wt\, u \equiv 1\ (2)$. The rest is easy. q.e.d.

LEMMA 19. *Under the assumptions and notation of Lemma* 14, *if* f_3 *contains* xy^2, *and if* $m > 1$ *is odd, then* $m \equiv 3$ *and after choosing a generator of* Z_m, *one has wts of* x, y, z, u, φ *equal to* $2, 2, 1, 0, 0 \bmod 3$ *and* f_3 *contains* xy^2, z^3.

Proof. (19.1) *Claim*: $wt\, x, wt\, y, wt\, z$ are prime to m.

It is enough to derive a contradiction from (e.g.) $wt\, x \equiv 0\ (m)$. By Remark 7, (1), some power of x appears in f, and $wt\, u \equiv (wt\, f)/2 \equiv 0\ (m)$, which contradicts Remark 7, (2).

(19.2) *Claim*: By (19.1), we may choose a generator σ of Z_m so that $\sigma\text{-}wt\, x \equiv 2$, $\sigma\text{-}wt\, y \equiv b$, $\sigma\text{-}wt\, z \equiv c\ (m)$, with $0 < b,\ c < m$. Then, for $v = (\sigma\text{-}wt)_m$, $v(f) = 2 + 2b - m$.

From $v(xy^2) = 2 + 2b \leqq 2m$ (19.1), follows $v(f) = 2 + 2b$ or $2 + 2b - m$. But if $v(f) = 2 + 2b$, then $v(f)$ is even and $2 + b + c - m - (2 + 2b)/2 = c + 1 - m \leqq 0$ by (19.1), which contradicts Corollary 2.2.

(19.3) *Claim*: Under the assumptions and notation of (19.2), if $2c < m$ and if $m \geqq 5$, then $2b > m$ and for $w = ((2\sigma)\text{-}wt)_m$, one has $w(x) = 4$, $w(y) = 2b - m$, $w(z) = 2c$, $w(f) = 4 + 4b - 2m$.

By (19.2), $v(f) = 2 + 2b - m \geqq 3$, whence $2b > m$. Thus $w(xy^2) = 4 + 4b - 2m$. If $w(f) \neq 4 + 4b - 2m$, then $w(f) = 4 + 4b - 3m$ because $b < m$. Then $w(f)$ is odd and $w(x)$, $w(z)$ are even. Thus $w(f)$ is attained by a term $y\cdot$(monomial of degree $\geqq 2$). But

$$\begin{aligned} &w(y\cdot(\deg \geqq 2)) - (4 + 4b - 3m) \\ &\qquad \geqq 2b - m + 2 - (4 + 4b - 3m) = 2(m - 1 - b) \geqq 0\,, \end{aligned}$$

https://doi.org/10.1017/S0027763000021358 Published online by Cambridge University Press

and if the equality holds, then $b = m - 1$ and $w((\deg \geqq 2)) = 2$. This is impossible by $m \geqq 5$. Thus (19.3) is proved.

(19.4) *Claim*: Under the situation of (19.2), if $m \geqq 5$, then one has $c \geqq (m-1)/2$.

If $c < (m-1)/2$, then by (19.3), Corollary 2.2 applied to $w(f)$ gives $0 < 4 + 2b - m + 2c - m - (4 + 4b - 2m)/2 = 2 + 2c - m \leqq 2 + (m-2) - m = 0$, which is a contradiction. Thus (19.4) is proved.

(19.5) *Claim*: Under the situation of (19.2), one has (1) if $m \geqq 5$ then c is odd and $c \leqq m - 4$, and (2) if $m = 3$ then $c = 1$ and $b = 2$.

Since $v(y\cdot(\deg \geqq 2)) - (2 + 2b - m) = b + 2 - (2 + 2b - m) = m - b > 0$ by $b < m$ (19.2), $v(f)$ is attained by a term containing only x and z. Since $v(f)$ is odd, $v(z) = c$ is odd. Since $2 + 2b - m \leqq m$ and $3c > m$ (19.4), f must contain xz if $c \geqq m - 2$ and $m \geqq 5$. This is absurd, and hence $c \leqq m - 4$. The case $m = 3$ is similar.

Let $c = m - 2e$, $e \geqq 2$. Then e is prime to m by (19.1).

(19.6) *Claim*: Under the situation of (19.2), there is no integer i such that

$$(*) \qquad m/(2e-1) \leqq i \leqq 2m/(2e+1)\,.$$

Assume that such i exists. Then $w' = ((i\cdot\sigma)\text{-}wt)_m$ satisfies $w'(x) = 2i$, $w'(y) = b'$, $w'(z) = 2m - 2ie$, where $b' = (bi)_m$ satisfies $0 < b' < m$ by (19.1). We will derive a contradiction to prove (19.6). From $w'(f) \equiv 2i + 2b'$, $2i < m$, and $2b' < 2m$, follows that $2i + 2b' < 3m$. Thus $w'(f) = 2i + 2b'$, $2i + 2b' - m$, or $2i + 2b' - 2m$. If $w'(f) = 2i + 2b' - m$ (odd), then $w'(f)$ is attained by $x^p y^q z^r$ with $q > 0$ ($w'(x)$, $w'(z)$ are even). Hence

$$2i + 2b' - m = p(2i) + qb' + r(2m - 2ie)\,, \quad \text{or}$$
$$2i + b' = m + (q-1)b' + p\cdot 2i + r(2m - 2ie)\,.$$

Since $2i \leqq m - 1$, $b' \leqq m - 1$ (19.1), one has $q = 1$, $p = 0$. By $p + q + r \geqq 3$, one gets $r \geqq 2$ and $2i + b' > b' + 2(2m - 2ie)$ contradicting $(*)$. If $w'(f) = 2i + 2b' - 2m$ (even), then it is attained by $x^p y^q z^r$, $p + q + r \geqq 3$, and

$$2i + 2b' - 2m = 2ip + b'q + r(2m - 2ie)\,.$$

Since $2i + 2b' - 2m < 2i$, one gets $p = 0$. Also from $2i + 2b' - 2m < 2b' - m < b'$, follows $q = 0$ and $r \geqq 3$. Hence $2i > 2i + 2b' - 2m = r(2m - 2ie)$

contradicting $(*)$ in (19.6). Thus $w'(f) = 2i + 2b'$. Now $w'(f)$ is even, and Corollary 2.2 applied to w' gives a contradiction to $(*)$:

$$0 < 2i + b' + (2m - 2ie) - m - (2i + 2b')/2 = m - (2e - 1)i .$$

(19.7) *Claim*: Under the situation of (19.2), if $m \geqq 5$, then one has $e = 2$, $m = 7$.

We note $m \geqq 4e - 1$ by (19.4), and hence

$$\begin{aligned}(2e - 1)(2e + 1)\{2m/(2e + 1) - m/(2e - 1) - 1\} \\ = (2e - 3)m - 4e^2 + 1 \geqq (2e - 3)(4e - 1) - 4e^2 + 1 = 4e^2 - 14e + 4 .\end{aligned}$$

Thus if $e \geqq 4$, then $2m/(2e + 1) > m(2e - 1) + 1$, and there exists i satisfying $(*)$ in (19.6). Hence $e \leqq 3$. Then it is easy to check (19.7) for $e \leqq 3$ and $m \geqq 5$.

(19.8) *Claim*: The case $e = 2$ and $m = 7$ does not give a terminal singularity.

One has $c = 3$. By (19.2), $2 + 2b - 7 \geqq 3$ and $b \geqq 4$, Since $v(x), v(y), v(z) \geqq 2$, one has by (19.2), $2 + 2b - 7 \geqq 3 \cdot 2$, whence $b = 1$. Then for $w'' = ((3\cdot\sigma)\text{-}wt)_m$, $w''(x) = 6$, $w''(y) = 4$, $w''(z) = 2$, and $w''(f) = 14$, whence $w''(x) + w''(y) + w''(z) - 7 - w''(f)/2 = -2 < 0$, which is a contradiction to Corollary 2.2.

(19.9) *Claim*: If $m = 3$, then f_3 contains z^3.

Otherwise, by (19.2) and (19.5), $v(x) = v(y) = 2$, $v(z) = 1$, one gets $v(f) = 6$, which contradicts (19.2). This proves Lemma 19.

PROPOSITION 20. *Under the situation of Lemma* 14, *if f_3 contains xy^2, then* (1) $m = 2$ *and wts of x, y, z, u, f are* $0, 1, 1, 1, 0 \bmod 2$, *or* (2) $m = 3$ *and after choosing generator σ of Z_3, one has σ-wts of x, y, z, u, f equal to* $2, 2, 1, 0, 0 \bmod 3$, *and f_3 contains xy^2, z^3.*

Proof. By Lemmas 18, 19, it remains to exclude $m = 6$. By Lemma 19, f_3 contains z^3. This implies $3\cdot wt\, z \equiv wt\, f$, which contradicts Lemma 18.

LEMMA 21. *Under the situation of Lemma* 14, *if f_3 does not contain cross terms (like xy^2, xyz), then $f_3 \in \boldsymbol{C}\cdot x^3 + \boldsymbol{C}\cdot y^3 + \boldsymbol{C}\cdot z^3$.*

This is obvious.

LEMMA 22. *Under the situation of Lemma* 21, *if $f_3 = x^3 + y^3 + z^3$ (resp. $x^3 + y^3$), then $m = 3$, and after choosing generator σ of Z_3 and permutation* *of* x, y, z *(resp.* x, y*) if necessary, one has* σ*-wts of* x, y, z, u, φ *equal to* 1, 2, 2, 0, 0 mod 3.

Proof. (22.1) *Claim*: 2 and 3 are the only possible prime factors of m.

Assuming that $(m, 6) = 1$, $m > 1$, we will derive a contradiction. From $wt\,x \equiv wt\,y\ (m)$, one has $(wt\,x, m) = 1$ by Remark 7, (2). Thus one can choose a generator ρ of Z_m such that $\rho\text{-}wt\,x \equiv 2$. Then $\rho\text{-}wts$ of x, y, z, u, φ are $2, 2, c, 3, 6$, where c is prime to m by $(wt\,\varphi, m) = 1$, and Remark 7, (1). Let $v = (\rho\text{-}wt)_m$. Since $v(\sigma) \geqq 2$, $m \geqq 5$, one has $v(\varphi) = 6$. Thus $v(x) + v(y) + v(z) + v(u) - m - v(\varphi) = c - m + 1 \leqq 0$, which contradicts Corollary 2.2.

(22.2) *Claim*: $(m, 2) = 1$.

Indeed if $m = 2$, then $wt\,x \equiv wt\,y \equiv 0\ (2)$. This contradicts Remark 7, (2).

(22.3) *Claim*: $m = 3$.

We will derive a contradiction assuming $m = 9$. From $2 \cdot wt\,u \equiv 3 \cdot wt\,y\ (9)$, follows $wt\,u \equiv 0\ (3)$. By Remark 7, (2), $wt\,x$, $wt\,y$, $wt\,z$ are prime to 3. Thus $wt\,u \not\equiv 0\ (9)$. If $wt\,x \equiv wt\,y\ (9)$, then Z_9-automorphism: $x \to x + \alpha y$, keeping y, z, u reduces the problem to Proposition 20, which contradicts $m = 9$. Thus $wt\,x \equiv wt\,y\ (3)$, $wt\,x \not\equiv wt\,y\ (9)$. Choosing a generator $\rho \in Z_9$, one may assume $\{\rho\text{-}wt\,x, \rho\text{-}wt\,y\} = \{2 \bmod 9, 5 \bmod 9\}$. By exchanging x, y if necessary, one may assume that $\rho\text{-}wts$ of x, y, z, u are $2, 5, c, 3$ mod 9 (c is prime to 3 by Remark 7, (2)). By applying Theorem 2 to $v = (\rho\text{-}wt)_9$, one gets $v(f) = 6$, $c > 5$. For $w = ((-\rho)\text{-}wt)_9$, $w(f) \equiv 3\ (9)$. By Theorem 2, one gets $w(x) + w(y) + w(z) + w(u) = 26 - c > 9 + w(f)$. Thus by $c > 5$, $w(f) = 3$. Then z^3 appears in f and $w(z) = 1$, $c = 8$. Thus $(-3\rho)\text{-}wts$ of x, y, z, u are $3, 3, 3, 0 \bmod 9$. Hence for $w' = ((-3\rho)\text{-}wt)_9$, $w'(\varphi) \geqq 9$, and $w'(x) + w'(y) + w'(z) + w'(u) \leqq 9 + w'(\varphi)$. This contradicts Theorem 2, and $m = 9$ is excluded.

(22.4) Since $wt\,u \equiv 0\ (3)$, $wt\,x$, $wt\,y$, $wt\,z$ are prime to 3. If $wt\,x \equiv wt\,y \equiv wt\,z\ (3)$, then we can choose wt so that $wt\,x \equiv 1$. Then setting orders of x, y, z, u equal to $1, 1, 1, 3$, one has an order v such that $v(\varphi) = 3$. Then $v(x) + v(y) + v(z) + v(u) = v(\varphi) + 3$, which contradicts Theorem 2. If $f_3 = x^3 + y^3 + z^3$, then after permutation of x, y, z and change of weight, one gets wts of x, y, z, u, φ equal to $1, 2, 2, 0, 0$. If $f_3 = x^3 + y^3$, then after permutation of x, y if necessary and change of weight, one gets wts of x, y, z, u, φ equal to $2, b, 1, 0, 0$, $(b = 1, 2) \bmod 3$. If $b = 2$, then $2, 2, 1, 3$

for x, y, z, u gives order w such that $w(\varphi) = 6$ because φ does not contain z^3. Then $w(x) + w(y) + w(z) + w(u) - 3 - w(\varphi) = -1$, which contradicts Theorem 2. Thus $b = 1$. Changing wt to $-wt$, one gets Lemma 22.

THEOREM 23. *Under the situation of Lemma 14, if f_3 is not a cube of a linear factor, then after permutation of x, y, z and choice of generator σ of Z_m, one of the following holds.*

(1) $m = 2$, wts of x, y, z, u, φ are $1, 1, 0, 1, 0 \bmod 2$, and xyz or y^2z appears in f_3.

(2) $m = 3$, σ-wts of x, y, z, u, are $1, 2, 2, 0, 0 \bmod 3$, and $f_3 = x^3 + y^3 + z^3$, $x^3 + yz^2$, or $x^3 + y^3$ modulo Z_3-automorphism of $C\{x, y, z\}$.

This follows from Propositions 17, 20, and Lemmas 21, 22.

Remark 23.1. In cases (1), (2) of Theorem 23, it is easy to see that modulo Z_m-automorphism of $C\{x, y, z, u\}$, one may put φ in one of the following forms by Theorem 3:

Case 1. $m = 2$, wts of x, y, z, u are $1, 1, 0, 1 \bmod 2$,

$$\varphi = u^2 + xyz + x^{2a} + y^{2b} + z^c\,, \tag{23.1.1}$$

$$\varphi = u^2 + y^2z + \lambda yx^{2a-1} + g\,, \tag{23.1.2}$$

$$(a, b \geqq 2,\ c \geqq 3,\ \lambda \in k,\ g \in (x^4, x^2z^2, z^3)C\{x^2, z\})\,,$$

Case 2. $m = 3$, wts of x, y, z, u are $1, 2, 2, 0 \bmod 3$,

$$\varphi = u^2 + x^3 + y^3 + z^3\,, \tag{23.2.1}$$

$$\varphi = u^2 + x^3 + yz^2 + xy^4\cdot\lambda + y^6\cdot\mu\,, \tag{23.2.2}$$

$$\varphi = u^2 + x^3 + y^3 + xyz^3\cdot\alpha + xz^4\cdot\beta + yz^5\cdot\gamma + z^6\cdot\delta\,, \tag{23.2.3}$$

$$(\lambda, \mu \in C\{y^3\},\ 4\lambda^3 + 27\mu^2 \neq 0,\ \alpha, \beta, \gamma, \delta \in C\{z^3\})\,.$$

For (23.1.1) and (23.2.1), (X, p) is terminal. (This follows from Corollary 2.1.) If φ is a general linear combination of a finite number of monomials as in (23.1.2), (23.2.2), or (23.2.3), and if φ has an isolated singularity at 0, then (X, p) is terminal. (This also follows from Corollary 2.1.)

The way to put φ in the standard forms above is as follows. If $m = 2$ and f_3 contains xyz, then after operating Z_2-automorphism, one may assume that $f_3 = xyz + \lambda z^3\ (\lambda \in k)$. If xyh (resp. if it is not reduced to (23.1.2) yzh, zxh) appears in f for some monomial $h \in (x, u, z)^2$, then Z_2-transformation $z \to z + \lambda h$ (resp. $x \to x + \lambda h$, $y \to y + \lambda h$) kills xyh (resp. yzh, zxh) for some λ. Thus for any $n \gg 0$, there is a Z_2-automorphism ψ such that $\varphi \equiv \psi(u^2 +$

$xyz + x^{2a}\alpha(x) + y^{2b}\beta(y) + z^c\gamma(z))$ modulo $(x, y, z, u)^n$, (α, β are even power series; $\alpha(0)\beta(0)\gamma(0) \neq 0$; $\infty \geqq a, b, c$; $a, b \geqq 2$; $c \geqq 3$). Thus, by Theorem 3, there is a Z_2-automorphism ψ such that

$$\varphi = \psi(u^2 + xyz + x^{2a}\alpha + y^{2b}\beta + z^c\gamma) .$$

Hence one may assume

$$\varphi = u^2 + xyz + x^{2a}\alpha + y^{2b}\beta + z^c\gamma .$$

Since $\varphi = 0$ has an isolated singularity, one sees $a, b, c < \infty$. It is easy to see that there are invariant units u_1, u_2, u_3, u_4 of $C\{x, y, z\}$ such that

$$u_4^2 = u_1u_2u_3 = u_1^{2a}\alpha = u_2^{2b}\beta = u_3\gamma .$$ Then Z_2-automorphism

τ of $C\{x, y, z, u\}$ such that $\tau x = u_1x$, $\tau y = u_2y$, $\tau z = u_3z$, $\tau u = u_4u$ satisfies

$$\tau\varphi = u_4^2(u^2 + xyz + x^{2a} + y^{2b} + z^c) .$$

Thus (23.1.1) is obtained, and other cases are similar.

§ 4. Terminal singularities of type *cE*

LEMMA 24. *Under the situation of Lemma* 14, *if* $f_3 = x^3$, *then modulo* Z_m-*automorphism*,

$$\varphi = u^2 + x^3 + g(y, z)x + h(y, z) ,$$

where $g, h \in C\{y, z\}$, $g \in (y, z)^3$, $h \in (y, z)^4$.

This is obvious.

THEOREM 25. *Under the situation of Lemma* 24, *one has* $m = 2$ *and wts of* x, y, z, u, φ *are* $0, 1, 1, 1, 0 \bmod 2$, *and* $h \notin (y, z)^5$.

Remark 25.1. If $m = 2$ and *wts* of x, y, z, u are $0, 1, 1, 1 \bmod 2$ and if an even polynomial g (resp. h) in y, z is a general linear combination of a finite number of monomials $\in (y, z)^4$ and $h \notin (y, z)^5$, and if

$$\varphi = u^2 + x^3 + g(y, z)x + h(y, z)$$

has an isolated singularity at 0, then (X, p) is terminal. (This follows from Corollary 2.1.)

Proof. For an integer $n \geqq 0$, let g_n (resp. h_n) be the homogeneous part of degree n of g (resp. h). We will treat two cases.

Case 1. m is an odd prime.

In this case, we will derive a contradiction in several steps.

(25.2) *Claim*: $wt\,x$, $wt\,y$, $wt\,z$ are prime to m.

If (*e.g.*) $wt\,x \equiv 0\ (m)$, then by Remark 7, (1), $wt\,f \equiv 0\ (m)$ and $wt\,u \equiv 0$, which contradicts Remark 7, (2).

(25.3) *Claim*: One may assume that m is a prime number > 7.

It is enough to derive a contradiction assuming $m \leqq 7$. By (25.2), let ρ be a generator Z_m of such that $\rho\text{-}wt\,x \equiv 2\ (m)$. Since $2\,wt\,u \equiv 3 \cdot wt\,x\ (m)$, $v = (\rho\text{-}wt)_m$ satisfies $v(f) = 6$. Let $b = v(y)$, $c = v(z)$. Then by Corollary 2.2, $2 + b + c - m - 3 > 0$ and $b + c \geqq m + 2$. Let $w = ((-\rho)\text{-}wt)_m$. One has $w(x) = m - 2$, $w(y) = m - b$, $w(z) = m - c$. Since $w(x^3) = 3m - 6$, $w(f) = 3m - 6$ (odd) or $2m - 6$ (even) because $w(f) \geqq 3$. Thus in any case, by Corollary 2.2, one has $(m - 2) + (m - b) + (m - c) > 2m - 3$. Hence $m + 1 > b + c \geqq m + 2$. This is a contradiction.

(25.4) *Claim*: $wt\,y - wt\,z$ is prime to m.

It is enough to derive a contradiction assuming $wt\,y \equiv wt\,z\ (m)$. Since φ defines a cDV point, one of h_4, g_3, h_5 is non-zero. If $h_4 \neq 0$, then after choosing generator σ of Z_m, one has σ-wts of x, y, z equal to $4, 3, 3 \bmod m$. Thus Corollary 2.2 applied to the induced order gives $0 < 4 + 3 + 3 - m - 6 = 4 - m$, which contradicts (25.3). The other cases are the same; if $g_3 \neq 0$ (resp. $h_5 \neq 0$), then one may assume that the wts of x, y, z are $3, 2, 2$ (resp. $5, 3, 3$) $\bmod m$. Hence for the induced order w, $w(f) = 9$ (resp. 15 by (25.3)), which contradicts (25.3), by Corollary 2.2.

(25.5) g_3, h_4, h_5 are monomials (or 0). (by (25.3) and (25.4))

(25.6) *Claim*: None of g_3, h_4, h_5 are powers of y, or z (up to constants).

By symmetry, enough to show that none of them are powers of y. If $h_4 = y^4$ (up to non-zero constants), then one can choose an order v such that $v(x) = 4$, $v(y) = 3$, $v(z) = c$ $(0 < c < m)$. Then $v(f) = 12$, because $12 < m + 3$. Then $0 < 3 + 4 + c - m - 6 = 1 + c - m$ (Corollary 2.2), which is a contradiction. If $g_3 = y^3$ (up to non-zero constants), then one can choose an order v such that $v(x) = 6$, $v(y) = 4$, $v(z) = c$ $(0 < c < m)$. Then $v(f) = 18$ or $18 - m$. If $v(f) = 18$, then Corollary 2.2 gives a contradiction; $0 < 6 + 4 + c - m - 9 = 1 + c - m$. If $v(f) = 18 - m$, then $18 - m\ (\leqq 7)$ is a sum of at least 4 numbers which are $4 = v(y)$ or $c = v(z)$. Thus $c = 1$. Hence one can choose an order w $(-2w \equiv v\,(m))$ such that $w(x) = m - 3$, $w(y) = m - 2$, $w(z) = (m - 1)/2$. Then $w(f) = 3m - 9$, $2m - 9$, or $m - 9$,

https://doi.org/10.1017/S0027763000021358 Published online by Cambridge University Press

and order of an arbitrary monomial in x, y, z of degree $\geqq 4$ is at least $4\cdot(m-1)/2 = 2m-2 > 2m-9$. Thus $w(f) = 3m-9$ (even). Corollary 2.2 gives $0 < (m-3)+(m-2)+(m-1)/2 - m - (3m-9)/2 = -1$, which is a contradiction. If $h_5 = y^5$ (up to constants), then one can choose an order v such that $v(x) = 10$, $v(y) = 6$, $v(z) = c$ $(0 < c < m)$. If $c \geqq 6$, then $v(f) = 30$ $(30 - m < 4\cdot 6)$ and Corollary 2.2 gives a contradiction; $0 < 10 + 6 + c - m - 15 = c + 1 - m$. If $c = 1, 3$, or 5, then one can choose an order w $(-2w \equiv v\,(m))$ such that $w(x) = m-5$, $w(y) = m-3$, $w(z) = (m-c)/2$. For a monomial M in x, y, z of degree $\geqq 4$, one has $w(M) \geqq 4\cdot(m-c)/2 = 2m-2c > 2m-15$. Thus $w(f) = 3m-15$ (even) by $w(x^3) = 3m-15$. Then Corollary 2.2 gives a contradiction; $0 < (m-5) + (m-3) + (m-c)/2 - m - (3m-15)/2 = -(c+1)/2$. If $c = 2$, or 4, then one can choose an order w $(2w \equiv v\,(m))$ such that $w(x) = 5$, $w(x) = 3$, $w(z) = c/2$. One has $w(x^3) = 15$ and $m \geqq 11$ by (25.3), hence $w(f) = 15$ (odd) or $15 - m$ (even). Then, in any case, Corollary 2.2 gives contradiction; $0 < 5 + 3 + c/2 - (m+15)/2 < (5-m)/2$.

(25.7) *Claim*: One chooses an integer $n \geqq 1$ such that $12n + 5 \geqq m \geqq 12n - 5$. Let v be an order such that $v(x) = 4n$, $v(y) = b$, $v(z) = c$ $(0 < b, c < m)$. Then $v(f) = 12n$.

Otherwise $v(f) = 12n - m \leqq 5$. Since $v(x) \geqq 4$, $w(f)$ is attained by h_4 or h_5. By (25.4), (25.5), (25.6), $w(f)$ must be attained by h_4, $12n - m = 5$, $(b, c) = (1, 2)$ or $(2, 1)$. Corollary 2.2 shows $0 < 4n + b + c - m/2 - 5/2 = b + c - 2n = 3 - 2n$. Thus $n = 1$ and $m = 7$, which is a contradiction to $m > 7$.

(25.8) *Claim*: $b + c > m + 2n$.

This follows from Corollary 2.2 applied to order v in (25.7); $0 < 4n + b + c - m - 6n = b + c - m - 2n$.

(25.9) *Claim*: Let w be an order $(w \equiv -v\ (m)$, with v in (25.7)) such $w(x) = m - 4n$, $w(y) = m - b$, $w(z) = m - c$. Then $w(f) = 3m - 12n$ of $2m - 12n$.

Otherwise, $w(f) = m - 12n \leqq 5$, because $w(x^3) = 3m - 12n$. Since $w(x) = m - 4n \geqq 12n - 5 - 4n = 8n - 5 \geqq 3$, $w(f)$ is attained by h_4 or h_5. Since $w(f)$ is odd, $m - 12n = 5$, and $(m-b) + (m-c) = 3$ by (25.4), (25.6). Let w' be an order $(w' \equiv 3w\,(m))$ such that $w'(x) = 5$, $w'(y) = 3(m-b)$, $w'(z) = 3(m-c)$. Then $w'(f) = 15$, and Corollary 2.2 shows $0 < 5 + 3(m-b) + 3(m-c) - m/2 - 15/2 = 5 + 9 - m/2 - 15/2 = (13-m)/2$. Thus $m = 11$

$= 12n + 5$, which is a contradiction.

(25.10) By (25.9), Corollary 2.2, applied to w shows that $0 < (m - 4n) + (m - b) + (m - c) - (2m - 6n) = m + 2n - b - c$. This contradicts (25.8). Thus Case 1 is finished and m is a power of 2.

Case 2. m is a power of 2.

(25.11) *Claim*: $wt\, x \equiv 0$, $wt\, u$, $wt\, y$, $wt\, z \equiv 1$ (2).

By $2 \cdot wt\, u \equiv 3 \cdot wt\, x$ (2), one has $wt\, x \equiv 0$ (2), and the rest follows from Remark 7, (2).

(25.12) *Claim*: $g_3 = h_5 = 0$, $h_4 \neq 0$.

Since $wt\, g_3 \equiv wt\, h_5 \equiv 0$ (2), one has $g_3 = h_5 = 0$ by (25.11). Since φ defines a cDV point, $h_4 \neq 0$.

(25.13) *Claim*: $m = 2$.

It is enough to derive a contradiction assuming $m = 4$. Since $h_4 \neq 0$, one has $3 \cdot wt\, x \equiv wt\, h_4 \equiv 2 \cdot wt\, u \equiv 2$ (4), hence $wt\, x \equiv 2$, $wt\, y \not\equiv wt\, z$ (4). Thus by (25.11), $wt\, y + wt\, z \equiv 0$ (4). Let ρ be a generator of Z_4 such that ρ-wts of x, y, z, u are $2, b, c, 3$ ($0 < b, c < 4$). Then $b + c = 4$. Let $v = (\rho\text{-}wt)_4$. Then $v(f) = 6$, and Theorem 2 gives contradiction; $0 < 2 + b + c + 3 - 4 - 6 = -1$. Thus Case 2 is finished and Theorem 25 is proved.

References

[1] M. Artin, Algebraic approximation of structures over complete local rings, Publication I.H.E.S., no. 36.

[2] M. Artin, Lectures on deformations of singularities, Bombay 1976.

[3] V. Danilov, Birational geometry of toric 3-folds, Izv. Akad. Nauk SSSR ser. Mat., **46** (1982), no. 5.

[4] D. Morrison and G. Stevens, Terminal quotient singularities in dimensions three and four, preprint.

[5] M. Reid, Canonical 3-folds, Journées de géometrie algébrique d'Angers, ed. A. Beauville, Sijthoff and Noordhoff, Alphen (1980), 273–310.

[6] ——, Minimal models of canonical 3-folds, Advanced studies in pure math. **1**, ed. S. Iitaka and H. Morikawa, Kinokuniya and North Holland, 1983, 131–180.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464
Japan